CAAM 335 Matrix Analysis: HW 9

Problem 1 (60 points) This exercise concerns the symmetric adjacency matrix

$$S = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that arose in Chapter 2 for the two-dimensional truss portrayed in Figure 2.2. The answers are all clean enough that you can (and should) do these computations by hand.

(a) (20 points)
Compute all the eigenvalues and eigenvectors of S.
(Hint: Note $\det(\lambda I - S) = (\lambda - 1)\det(\lambda I - S_1)$ where S_1 is the leading 3 by 3 block of S.)

(b) (10 points)
Find the eigenvalue decomposition $S = QDQ^T$ where Q is orthogonal ($Q^T Q = I$) and D is diagonal (write down Q and D). (Hint: You need an orthonormal basis for each eigenspace. use the Gram Schmidt whenever necessary).

(c) (10 points)
Find the spectral representation $S = \sum_{j=1}^{h} \lambda_j P_j$ (write down the projections P_j's).

(d) (10 points)
Keeping in mind the expressions for B^n and $\exp(B)$, guess a formula for the matrix square root, $S^{1/2}$. Use your formula to compute $S^{1/2}$ for the adjacency matrix S, then verify that $S^{1/2}S^{1/2} = S$. (You may compare `sqrt(S)` and `sqrtm(S)` in MATLAB.)

(e) (10 points) Since S has a zero eigenvalue, it is not invertible. Find its pseudo-inverse $S^{\dagger}$ (you may compare your result to `pinv(S)` in MATLAB), and use S^{+} to solve $Sx = f$ for $f = [0;2;0;1]^T$.

Problem 2 (10 points) Show that if λ is an eigenvalue of an orthogonal matrix Q, then $|\lambda| = 1$. (Hint: How does $\|Qx\|_2^2$ relate to $\|x\|_2^2$?)

Problem 3 (10 points) A matrix $B \in \mathbb{R}^{n\times n}$ is *skew symmetric* if $B^T = -B$. Show that any eigenvalue of a skew symmetric matrix is purely imaginary, i.e, of the form $\lambda = iy$ with $y \in \mathbb{R}$.