

# Computation of Natural Cubic Splines

- ▶ Let  $a = x_0 < x_1 < \dots < x_n = b$  and  $h_i = x_{i+1} - x_i$ .
- ▶ The cubic spline  $S$  satisfies the following properties

$$S(x) = S_i(x), \quad x \in [x_i, x_{i+1}] \quad \text{for } i = 0, \dots, n-1, \quad (1a)$$

$$S(x_i) = f(x_i) \quad \text{for } i = 0, \dots, n, \quad (1b)$$

$$S_{i+1}(x_{i+1}) = S_i(x_{i+1}) \quad \text{for } i = 0, \dots, n-2, \quad (1c)$$

$$S'_{i+1}(x_{i+1}) = S'_i(x_{i+1}) \quad \text{for } i = 0, \dots, n-2, \quad (1d)$$

$$S''_{i+1}(x_{i+1}) = S''_i(x_{i+1}) \quad \text{for } i = 0, \dots, n-2. \quad (1e)$$

where  $S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$ .

- ▶ (1b)  $\Rightarrow a_i = f(x_i)$ ,  $i = 0, \dots, n-1$ .
- ▶ (1e)  $\Rightarrow 2c_{i+1} = S''_{i+1}(x_{i+1}) = S''_i(x_{i+1}) = 2c_i + 6d_i h_i$ ,  $i = 0, \dots, n-2$ .  
At  $x_n$ :  $2c_n \stackrel{\text{def}}{=} S''_{n-1}(x_n) = 2c_{n-1} + 6d_{n-1} h_{n-1}$ .  
 $\Rightarrow d_i = \frac{1}{3h_i}(c_{i+1} - c_i)$ ,  $i = 0, \dots, n-1$ .
- ▶ (1a-e) and the previous identities  $\Rightarrow$  (after a few calculations)  
 $b_i = \frac{1}{h_i}(a_{i+1} - a_i) - \frac{h_i}{3}(2c_i + c_{i+1})$ ,  $i = 0, 1, \dots, n-1$ .
- ▶ (1a-e), the previous identities and  $S''_0(x_0) = S''_{n-1}(x_n) = 0$  (after a few calculations) lead to a linear system for  $c_0, \dots, c_n$ .

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- ▶ If  $S$  satisfies (1) and has zero second derivatives at both ends, then  $2c_0 = S''_0(x_0) = 0$ ,  $2c_n \stackrel{\text{def}}{=} S''_{n-1}(x_n) = 0$ , and
- ▶  $Tc = r$ ,
- ▶ where  $T \in \mathbb{R}^{(n-1) \times (n-1)}$  is the symmetric tridiagonal matrix

$$T = \begin{pmatrix} 2(h_0 + h_1) & h_1 & & & & \\ h_1 & 2(h_1 + h_2) & h_2 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-3} & 2(h_{n-3} + h_{n-2}) & h_{n-2} \\ & & & & h_{n-2} & 2(h_{n-2} + h_{n-1}) \end{pmatrix}$$

and  $c, r \in \mathbb{R}^{n-1}$  are given by

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix}, \quad r = \begin{pmatrix} \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) - \frac{3}{h_{n-3}}(a_{n-2} - a_{n-3}) \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \end{pmatrix}.$$

- ▶  $T$  is symmetric positive definite.