CAAM 335: Matrix Analysis
Solutions to Examination 1

Problem 1 (12 points)

Which of the following sets $\mathcal{M} \subset \mathbb{R}^3$ are subspaces? (Justify your answer!)

(a) (4 points) $\mathcal{M} = \{(x_1,x_2,x_3)^T \,:\, x_1 = x_2 = x_3\}$.
(b) (4 points) $\mathcal{M} = \{(x_1,x_2,x_3)^T \,:\, x_1 = x_2, x_3 = 1\}$.
(c) (4 points) $\mathcal{M} = \{(x_1,x_2,x_3)^T \,:\, x_1^2+x_2^2+x_3^2 \le 0\}$.

(a) $\mathcal{M} = \{(x_1,x_2,x_3)^T \,:\, x_1 = x_2 = x_3\}$ is a subspace.
Let $x = (x_1,x_2,x_3)^T$ and $y = (y_1,y_2,y_3)^T$ be vectors in $\mathcal{M}$, i.e., $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$. The vector $x+y = (x_1+y_1, x_2+y_2, x_3+y_3)^T$ satisfies $x_1+y_1 = x_2+y_2 = x_3+y_3$, i.e., $x+y \in \mathcal{M}$. Let $x = (x_1,x_2,x_3)^T \in \mathcal{M}$, i.e., $x_1 = x_2 = x_3$ and $t \in \mathbb{R}$. The vector $tx = (tx_1,tx_2,tx_3)^T$ satisfies $tx_1 = tx_2 = tx_3$, i.e., $tx \in \mathcal{M}$.

(b) $\mathcal{M} = \{(x_1,x_2,x_3)^T \,:\, x_1 = x_2, x_3 = 1\}$ is NOT a subspace, since $x = (1,1,1) \in \mathcal{M}$, but $2x = (2,2,2) \notin \mathcal{M}$ $(2x_3 = 2 \ne 1)$.

(c) $\mathcal{M} = \{(x_1,x_2,x_3)^T \,:\, x_1^2+x_2^2+x_3^2 \le 0\}$ is a subspace.
The zero vector, $(0,0,0)^T$ is the only vector in $\mathbb{R}^3$ that satisfies $x_1^2+x_2^2+x_3^2 \le 0$. The set of zero vector trivially satisfies the two properties of a subspace.

Problem 2 (20 points)

(a) (10 points) Compute the four subspaces $\mathcal{R}(A), \mathcal{N}(A), \mathcal{R}(A^T), \mathcal{N}(A^T)$ associated with

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

(b) (5 points) Find all solutions of $Ax = b$, where $b = (1,1)^T$.
(c) (5 points) Give a d so that $A^T y = d$ does not have a solution. (Justify your answer!)

(a) The matrix is already in reduced form. Rows 1 and 2 are pivot rows, columns 1 and 2 are pivot columns. x_3 is the free variable. Hence $\mathcal{R}(A) = \mathbb{R}^2$, or following our algorithm,

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{R}(A). \qquad \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(A).$$

$$A^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = (A^T)_{\text{red}} \quad \text{(this step is not necessary)}$$

Rows 1 and 2 are pivot rows, columns 1 and 2 are pivot columns. There are no free variables

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{R}(A^T), \quad \text{ and } \mathcal{N}(A^T) = \{0\}.$$

(b) (3 points)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right)$$

x_3 is free. Given x_3, the variables x_1, x_2 are determined by

$$x_1 + 2x_3 = 0, \quad x_2 - x_3 = 1.$$

Hence the set of all solutions is given by

$$\left\{ (-2x_3, 1 + x_3, x_3)^T \ : \ x_3 \in \mathbb{R} \right\} = \left\{ (0,1,0)^T + x_3(-2,1,1)^T \ : \ x_3 \in \mathbb{R} \right\}.$$

(c) The system $A^T y = d$ has a solution if and only if $d \in \mathcal{R}(A^T)$. By the Fundamental Theorem of Linear Algebra, $\mathbb{R}^3 = \mathcal{R}(A^T) \oplus \mathcal{N}(A)$ and $\mathcal{R}(A^T) \perp \mathcal{N}(A)$. Hence $A^T y = d$ does not have a solution for all d with $(-2,1,1)d \neq 0$. In particular, $A^T y = d$ does not have a solution if $d = (-2,1,1)^T$.

