CAAM 335: Matrix Analysis

Examination 2

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Total	

Posted Thursday, December 3rd, 2009.

Due 5pm on Wednesday, December 9th, 2009

(accepted until 5pm Friday, December 11th, 2009)

Instructions

1. Please do not look at the questions until you begin the exam. There are 5 questions worth a total of 100 points.
2. You may not use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
3. Please answer the questions thoroughly and justify all your answers. Show all your work to maximize partial credit.
4. Time limit: **200 minutes including a possible break of 20 minutes or less**.
5. Print your name on the line below:

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6. Time started: ____________________ Time completed: ____________________
7. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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8. Staple this page to the front of your exam.

Read each question very carefully before you start answering.

Problem 1 (15 points)

The standard linear least squares problem is $\min_x \|Ax-b\|^2$ where $A \in \mathbb{R}^{m\times n}$ and $m \geq n$. A vector $x \in \mathbb{R}^n$ is a solution if and only if it satisfies the normal equations $A^T Ax = A^T b$.

Now given $C \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{k\times n}$ ($k < n$), consider the matrix least squares problem

$$\min_{X \in \mathbb{R}^{m\times k}} \|C - XB\|_F^2 \tag{1}$$

where $\|\cdot\|_F$ is the Frobenius norm (square root of sum of squares of all elements).

1. Derive *a matrix equation* involving only B, C and X such that X is a solution to (1) if and only if it satisfies the derived matrix equation (i.e., a matrix form of the "normal equations" for (1)).
2. Write down an expression for the "shortest solution" (in the form $X = \cdots$) that has the minimum Frobenius-norm among all possible solutions to (1). Justify your formula.
3. Express the residue $C - XB$ at a solution X as a product of C with an orthogonal projection matrix. Which subspace does this projection matrix project onto?

(Hint: $\|A\|_F = \|A^T\|_F$ and $(A^T)^\dagger = (A^\dagger)^T$. The sum of squares of a matrix can be done either row by row or column by column.)

Problem 2 (20 points) Recall that the Cauchy Riemann equations for $f(z) = u(x,y) + iv(x,y)$ are

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y), \qquad \frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y),$$

or in a short form $u_x = v_y$ and $v_x = -u_y$, which are necessary for the differentiability of $f(z)$ at $z = x + iy$, but not sufficient.

Let

$$f(z) = \begin{cases} (\bar{z})^2/z, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

1. Derive real functions $u(x,y)$ and $v(x,y)$ so that $f(z) = u(x,y) + iv(x,y)$.
2. Determine whether or not $f(z)$ satisfies the Cauchy-Riemann equations at $z = 0$. (Hint: $u_x(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$ and so on.)
3. Compute the limit of $f(z)/z$ as $z \to 0$ along the line $x = y$.
4. Is the function $f(z)$ differentiable at $z = 0$? Why or why not?

Problem 3 (25 points) Let $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_1 \neq \lambda_2$,

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We want to solve the dynamical system $x'(t) = Bx(t)$, $t > 0$, $x(0) = x_0$ using the Laplace transform and inverse Laplace transform. Let us denote the Laplace transform of $x(t)$ by

$$X(s) \equiv \mathcal{L}(x)(s) = \int_0^\infty e^{-st} x(t) dt.$$

For a polynomial g and a differentiable function f the inverse Laplace transform of f/g is

$$\mathcal{L}^{-1}\left(\frac{f}{g}\right)(t) = \frac{1}{2\pi i} \int_C \frac{f(z)}{g(z)} e^{zt} dz,$$

where C is a closed curve that encloses each of the roots of g.

Show all details of your work!

1. Show that the Laplace transform $X(s)$ satisfies

$$X(s) = R(s) x_0.$$

   where $R(s) = (sI - B)^{-1}$ is the resolvent of B.

2. Find the resolvent, and observe its partial fraction expansion,

$$R(s) = \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{R_{jk}}{(s - \lambda_j)^k}$$

   (recall that h is the number of distinct eigenvalues and m_j's are algebraic multiplicities). Then express

$$X(s) = \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{1}{(s - \lambda_j)^k} R_{jk} x_0.$$

3. Use the residue theorem to compute the inverse Laplace transforms

$$\mathcal{L}^{-1}\left(\frac{1}{(s - \lambda)^k}\right)(t), \;\; k = 1, 2.$$

4. Use the results derived above to assemble the solution $x(t)$.
5. (4 points) Under what conditions $\lim_{t \to \infty} x(t) = \mathbf{0}$ for all $x(0)$?

Problem 4 (20 points)

It is known that every square matrix A has a Schur decomposition $A = UTU^*$ where T is upper triangular ($t_{ij} = 0, i > j$) and U is unitary ($U^*U = I$). If T is diagonal, then A is diagonalizable by a unitary transformation.

Now we consider skew-symmetric matrices that satisfy $A^T = -A$.

1. (a) Construct a nonzero, skew-symmetric matrix $A \in \mathbb{R}^{2\times 2}$ using only ± 1 and 0, find its eigenvalues λ_1, λ_2 and corresponding unit eigenvectors u_1 and u_2, and then form its eigen-decomposition $A = UDU^*$ where D is diagonal and U is unitary.
   (b) Give the 2 by 2 matrix e^A (either in an analytic or a numeric form), and determine whether it is unitary or not.
2. (a) Show that any skew-symmetric $A \in \mathbb{R}^{n\times n}$ is diagonalizable by a unitary transformation; i.e., $A = UDU^*$ where D is diagonal and U is unitary. Moreover, the eigenvalues of A are all purely imaginary.
   (b) Prove that for any skew-symmetric A the matrix exponential function $\exp(At)$ is unitary for $t \in \mathbb{R}$.

Problem 5 (20 points) Let $A \in \mathbb{R}^{10\times 5}$ have singular value decomposition $A = U\Sigma V^T$ where the first 3 diagonal entries of Σ are $\sigma_1 = 3$, $\sigma_2 = 2$ and $\sigma_3 = 1$, and the rest of the entries in Σ are all zeros. Let the columns of U be $u_1, u_2, \cdots, u_{10}$, and the columns of V be $v_1, v_2, \cdots, v_5$. Let $A^\dagger$ be the pseudo-inverse of A and I be an identity matrix of appropriate size.

Answer the following questions (justification unnecessary).

1. What is the dimension of $\mathcal{R}(A^T)$?
2. What is the dimension of $\mathcal{N}(A^T)$?
3. What is the value of $u_1^T AA^T u_1$?
4. What is the value of $u_4^T AA^T u_4$?
5. Give an orthonormal basis for $\mathcal{N}(A)$.
6. What is the trace of AA^T?
7. What is the trace of $(A^T A + I)^{-1}$?
8. What is the Frobenius norm $\|A\|_F$?
9. What is the Frobenius norm $\|A^\dagger\|_F$?
10. Express $I - A^\dagger A$ using the columns of V only.