Prob 1. Prove in detail that GS converges for any initial guess if the matrix $A$ is column strictly diagonally dominant.

Proof. Let $A = (a_{ij})_{n \times n}$ and $A = D - L - U$, where $D$, $L$, and $U$ are respectively diagonal, strictly lower triangular and strictly upper triangular matrices. Since $A$ is column strictly diagonally dominant, $D$ and $D - L$ are invertible. GS scheme is written as: $x \leftarrow (D - L)^{-1}Ux + (D - L)^{-1}b$. To prove GS converges, just need to show that $\rho ((D - L)^{-1}U) < 1$.

Since a matrix, its similar matrices and transpose have the same eigenvalues, then

$$\rho ((D - L)^{-1}U) = \rho (U(D - L)^{-1})$$
$$= \rho ((U(D - L)^{-1})^T) = \rho ((D - L^T)^{-1}U^T).$$

Let $\lambda$ be any eigenvalue of $(D - L^T)^{-1}U^T$ and $x$ be one of the corresponding eigenvectors. Then

$$(D - L^T)^{-1}U^T x = \lambda x$$
$$U^T x = \lambda (D - L^T)x$$
$$\lambda Dx = U^T x + \lambda L^T x$$

i.e., $\lambda a_{ii}x_i = \sum_{j=1}^{i-1} a_{ji}x_j + \lambda \sum_{j=i+1}^{n} a_{ji}x_j$ for $i = 1, 2, \ldots, n$.

Let $|x_{i_0}| = \|x\|_\infty$, then for $i = i_0$, we have

$$|\lambda a_{i_0 i_0}x_{i_0}| = \left| \sum_{j=1}^{i_0-1} a_{j i_0} x_j + \lambda \sum_{j=i_0+1}^{n} a_{j i_0} x_j \right|$$
$$\leq \sum_{j=1}^{i_0-1} |a_{j i_0}| |x_j| + \lambda \sum_{j=i_0+1}^{n} |a_{j i_0}| |x_j|$$
$$\leq \left( \sum_{j=1}^{i_0-1} |a_{j i_0}| + \lambda \sum_{j=i_0+1}^{n} |a_{j i_0}| \right) |x_{i_0}|.$$

Hence, $|\lambda|(|a_{i_0 i_0}| - \sum_{j=i_0+1}^{n} |a_{ji_0}|) \leq \sum_{j=1}^{i_0-1} |a_{j i_0}|$, and $|\lambda| \leq \frac{\sum_{j=1}^{i_0-1} |a_{ji_0}|}{|a_{i_0 i_0}| - \sum_{j=i_0+1}^{n} |a_{ji_0}|} < 1$, because $A$ is column strictly diagonally dominant. Thus, $\rho ((D - L)^{-1}U) = \rho ((D - L^T)^{-1}U^T) < 1$, and GS converges. ■
Prob 2. Prove that the scheme \( x \leftarrow x + a \ast (b - Ax) \) converges for any initial guess if the matrix \( A \) is symmetric positive definite and \( a > 0 \) is less than 2 divided by the maximum eigenvalue of \( A \).

Proof. Suppose \( A \) is symmetric positive definite and \( 0 < a < 2/\rho(A) \), let's show the convergence of the scheme \( x \leftarrow x + a \ast (b - Ax) \).

The scheme can be written as \( x \leftarrow (I - aA)x + ab \). To show the convergence, just need to show that \( \rho(I - aA) < 1 \).

Let \( \lambda \) be any eigenvalue of \( (I - aA) \), and \( x \) be one of the corresponding eigenvectors.

Then, \( \lambda x = (I - aA)x = x - aAx \). Thus, \( Ax = \frac{1-\lambda}{a} x \), i.e., \( \frac{1-\lambda}{a} \) is an eigenvalue of \( A \).

Based on the given conditions, we have \( 0 < \frac{1-\lambda}{a} \leq \rho(A) \) and \( 0 < a < 2/\rho(A) \).

Then, it is easy to see that \( \lambda < 1 \) and \( \lambda \geq 1 - a\rho(A) > 1 - 2 = -1 \).

Hence, \( |\lambda| < 1 \), i.e., the scheme converges. ■

Prob 3. Prove that in any matrix norm \( ||.|| \), \( ||A^k|| \rightarrow \rho(A) \) as \( k \) goes to infinity.