CAAM 335 Matrix Analysis – Planar Trusses

September 11, 2009

We consider trusses with m bars and n nodes. Each node can be displaced in horizontal and vertical discrection. If the node number is j, then its horizontal displacement is denoted by x_{2j-1} and its vertical displacement is denoted by x_{2j}.

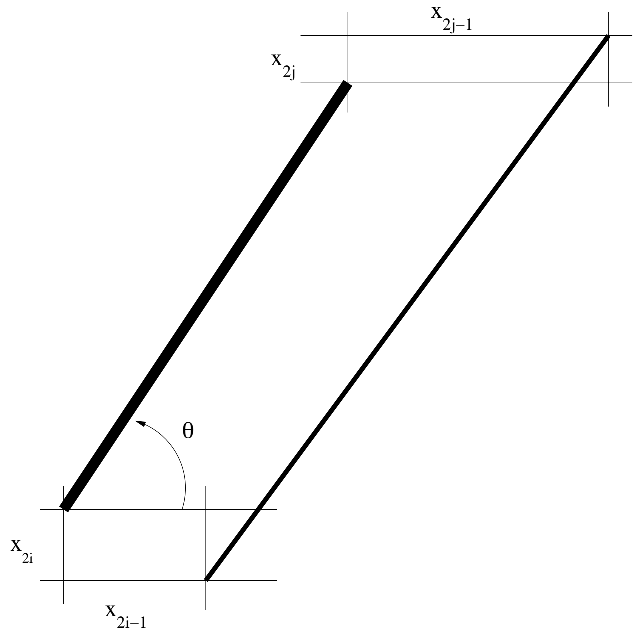

Figure 1: Elongation of a generic bar.

Now consider the generic bar in Figure 1. If ℓ is the length of the undeformed bar and $\ell + e$ is the length of the deformed bar, then, by Pythagoras' theorem (and noting that positive vertical displacements are downward),

$$\begin{aligned}(\ell+e)^2 &= (\ell\cos(\theta)+x_{2j-1}-x_{2i-1})^2+(\ell\sin(\theta)+x_{2i}-x_{2j})^2\\ &= \ell^2+2\ell(\cos(\theta)x_{2j-1}-\cos(\theta)x_{2i-1}+\sin(\theta)x_{2i}-\sin(\theta)x_{2j})\\ &\quad +(x_{2j-1}-x_{2i-1})^2+(x_{2i}-x_{2j})^2\\ &= \ell^2\Big(1+2(\cos(\theta)x_{2j-1}-\cos(\theta)x_{2i-1}+\sin(\theta)x_{2i}-\sin(\theta)x_{2j})/\ell\\ &\qquad +[(x_{2j-1}-x_{2i-1})^2+(x_{2i}-x_{2j})^2]/\ell^2\Big)\end{aligned}$$

Now we take square roots, use the Taylor expansion $\sqrt{1+t}=1+t/2+O(t^2)$, and neglect the terms $x_\mu x_\nu/\ell$ for $\mu,\nu\in\{2i-1,i,2j-1,2j\}$, to obtain

$$\ell+e\approx\ell+\cos(\theta)x_{2j-1}-\cos(\theta)x_{2i-1}+\sin(\theta)x_{2i}-\sin(\theta)x_{2j}.$$

If the displacements are small relative to the length of the undeformed bar, then the elongation of bar j is (approximately) given by

$$e=\cos(\theta)(x_{2j-1}-x_{2i-1})+\sin(\theta)(x_{2i}-x_{2j}). \tag{1}$$

We make the following assumptions:

- $x_{2j-1}>0$ means that node j is displaced to the right,
- $x_{2j}>0$ means that node j is displaced down,
- θ is the angle of the bar with the positive horizontal axis taken at the node i,
- the angle takes a positive sign unlike formula (2.14) in lecture notes where a negative sign is taken.

The relation (1) between elongation and displacement is used for all bars. This gives the linear system

$$Ax=e. \tag{2}$$

For example, consider the truss in Figure 2. The matrix A for this example is

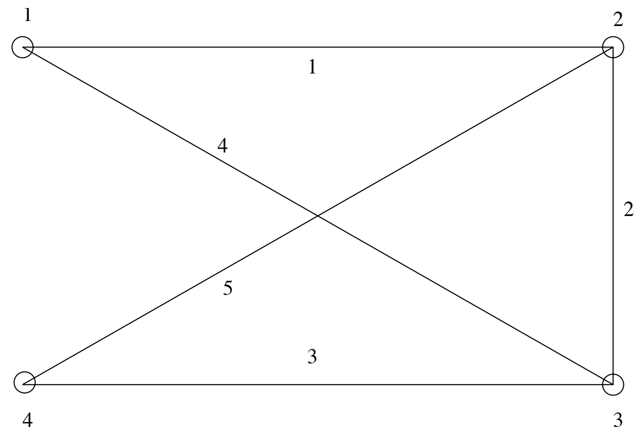

Figure 2: Truss with $N = 4$ nodes and $m = 5$ bars.

