CAAM 335: Matrix Analysis

Solutions to HW 1

Problem 1 (20 points) Let

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 3 \\ 6 & 2 & 3 \end{pmatrix}.$$

1.1 Compute Ax where $x = (2, 1, -3)^T$.

1.2 Write Ax as a linear combination of columns of A.

1.3 Compute the square root of the inner product of Ax with itself. This quantity is called the 2-norm of the vector (Ax, in this case) which measures the magnitude of the vector.

Solution 1.1 $\begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 3 \\ 6 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ -7 \\ 5 \end{pmatrix}.$

1.2 $Ax = 2 \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}.$

1.3 $\|Ax\|_2 = \sqrt{(Ax)^T(Ax)} = \sqrt{8^2 + 7^2 + 5^2} = \sqrt{138} = 11.7473.$

Problem 2 (30 points) Let us consider the following matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 0 \\ -6 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -3 & 4 \\ 2 & 6 & -5 \end{pmatrix}.$$

2.1 Find $5A - 2B$.

2.2 Find: AB and BA. Note that $AB \neq BA$.

2.3 Find: (a) AB and $(AB)C$; (b) BC and $A(BC)$. Note that $(AB)C = A(BC)$.

2.4 Find A^T, B^T, and $A^T B^T$. Note that $A^T B^T \neq (AB)^T$.

2.5 Find $AA = A^2$, and AC.

Solution 2.1 $5A - 2B = \begin{pmatrix} -5 & 10 \\ 27 & -34 \end{pmatrix}$

2.2

$$AB = \begin{pmatrix} -7 & 14 \\ 39 & -28 \end{pmatrix} \neq BA = \begin{pmatrix} 5 & 10 \\ 15 & -40 \end{pmatrix}.$$

2.3

$$(AB)C = \begin{pmatrix} -7 & 14 \\ 39 & -28 \end{pmatrix} \begin{pmatrix} 1 & -3 & 4 \\ 2 & 6 & -5 \end{pmatrix} = \begin{pmatrix} 21 & 105 & -98 \\ -17 & -285 & 296 \end{pmatrix}.$$

$$A(BC) = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 5 & -15 & 20 \\ 8 & 60 & -59 \end{pmatrix} = \begin{pmatrix} 21 & 105 & -98 \\ -17 & -285 & 296 \end{pmatrix}.$$

2.4

$$A^T = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}, \quad B^T = \begin{pmatrix} 5 & -6 \\ 0 & 7 \end{pmatrix}, \quad A^T B^T = \begin{pmatrix} 5 & 15 \\ 10 & -40 \end{pmatrix} \neq (AB)^T = \begin{pmatrix} -7 & 39 \\ 14 & -28 \end{pmatrix}.$$

2.5

$$AA = A^2 = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}, \quad AC = \begin{pmatrix} 5 & 9 & -6 \\ -5 & -33 & 32 \end{pmatrix}.$$

Problem 3 (10 points) Consider A and B in Problem 2. Write AB into a sum of vector outer products. (Hint: partition A by columns and B by rows.)

Solution

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -6 & 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 5 & 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} \begin{pmatrix} -6 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 15 & 0 \end{pmatrix} + \begin{pmatrix} -12 & 14 \\ 24 & -28 \end{pmatrix}.$$

Problem 4 (20 points) (Problem 1 on page 9 of the Lecture Notes)
In order to refresh your matrix-vector multiply skills please calculate, by hand, the product $A^T GA$ in the 3 compartment case and write out the 4 equations in (1.1). The second equation should read

$$\frac{-x_1 + 2x_2 - x_3}{R_i} + \frac{x_2}{R_m} = 0.$$

Solution Recall

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, G = \begin{pmatrix} 1/R_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/R_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/R_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/R_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/R_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/R_m \end{pmatrix}$$

$$GA = \begin{pmatrix} -1/R_i & 1/R_i & 0 & 0 \\ 0 & -1/R_m & 0 & 0 \\ 0 & -1/R_i & 1/R_i & 0 \\ 0 & 0 & -1/R_m & 0 \\ 0 & 0 & -1/R_i & 1/R_i \\ 0 & 0 & 0 & -1/R_m \end{pmatrix},$$

$$A^T GA = \begin{pmatrix} 1/R_i & -1/R_i & 0 & 0 \\ -1/R_i & 2/R_i + 1/R_m & -1/R_i & 0 \\ 0 & -1/R_i & 2/R_i + 1/R_m & -1/R_i \\ 0 & 0 & -1/R_i & 1/R_i + 1/R_m \end{pmatrix}$$

Hence $A^T GAx = f$ is given by

$$\begin{pmatrix} 1/R_i & -1/R_i & 0 & 0 \\ -1/R_i & 2/R_i + 1/R_m & -1/R_i & 0 \\ 0 & -1/R_i & 2/R_i + 1/R_m & -1/R_i \\ 0 & 0 & -1/R_i & 1/R_i + 1/R_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} i_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e,

$$\begin{aligned} \frac{x_1 - x_2}{R_i} &= i_0, \\ \frac{-x_1 + 2x_2 - x_3}{R_i} + \frac{x_2}{R_m} &= 0, \\ \frac{-x_2 + 2x_3 - x_4}{R_i} + \frac{x_3}{R_m} &= 0, \\ \frac{-x_3 + x_4}{R_i} + \frac{x_4}{R_m} &= 0. \end{aligned}$$