# CAAM 335 Matrix Analysis
# Additional Notes for Symmetric Eigenvalue Problem

A matrix $U \in \mathbb{C}^{n\times n}$ is called *unitary* if it satisfies

$$U^*U = I, \tag{1}$$

which implies that $U^{-1} = U^*$, and that the columns (and rows) of $U$ are orthogonal to each other and are of unit length (in 2-norm). The transformation $U(\cdot)U^*$, or $U^*(\cdot)U$, is called a unitary transformation which is a special case of similarity transformations.

**Theorem 1** (Schur Decomposition)**.** *For any square matrix $A \in \mathbb{C}^{n\times n}$, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = UTU^* \quad (\text{or } U^*AU = T) \tag{2}$$

*where $T \in \mathbb{C}^{n\times n}$ is upper triangular.*

This result says that if we restrict similarity transformations to unitary transformations, then we will not be able to transform a general square matrix to a Jordan form but to a triangular form which does not look as pretty as a Jordan form. However, a Schur decomposition still serves the purpose of exposing the eigenvalues of $A$ on the diagonal of $T$.

The computation of the Schur decomposition is numerically stable in the sense that one can manage to control the growth of error, which is not the case for computing the Jordan form. For this and other reasons, unitary transformations are the main tool in computing eigenvalues on computer.

A matrix $A \in \mathbb{C}^{n\times n}$ is *normal* if it satisfies

$$A^*A = AA^*. \tag{3}$$

For example, Hermitian matrices ($A^* = A$, including real symmetric matrices) are normal; so are skew symmetric matrices and unitary matrices.

A square matrix is unitarily diagonalizable if $T = D$ in its Schur decomposition where $D$ is diagonal.

**Theorem 2** (Normality and Unitary Diagonalization)**.** *A matrix $A \in \mathbb{C}^{n\times n}$ is unitarily diagonalizable if and only if it is normal.*

*Proof.* The necessity can be verified directly using the decomposition $A = UDU^*$ and noting that diagonal matrices commute in multiplications.

We verify the sufficiency for Hermitian matrices only. The Schur form $A = UTU^*$ and $A^* = A$ together imply that $T = T^*$; hence not only $T$ must be diagonal but the diagonal must be real. □

Consequently, we see that real symmetric matrices enjoy the following nice properties:

1. They are unitarily diagonalizable.

2. Their eigenvalues are all real.

3. They have a set of orthonormal eigenvectors.

In fact, the eigenvectors can be chosen to be real, and the unitary matrix $U$ can be replaced by a real orthogonal matrix $Q$ so that $Q^TQ = I$ or $q_i^T q_j = \delta_{ij}$ where $q_j$'s are the columns of $Q$.

For a real symmetric matrix $A$, we can expand the expression $A = QDQ^T$ into

$$A = \sum_{k=1}^{n} \lambda_k q_k q_k^T = \sum_{j=1}^{h} \lambda_j \left( \sum_{\lambda_k = \lambda_j} q_k q_k^T \right) = \sum_{j=1}^{h} \lambda_j P_j. \tag{4}$$

Unlike for the general case, this time the matrices $P_j$'s are *orthogonal* projections because they are not only idempotent ($P_j^2 = P_j$) but also symmetric ($P_j^T = P_j$). Naturally, we still have $P_iP_j = 0$ for $i \neq j$ and $\sum_{j=1}^{h} P_j = I$.

Using the above properties of $P_j$'s, we can decompose $\Re^n$ into an orthogona direct sum of the eigenspaces $\mathcal{R}(P_j)$'s (each of which is spanned by eigenvectors associated with an distinct eigenvalue)

$$\Re^n = \mathcal{R}(P_1) \oplus \mathcal{R}(P_2) \oplus \cdots \oplus \mathcal{R}(P_h). \tag{5}$$

These eigenspaces are *invariant* subspaces of $A$ defined by the property

$$A\mathcal{R}(P_j) \subseteq \mathcal{R}(P_j);$$

i.e., if $x \in \mathcal{R}(P_j)$, then so is $Ax$, which is obvious since $Ax = \lambda_j x$.

Recall that for a defective matrix $A$ and its Jordan canonical form, we may have

$$AP_j = \lambda_j P_j + D_j$$

for $D_j \neq 0$. Therefore, $\mathcal{R}(P_j)$ is not necessarily an eigenspace (though it can be called a generalized eigenspace) for a defective matrix. This can never happen to symmetric matrices because they are all nondefective (diagonalizable).