Problem 3 (16 points) Always try to justify your answers concisely and rigorously.

(a) (5 points) Compute the inverse of

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & -3 & 1 \end{pmatrix}.$$

(b) (5 points) Let I be the identity in $\mathbb{R}^n$. Is it true that the inverse of the matrix

$$I - 2\frac{uu^T}{u^T u}$$

is itself for any nonzero (column) vector $u \in \mathbb{R}^n$.

(c) (6 points) Let $L \in \mathbb{R}^{n\times n}$ be a triangular matrix and let

$$t > \max_{1\le i\le n} |L_{ii}|$$

where L_{ii} is the i-th diagonal element of L. What is the rank of $tI - L$? What are the subspaces $\mathcal{R}(tI-L)$ and $\mathcal{N}(tI-L)$?

(a)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 & 1 & 0 \\ -3 & -3 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & -3 & 1 & 3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 12 & 3 & 1 \end{array}\right)$$

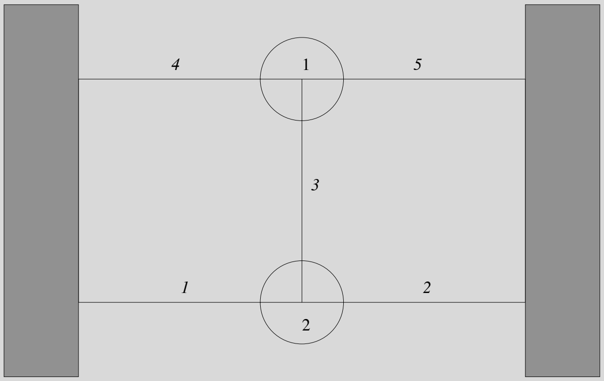
The inverse of $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & -3 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 12 & 3 & 1 \end{pmatrix}$

(b) The matrix is the inverse of itself for any nonzero (column) vector $u \in R^n$.

$$\begin{aligned} \left(I - 2\frac{uu^T}{u^T u}\right)\left(I - 2\frac{uu^T}{u^T u}\right) &= I - 2\frac{uu^T}{u^T u} - 2\frac{uu^T}{u^T u} + 4\frac{uu^T uu^T}{(u^T u)^2} \\ &= I - 4\frac{uu^T}{u^T u} + 4\frac{u(u^T u)u^T}{(u^T u)^2} \\ &= I - 4\frac{uu^T}{u^T u} + 4\frac{uu^T}{u^T u} \\ &= I \end{aligned}$$

(c) Since t is strictly greater than $\max_{1\le i\le n}|L_{ii}|$, the diagonally entries of $tI - L$ will all be strictly greater than zero, hence $tI - L$ is non-singular and must have rank n. Thus $\mathcal{R}(tI-L) = \mathbb{R}^n$ and $\mathcal{N}(tI-L) = \{0\}$ (not $\mathcal{N}(tI-L) = 0$).

Problem 4 (18 points)



(a) (6 points) Compute the adjacency matrix A for the truss shown above.

(b) (6 points) Compute $\mathcal{R}(A)$ and $\mathcal{N}(A)$.

(c) (6 points) Does the linear system $A^TAx = f$ have solutions for the two right-hand sides below? Justify your answer!

$$f^{(1)} = (1,1,1,-1)^T,$$
$$f^{(2)} = (-1,0,1,0)^T.$$

(a)

$$\begin{aligned} e_1 &= x_3 \\ e_2 &= -x_3 \\ e_3 &= -x_2 + x_4 \\ e_4 &= x_1 \\ e_5 &= -x_1 \end{aligned}$$

In matrix-vector notation $e = Ax$ with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_{\text{red}}$$

Rows $1,3,4$ of A_{red} are pivot rows. Columns $1,2,3$ are pivot columns. Free variable$= x_4$.

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{R}(A). \qquad \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{N}(A).$$

(c) By the Fundamental Theorem of Linear Algebra, $\mathbb{R}^4 = \mathcal{R}(A^TA) \oplus \mathcal{N}(A^TA)$ and $\mathcal{R}(A^TA) \perp \mathcal{N}(A^TA)$. Furthermore $\mathcal{N}(A^TA) = \mathcal{N}(A)$. Hence $A^TAx = f$ has a solution if and only if $f \perp \mathcal{N}(A)$, i.e., if and only if $f^Tv = 0$ for all vectors in $\mathcal{N}(A)$, that is $f^Tv = 0$ for v being a basis vector of $\mathcal{N}(A)$ which is one-dimensional.

