

# CAAM 335 Matrix Analysis

## Solutions to Homework 9

**Problem 1 (60 points)** This exercise concerns the symmetric adjacency matrix

$$S = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that arose in Chapter 2 for the two-dimensional truss portrayed in Figure 2.2. The answers are all clean enough that you can (and should) do these computations by hand.

(a) (20 points)

Compute all the eigenvalues and eigenvectors of  $S$ .

(Hint: Note  $\det(\lambda I - S) = (\lambda - 1) \det(\lambda I - S_1)$  where  $S_1$  is the leading 3 by 3 block of  $S$ .)

(b) (10 points)

Find the eigenvalue decomposition  $S = QDQ^T$  where  $Q$  is orthogonal ( $Q^T Q = I$ ) and  $D$  is diagonal (write down  $Q$  and  $D$ ). (Hint: You need an orthonormal basis for each eigenspace. use the Gram Schmidt whenever necessary).

(c) (10 points)

Find the spectral representation  $S = \sum_{j=1}^h \lambda_j P_j$  (write down the projections  $P_j$ 's).

(d) (10 points)

Keeping in mind the expressions for  $B^n$  and  $\exp(B)$ , guess a formula for the matrix square root,  $S^{1/2}$ . Use your formula to compute  $S^{1/2}$  for the adjacency matrix  $S$ , then verify that  $S^{1/2} S^{1/2} = S$ . (You may compare `sqrt(S)` and `sqrtm(S)` in MATLAB.)

(e) (10 points) Since  $S$  has a zero eigenvalue, it is not invertible. Find its pseudo-inverse  $S^\dagger$  (you may compare your result to `pinv(S)` in MATLAB), and use  $S^\dagger$  to solve  $Sx = f$  for  $f = [0; 2; 0; 1]^T$ .

### Solution:

(a) Based upon the hint, we need only determine the characteristic equation of  $S_1$ , the

leading 3 by 3 block of  $S$ . Computing the characteristic polynomial of  $S_1$ ,

$$\begin{aligned}\det(\lambda I - S_1) &= \det \begin{pmatrix} \lambda - 1 & 0 & 1 \\ 0 & \lambda - 1 & 0 \\ 1 & 0 & \lambda - 1 \end{pmatrix} \\ &= (\lambda - 1)(\lambda - 1)^2 + (1)(-(\lambda - 1)) \\ &= (\lambda - 1)(\lambda^2 - 2\lambda + 1 - 1) \\ &= (\lambda - 1)(\lambda - 2)\lambda,\end{aligned}$$

thus  $p_S(\lambda) = (\lambda - 2)(\lambda - 1)^2\lambda$ . Computing the eigenvectors associated with each eigenvector. For  $\lambda = 2$ , we seek  $x$  such that

$$(S - 2I)x = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = -x_3 \\ x_2 = 0 \\ x_4 = 0 \end{matrix},$$

hence an eigenvector for  $\lambda = 2$  is  $(1, 0, -1, 0)^T$ .

For  $\lambda = 1$ , we seek  $x$  such that

$$(S - I)x = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_3 = 0 \end{matrix},$$

hence eigenvectors for  $\lambda = 1$  are  $(0, 1, 0, 0)^T$  and  $(0, 0, 0, 1)^T$ .

For  $\lambda = 0$ , we seek  $x$  such that

$$Sx = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = x_3 \\ x_2 = 0 \\ x_4 = 0 \end{matrix},$$

hence an eigenvector for  $\lambda = 0$  is  $(1, 0, 1, 0)^T$ .

- (b) As the eigenvectors above are already orthogonal, to form  $Q$  we need only normalize the eigenvectors, thus

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Now let  $D = \text{diag}(2, 1, 1, 0)$ , then

$$S = QDQ^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

(c) Since  $S$  is symmetric the spectral projectors are simply the outerproducts of the corresponding normalized eigenvectors.

