

Matrix Inverse

A square matrix $S \in \mathbb{R}^{n \times n}$ is invertible if there exists a matrix $S^{-1} \in \mathbb{R}^{n \times n}$ such that

$$S^{-1}S = I \quad \text{and} \quad SS^{-1} = I.$$

The matrix S^{-1} is called the inverse of S .

- ▶ An invertible matrix is also called non-singular.
A matrix is called non-invertible or singular if it is not invertible.
- ▶ A matrix $S \in \mathbb{R}^{n \times n}$ cannot have two different inverses.
In fact, if $X, Y \in \mathbb{R}^{n \times n}$ are two matrices with $XS = I$ and $SY = I$, then

$$X = XI = X(SY) = (XS)Y = IY = Y.$$
- ▶ If $S \in \mathbb{R}^{n \times n}$ is invertible, then $Sx = f$ implies $x = S^{-1}Sx = S^{-1}f$, i.e., for every f the linear system $Sx = f$ has a solution $x = S^{-1}f$.
The linear system $Sx = f$ cannot have more than one solution because $Sx = f$ and $Sy = f$ imply $S(x - y) = Sx - Sy = f - f = 0$ and $x - y = S^{-1}0 = 0$.
Hence if S is invertible, then for every f the linear system $Sx = f$ has the unique solution $x = S^{-1}f$.
- ▶ We will see later that if for every f the linear system $Sx = f$ has a unique solution x , then S is invertible.

Computation of the Matrix Inverse

We want to find the inverse of $S \in \mathbb{R}^{n \times n}$, that is we want to find a matrix $X \in \mathbb{R}^{n \times n}$ such that $SX = I$.

- ▶ Let $X_{:,j}$ denote the j th column of X , i.e., $X = (X_{:,1}, \dots, X_{:,n})$. Consider the matrix-matrix product SX . The j th column of SX is the matrix-vector product $SX_{:,j}$, i.e., $SX = (SX_{:,1}, \dots, SX_{:,n})$. The j th column of the identity I is the j th unit vector $e_j = (0, \dots, 0, 1, 0, \dots, 0)^T$. Hence $SX = (SX_{:,1}, \dots, SX_{:,n}) = (e_1, \dots, e_n) = I$ implies that we can compute the columns $X_{:,1}, \dots, X_{:,n}$ of the inverse of S by solving n systems of linear equations

$$SX_{:,1} = e_1,$$

$$\vdots$$

$$SX_{:,n} = e_n.$$

Note that if for every f the linear system $Sx = f$ has a unique solution x , then there exists a unique $X = (X_{:,1}, \dots, X_{:,n})$ with $SX = I$.

Want inverse of

$$S = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}.$$

Use Gaussian Elimination to solve the systems

$SX_{:,1} = e_1, SX_{:,2} = e_2, SX_{:,3} = e_3$ for the three columns of $X = S^{-1}$

$$\begin{aligned} & \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array} \right) \\ & \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array} \right) \\ & \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array} \right). \end{aligned}$$

$$S^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & -1 & 0 \\ 6 & -1 & -1 \end{pmatrix}.$$

LU-Decomposition

Consider

$$S = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}.$$

Express Gaussian Elimination using Matrix-Matrix-multiplications

$$\begin{aligned} & \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}}_{=E_1} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{=E_1S} \\ & \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{=E_2} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{=E_1S} = \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=E_2E_1S=U} \end{aligned}$$

Inverses of E_1 and E_2 can be easily computed:

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

We have

$$E_2 E_1 S = U$$

Hence

$$E_1 S = E_2^{-1} U, \quad \text{and} \quad S = E_1^{-1} E_2^{-1} U.$$

$$\begin{aligned} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}}_{=S} &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}}_{=E_1^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{=E_2^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U} \\ &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}}_{=E_1^{-1} E_2^{-1} = L} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U} \end{aligned}$$

Hence we have the **LU-Decomposition** of S ,

$$S = LU,$$

where L is a lower triangular matrix and U is an upper triangular matrix.

In Matlab compute using `[L,U]=lu(S)`.

If we have computed the LU decomposition

$$S = LU,$$

then we can use it to solve

$$Sx = f.$$

We replace S by LU ,

$$LUx = f,$$

and introduce $y = Ux$. This leads to the two linear systems

$$Ly = f \quad \text{and} \quad Ux = y.$$

Since L is lower triangular and U is upper triangular, these two systems can be easily solved.

Example:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}}_{=L} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U}, \quad f = \begin{pmatrix} -4 \\ -6 \\ -15 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}}_{=L} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -4 \\ -6 \\ -15 \end{pmatrix}}_{=f} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}}_{=y} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

In Matlab the matrix inverse is computed using the LU decomposition.

Given S , we want to compute S^{-1} . Recall that the columns $X_{:,1}, \dots, X_{:,n}$ of the inverse $S^{-1} = X$ are the solutions of

$$\begin{aligned} SX_{:,1} &= e_1, \\ &\vdots \\ SX_{:,n} &= e_n. \end{aligned}$$

If we have computed the LU decomposition

$$S = LU,$$

then we can use it to solve the n linear systems $SX_{:,j} = e_j, j = 1, \dots, n$.

Use the LU decomposition to compute the inverse of

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}}_{=L} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U}$$

What do you think of the following approach to solve $Sx = f$:

```
>> Sinv = inv(S);
>> x = Sinv*f;
```

Execute the following:

```
n = input('Problem size = ');
S = rand(n,n);
f = rand(n,1);
```

```
ntry = 50;
```

```
tic
for i = 1:ntry
    x = S\f;
end
toc
```

```
tic
for i = 1:ntry
    Sinv = inv(S);
    x = Sinv*f;
end
toc
```

```
tic
[L,U] = lu(S);
for i = 1:ntry
    x = U\(L\f);
end
toc
```