Since

$$(f^{(1)})^T(0,1,0,1) = (1,1,1,-1)(0,1,0,1)^T = 0$$

$A^TAx = f^{(1)}$ does have a solution. Since

$$(f^{(2)})^T(0,1,0,1) = (-1,0,1,0)(0,1,0,1)^T = 0$$

$A^TAx = f^{(2)}$ does have a solution.

Problem 5 (16 points) Let A and B be square matrices.

(a) (5 points) Is it necessarily true that $\mathcal{N}(A) \cap \mathcal{N}(B) \subset \mathcal{N}(A+B)$?

(b) (5 points) Show that $\mathcal{N}(B) \subset \mathcal{N}(AB)$.

(c) (6 points) Let A be invertible. Show that $\mathcal{N}(AB) \subset \mathcal{N}(B)$.

(a) Let $x \in \mathcal{N}(A) \cap \mathcal{N}(B)$. Then $Ax = 0$ and $Bx = 0$. Thus $(A+B)x = Ax + Bx = 0 + 0 = 0$. This proves $\mathcal{N}(A) \cap \mathcal{N}(B) \subset \mathcal{N}(A+B)$.

(b) Let $x \in \mathcal{N}(B)$. Then $Bx = 0$. Hence $ABx = A(Bx) = A(0) = 0$. This proves $\mathcal{N}(AB) \subset \mathcal{N}(B)$.

(c) Let $x \in \mathcal{N}(AB)$. Then $ABx = 0$. Since A is invertible

$$0 = A^{-1}0 = A^{-1}ABx = Bx.$$

This proves that $\mathcal{N}(AB) \subset \mathcal{N}(B)$.

Problem 6 (18 points) Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times n}$ be non-singular. Answer the following questions as thoroughly as possible (proof or counter example).

(a) (6 points) If $\mathcal{R}(A) = \{0\}$ is $A = 0$?

(b) (6 points) Is $\mathcal{R}(A) = \mathcal{R}(AB)$?

(c) (6 points) Is $\mathcal{R}(A) = \mathcal{R}(\beta A)$ for a fixed scalar β?

(a) If $\mathcal{R}(A) = \{0\}$ then $A = 0$. Ture.

Suppose that $A \neq 0$. Then there exists a matrix entry $a_{ij} \neq 0$. If we multiply A by the jth unit vector e_j, then

$$\mathcal{R}(A) \ni Ae_j = (a_{1j}, \ldots, \underbrace{a_{ij}}_{\neq 0}, \ldots, a_{nj})^T \neq 0,$$

which contradicts the assumption that $\mathcal{R}(A) = \{0\}$.

(b) $\mathcal{R}(A) = \mathcal{R}(AB)$. True since B is nonsingular.

$\mathcal{R}(AB) \subset \mathcal{R}(A)$ is immediate by the definition and the fact $x = (AB)y = A(By) = Az$ for $z = By$. On the other hand, $Az = AB(B^{-1}z) = (AB)y$ for $y = B^{-1}z$. So the reverse inclusion also holds.

(c) This is only true for $\beta \neq 0$.

For $\beta = 0$ and A nonzero $\mathcal{R}(\beta A) = \{0\}$ whereas $\mathcal{R}(A) = \{0\}$ only if $A = 0$.

Assume $\beta \neq 0$. Let $y \in \mathcal{R}(A)$. There exists x with $y = Ax$. We have $y = \beta A(\frac{1}{\beta}x)$. Hence $y = \beta Az$ with $z = \frac{1}{\beta}x$, which means that $y \in \mathcal{R}(\beta A)$.

Let $y \in \mathcal{R}(\beta A)$. There exists x with $y = \beta Ax$. We have $y = A(\beta x)$. Hence $y = Az$ with $z = \beta x$, which means that $y \in \mathcal{R}(A)$.

Or simply, $\mathcal{R}(\beta A)$ consists of all vectors that are linear combinations of the columns of βA, but they are also all linear combinations of the columns of A.