$$\begin{pmatrix} -\cos(0) & -\sin(0) & \cos(0) & \sin(0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & -\cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(0) & \sin(0) & -\cos(0) & -\sin(0) \\ \cos(\frac{3\pi}{4}) & -\sin(\frac{3\pi}{4}) & 0 & 0 & -\cos(\frac{3\pi}{4}) & \sin(\frac{3\pi}{4}) & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) & 0 & 0 & -\cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \end{pmatrix}$$

and equation (2) for the truss in Figure 2 is given by

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}. \tag{3}$$

The displacement of the bars $i = 1, \ldots, m$ generate internal forces y_i, $i = 1, \ldots, m$, acting on the nodes. The forces can be composed into forces in horizontal and vertical direction. Our indexing of the forces acting on the nodes follows the indexing of the nodes, i.e., the horizontal component of the force acting on node j is f_{2j-1} and the vertical component is f_{2j}. The equilibirum equations state that the internal forces acting on a node equal the external forces acting on the same node. One can show that the corresponding equation is given by

$$A^T y = f. \tag{4}$$

Finally, we need a constitutive law that relates elongations to the internal forces. Let a_i be the cross sectional area of the undeformed bar. We define the stress σ_i and the strain ε_i as

$$\sigma_i = \frac{y_i}{a_i}, \quad \varepsilon_i = \frac{e_i}{\ell_i}.$$

The assumption of linear elasticity states that

$$\sigma_i = E_i \varepsilon_i,$$

where E_i is Young's modulus for the material of the i-th bar. Hence

$$y_i = \underbrace{\frac{a_i E_i}{\ell_i}}_{=k_i} e_i.$$

If we set

$$K = \text{ diag } (\ldots, a_i E_i / \ell_i, \ldots),$$

then

$$Ke = y. \tag{5}$$

If we combine equations (2), (4), and (5) we obtain the equation

$$A^T K A u = f. \tag{6}$$

The matrix

$$S = A^T K A$$

is called the stiffness matrix of the structure.

If nodes in a truss are supported, then the corresponding displacement is deleted from the system. For example, consider the truss in Figure 3. The nodes 1 and 4 are supported. Equation (2) for this truss is given

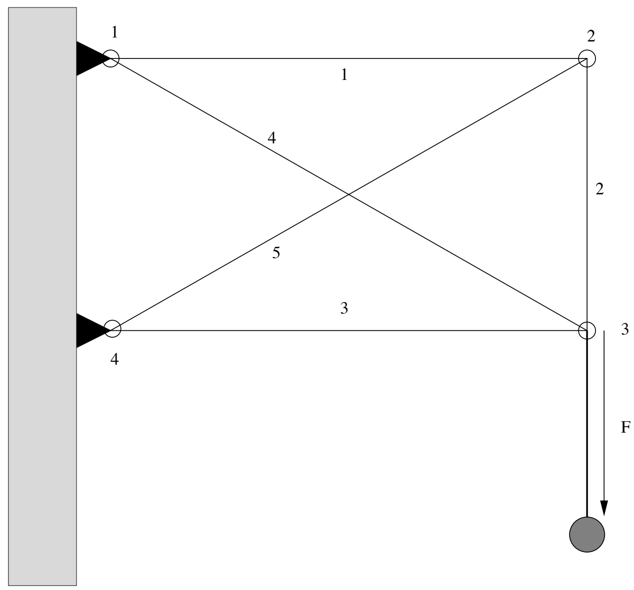

Figure 3: Truss with two supported and two unsupported nodes and five bars.

by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix} \tag{7}$$

and the equilibrium equation (4) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F \end{pmatrix}. \tag{8}$$

With our sign conventions, forces pointing left or down have positive signs.

We use `truss.m` to specify the truss topology of the truss in Figure 3 and `stiff.m` to compute the stiffness matrix S.

Figure 4 shows the original truss (solid lines) and deformed truss (dashed lines) after a force $F = 100 * 9.80665$[N] has been applied. All bars are made of the same material with Young's modulus $E_i = 195$ [GPa] and have the same cross sectional area $a_i = 0.0025^2\pi$ [m^2]. In Figure 4 the actual displacements are magnified by a factor 100 to increase visibility.

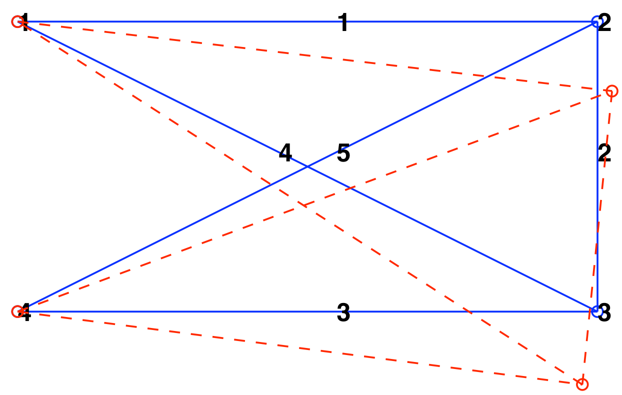

Figure 4: Original truss (solid lines) and deformed truss (dashed line) due to a load attached to node 3.