CAAM 335 Matrix Analysis
Additional Notes for Eigenvalue Problem

Consider $A \in \mathbb{C}^{n\times n}$ and its characteristic polynomial

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{a_1} \cdots (\lambda - \lambda_j)^{a_j} \cdots (\lambda - \lambda_h)^{a_h} \tag{1}$$

where $a_j \geq 1$ are integers so that $\sum_{j=1}^{h} a_j = n$. An eigenvalue-vector pair of A satisfies

$$Ax_j = \lambda_j x_j \tag{2}$$

where $\lambda_j \in \mathbb{C}$, and $x_j \in \mathbb{C}^n$ and $x_j \neq 0$. The vectors x_j's are also called *right eigenvectors* of A since they appear on the right side of A in the above equation.

The *algebraic multiplicity* of a distinct eigenvalue λ_j is $a_j \geq 1$. The *geometric multiplicity* of λ_j, say g_j, is the number of independent eigenvectors associated with λ_j. It holds that

$$1 \leq g_j \leq a_j, \quad j = 1, 2, \cdots, h.$$

If A has any eigenvalue λ_j with $g_j < a_j$, then A is called *defective*; otherwise, it is *nondefective*. Clearly, A is nondefective when it has n distinct eigenvalues, i.e., $a_j = 1$ for all j; though A still can be nondefective even $a_j > 1$ for some or all j.

The equations in (2) can be collectively written as

$$AX = [x_1\lambda_1 \; x_2\lambda_2 \; \cdots \; x_n\lambda_n] = XD$$

where $X = [x_1 \; x_2 \; \cdots \; x_n]$, and $D = \text{diag}(\lambda_j)$ is the diagonal matrix with the eigenvalues on its diagonal (with λ_j repeating a_j times).

A nondefective matrix has n independent eigenvectors in X, which is necessary and sufficient for A to be *diagonalizable* by a similarity transformation

$$X^{-1}AX = D. \tag{3}$$

In the sequel, we assume that A is diagonalizable. Let $Y^* = X^{-1}$, then (3) becomes

$$Y^*A = DY^*, \tag{4}$$

which in turn is equivalent to, upon denoting $Y = [y_1 \; y_2 \; \cdots \; y_n]$,

$$y_i^*A = \lambda_i y^*. \tag{5}$$

For obvious reason, we call the vectors $y_j \in \mathbb{C}^n$ the *left-eigenvectors* of A. The conjugate transposes of the left-eigenvectors are simply the rows of X^{-1}. Taking a conjugate transpose of (5), we have

$$A^* y_i = \bar{\lambda}_i y_i. \tag{6}$$

Therefore, the left-eigenvectors of A are the right-eigenvectors of A^* and the eigenvalues of A^* are $\bar{\lambda}_i$'s. If $A^* = A$ (Hermitian or real symmetric), then the left-eigenvectors of A are just the right-eigenvectors of A, and the eigenvalues must be all real, i.e., $\lambda_i = \bar{\lambda}_i$ for all j.

Since $Y^*X = X^{-1}X = I$, we have

$$y_i^* x_j = \delta_{ij} \equiv \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{7}$$

For simplicity we first assume that all the eigenvalues are distinct. Now we expand the equation

$$A = XDX^{-1} = XDY^* = \sum_{j=1}^{n} \lambda_j x_j y_j^* = \sum_{j=1}^{n} \lambda_j P_j, \tag{8}$$

where

$$P_j = x_j y_j^*, \quad j = 1, \cdots, n. \tag{9}$$

It follows from (7) and recall that x_j's and y_j's the right- and left-eigenvectors of A that

$$P_i P_j = \delta_{ij} P_i, \quad \sum_{j=1}^{n} P_j = I, \quad AP_j = \lambda_j P_j = P_j A. \tag{10}$$

(**Compare the above results with the spectral representation (9.20) in the Lecture Notes for the case of all distinct eigenvalues where all** $D_j = 0$.)

If an eigenvalue λ_j has algebraic multiplicity $a_j > 1$, but A is still nondefective, we can collect all P_k's associated with λ_j to form a new P_j that is not rank-1 but rank-a_j. Yet, one can verify that the above properties still hold for such P_j's.

Now let A be defective with h distinct eigenvalues. Then with similarity transformations the closest thing one can get is to transform A to a block-diagonal matrix J of the form

$$X^{-1}AX = J \equiv \begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_p \end{bmatrix} \tag{11}$$

where J_k's are called Jordan blocks each of which has an eigenvalue on the diagonal and 1's on the super-diagonal (unless the block size is 1); e.g.,

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix},$$

and the total number of Jordan blocks p equal to the number of independent eigenvectors, i.e., the sum of the geometric multiplicities, $p = \sum_{j=1}^{h} g_j \leq n$. In this form, each eigenvalue λ_j is associated with g_j Jordan blocks with aggregated size of $a_j \times a_j$.

The equation in (11) is called the *Jordan canonical (or normal) form* for a general matrix $A \in \mathbb{C}^{n\times n}$. When A is nondefective, then since $a_j = g_j$ for all j, λ_j has g_j Jordan blocks and all the blocks must be 1 by 1. Hence, all the 1's on the super-diagonal all disappear, and J is actually diagonal. Consequently, a nondefective matrix is diagonalizable.

For example, if eigenvalue λ_1 has algebraic multiplicity $a_1 = 3$ and geometric multiplicity $g_1 = 1$, then there is only 1 Jordan block in J that belongs to λ_1, say,

$$J_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $g_1 = 2$, then λ_1 will appear in 2 blocks; one is 1 by 1, another 2 by 2; say,

$$J_1 = [\lambda_1], \quad J_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

where the order of the blocks are not unique.

Now assume that $A \in \mathbb{C}^{3\times 3}$ have the Jordan canonical form $A = XJX^{-1}$ where J is specified below. Let $Y^* = X^{-1}$, $X = [x_1\ x_2\ x_3]$ and $Y = [y_1\ y_2\ y_3]$ where $x_j, y_j \in \mathbb{C}^3$. We do the expansion

$$A = XJX^{-1} = X \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix} Y^* = \lambda_1(x_1 y_1^*) + \lambda_2(x_2 y_2^* + x_3 y_3^*) + (x_2 y_3^*),$$

or

$$A = \lambda_1 P_1 + \lambda_2 P_2 + D_1 + D_2, \tag{12}$$

where

$$P_1 = x_1 y_1^*,\ P_2 = x_2 y_2^* + x_3 y_3^*,\ D_1 = 0,\ D_2 = x_2 y_3^*. \tag{13}$$

Again,

$$Y^* X = I \quad \Leftrightarrow \quad y_i^* x_j = \delta_{ij}, \quad i, j = 1, 2, 3.$$

Using the above, we can verify the following properties: for $i, j = 1, 2$,

$$P_i P_j = \delta_{ij} P_i,\ P_1 + P_2 = I,\ D_i D_j = 0,\ D_j^j = 0 \tag{14}$$

$$P_j D_j = D_j = D_j P_j,\ P_1 D_2 = P_2 D_1 = 0, \tag{15}$$

$$AP_1 = \lambda_1 P_1, \quad AP_2 = \lambda_2 P_2 + D_2. \tag{16}$$

(**Again compare with (9.20)**)

Detailed treatments of Jordan canonical form can be found in most advanced linear algebra books. Nevertheless, diagonalizable matrices are numerically more tractable and practically more useful in most applications.