For  $\lambda = 2$ ,

$$P_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For  $\lambda = 1$ ,

$$P_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For  $\lambda = 0$ ,

$$P_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(d) Recall the following formula for  $B^n$  and  $\exp(B)$  for  $B \in \mathbb{R}^{m \times m}$ ,

$$B^n = \sum_{i=1}^m \lambda_i^n P_i \quad \exp(B) = \sum_{i=1}^m e^{\lambda_i} P_i$$

Based upon these expression, a similar expression for  $S^{1/2}$  is

$$S^{1/2} = \sum_{i=1}^m \lambda_i^{1/2} P_i.$$

Computing  $S^{1/2}$ ,

$$S^{1/2} = \sqrt{2}P_1 + P_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and indeed  $S = S^{1/2}S^{1/2}$ . MATLAB's `sqrtm` command gives

```
sqrtm(S)
ans =
    0.7071         0   -0.7071         0
         0    1.0000         0         0
   -0.7071         0    0.7071         0
         0         0         0    1.0000
```

which is what we have above. Note that MATLAB's `sqrt` command gives

```
sqrt(S)
ans =
    1.0000         0         0 + 1.0000i         0
         0         1.0000         0         0
    0 + 1.0000i         0         1.0000         0
         0         0         0         1.0000
```

which is the entrywise square root of  $S$ .

- (e) Since  $S$  is real symmetric and all the eigenvalues are greater than or equal to zero, the eigenvalue decomposition above is also the singular value decomposition of  $S$  with  $V = Q$  and  $\Sigma = D$ . Thus  $S^\dagger = V\Sigma^\dagger V^T$ , where  $\Sigma^\dagger = \text{diag}(1/2, 1, 1, 0)$ . Thus

$$\begin{aligned} S^\dagger &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Note that  $f$  is in the span of the eigenvectors associated with  $\lambda = 1$ ,  $x = S^\dagger f = (0, 2, 0, 1)^T$  and  $Sx = f$ , thus  $S^\dagger f$  gives the minimal norm solution to  $Sx = f$ .

MATLAB's `pinv` command gives

```
pinv(S)
ans =
    0.2500         0   -0.2500         0
         0    1.0000         0         0
   -0.2500         0    0.2500         0
         0         0         0    1.0000
```

which is what we have above.

**Problem 2 (10 points)** Show that if  $\lambda$  is an eigenvalue of an orthogonal matrix  $Q$ , then  $|\lambda| = 1$ . (Hint: How does  $\|Qx\|_2^2$  relate to  $\|x\|_2^2$ ?)  
From the hint, for any  $x$

$$\|Qx\|_2^2 = x^T Q^T Qx = x^T Ix = x^T x = \|x\|_2^2,$$

where we have used the fact that  $Q^T Q = I$  for an orthogonal matrix. Now let  $x$  be an eigenvector,

$$\|Qx\|_2^2 = \|\lambda x\|_2^2 = |\lambda|^2 \|x\|_2^2,$$

hence

$$|\lambda|^2 \|x\|_2^2 = \|x\|_2^2$$

Since eigenvectors are nonzero by definition, every eigenvalue  $\lambda$  of an orthogonal matrix satisfies  $|\lambda| = 1$ .

**Problem 3 (10 points)** A matrix  $B \in \mathbb{R}^{n \times n}$  is *skew symmetric* if  $B^T = -B$ . Show that any eigenvalue of a skew symmetric matrix is purely imaginary, i.e., of the form  $\lambda = iy$  with  $y \in \mathbb{R}$ . Let  $x$  be an eigenvector of  $B$  with eigenvalue  $\lambda$ , then

$$\lambda x^* x = x^* (Bx) = x^* (-B^T x) = -(Bx)^* x = -(\lambda x)^* x = -\bar{\lambda} x^* x.$$

Thus  $\lambda = -\bar{\lambda}$  which suggests that the real part of lambda must be zero, hence  $\lambda$  is purely imaginary.