On the Chi-Squaredness of the Quadratic Form $\hat{y}'V^{-1}\hat{y}$

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SUMMARY. In this note we give necessary and sufficient conditions on a symmetric $g$-inverse $V^{-}$ of $V$ such that the quadratic form $\hat{y}' V^{-} \hat{y}$ is distributed chi-squared, where $\hat{y} = X\hat{\beta}$ is the BLUE for $X\beta$ under the linear model $(Y, X\beta, \sigma^2 V)$. Further, an ANOVA result for the general linear model is obtained by decomposing the total sum of squares under the model into independent sums of squares which are distributed chi-squared, and a nonstochastic sum of squares.

1. Chi-squaredness of $\hat{y}' V^- \hat{y}$

Rao (1971) showed that the best linear unbiased estimator (BLUE) of $X\beta$ under the linear model $(Y, X\beta, \sigma^2 V)$ is given by

$$X\hat{\beta} = X(X'V'X)^{-1}X'V'y$$

$$= XA y,$$  \hspace{1cm} (1.1)

where $A$ is defined by collection of terms, $V' = (V + VXU'X')^{-1}$ is a $g$-inverse of $V + VXU'$, and $U$ is an arbitrary conformable matrix such that $C(V + VXU') = C([X:V])$ and $C(V) \cap C(XUX') = \{0\}$. Then $V'$ is a $g$-inverse also of $V$ and of $XUX'$. Rao (1971) also investigated quadratic forms $y'My$ for suitable choices of $M$.

When $Y$ follows a normal distribution, the quadratic form $\hat{y}' V^- \hat{y}$ is not in general distributed chi-squared, i.e. not for an arbitrary choice of a $g$-inverse $V^-$, of $V$. In the following we determine the class of symmetric $g$-inverses $V^-$ of $V$ for which the quadratic form $\hat{y}' V^- \hat{y}$ is distributed chi-squared, and we show that this class is not empty. It will transpire to include all symmetric matrices $V'VV'$.

*Theorem:* If $V^-$ is a symmetric $g$-inverse of $V$, then the quadratic form $\hat{y}' V^- \hat{y}$ is distributed chi-squared under the normal linear model $(Y, X\beta, \sigma^2 V)$ if and only if
\[ X'V'VV'X = X'V'X, \text{ and} \] \[ VV'X = XB, \text{ for some } B. \] \[ (1.2) \]
\[ (1.3) \]

Proof: Khatri (1962, 1963) showed that a quadratic form \( y'Qy \), with \( Q \) symmetric, in a normal variate \( Y \sim N(\mu, V) \), is distributed chi-squared if and only if

\[ VQVQ = VQV, \] \[ (1.4) \]
\[ VQVQ\mu = VQ\mu, \text{ and} \] \[ (1.5) \]
\[ \mu'QVQ\mu = \mu'Q\mu. \] \[ (1.6) \]

We can write

\[ \hat{y}'V^{-1}\hat{y} = y'A'X'V'XAy, \text{ from (1.1)} \]
\[ = y'Qy \]

so that \( Q = A'X'V'XA \). To prove the sufficiency of (1.2) and (1.3) it is sufficient to show that \( QVQ = Q \). But first we note that it is well-known that

\[ XAX = X, \] \[ (1.7) \]
\[ XAVA'X' = XAV, \text{ and} \] \[ (1.8) \]
\[ XAVZ = 0, \] \[ (1.9) \]
where \( Z \) is a matrix of maximum rank such that \( X'Z = 0 \). Then

\[ QVQ = A'X'V'XAVA'X'V'XA \]
\[ = A'X'V'XAVV'XA, \text{ from (1.8)} \]
\[ = A'X'V'XAXB, \text{ from (1.3)} \]
\[ = A'X'V'XBA, \text{ from (1.7)} \]
\[ = A'X'V'V'XA, \text{ from (1.3)} \]
\[ = A'X'V'XA, \text{ from (1.2)} \]
\[ = Q. \]

This sufficiency of (1.2) and (1.3) is shown. To prove necessity, we consider condition (1.6). It is necessary for the chi-squaredness of \( \hat{y}'V^{-1}\hat{y} \) that

\[ \beta'X'A'X'V'XA\beta = \beta'X'A'X'V'XAVA'X'V'XA\beta, \text{ for all } \beta \]
\[ \iff X'V'X = X'V'XAVV'X, \text{ from (1.7), (1.8)}. \] \[ (1.10) \]
But $VV^\top X$ can always be written as

$$VV^\top X = XB + VZC,$$

for some $B$ and $C$ \hspace{1cm} (1.11)

so that we can write

$$X^\top V^\top XAVV^\top X = X^\top V^\top XAXB + X^\top V^\top XAVZC,$$

from (1.11)

$$= X^\top V^\top XB,$$

from (1.7), (1.9)

$$= X^\top V^\top VV^\top X - X^\top V^\top VZC,$$

from (1.11)

$$= X^\top V^\top VV^\top X - C^\top Z^\top VZC,$$

from (1.11).

But any quadratic form which is distributed chi-squared is nonnegative definite with probability 1 (w.p.1) as pointed out by Mitra (1968), so that $X^\top V^\top X \geq X^\top V^\top VV^\top X$, and clearly $C^\top Z^\top VZC \geq 0$. Thus for (1.10) to hold it is necessary that $C^\top Z^\top VZC = 0$, or equivalently $VZC = 0$, which noting (1.11) is in turn equivalent with $VV^\top X = XB$. Now using $VV^\top X = XB$ in (1.10) leads to $X^\top V^\top X = X^\top V^\top VV^\top X$.

We note in passing that the degrees of freedom under the conditions of the theorem are expressible as

$$r(VQV) = r(XAVA^\top X^\top) = r(XAV) = r(XAVV^\top)$$

$$= r[X(X^\top V^\top X)^{-1}X^\top - UXU^\top]$$

$$= r[X(X^\top V^\top X)^{-1}X^\top] - r(AXU^\top)$$

$$= r(X) - r(XUX^\top)$$

or alternatively as

$$tr(VQ) = tr(VA^\top X^\top V^\top XA) = tr(XAVV^\top XA)$$

$$= tr(VV^\top XA) = tr(XBA)$$

$$= tr[(AX - UXU^\top X)A]$$

$$= tr(XA) - tr(XUX^\top V^\top XA)$$

$$= r(X) - r(XUX^\top)$$

$$= r(X) + r(V) - r[V^\top X]$$

Further, condition (1.2) is equivalent to the chi-squaredness of $y^\top V^\top y$, and (1.3) is equivalent to $(y - \hat{y})^\top V^\top \hat{y} = \hat{e}^\top V^\top \hat{y} = 0$, since with $\hat{e} \in C(VZ)$ and $\hat{y} \in C(X)$ we have that $\hat{e} = VZ\gamma$ and $\hat{y} = X\lambda$ for some $\gamma$ and $\lambda$, so that $\hat{e}^\top V^\top \hat{y} = \lambda^\top Z^\top VV^\top X\lambda = \lambda^\top Z^\top XB\lambda = 0$. Thus
(1.2) is responsible for the chi-squaredness of $y'V^{-}y$, and (1.3) allows for the decomposition $y'V^{-}y = \hat{y}'V^{-}\hat{y} + \hat{\epsilon}'V^{-}\hat{\epsilon}$. But $\hat{\epsilon}'V^{-}\hat{\epsilon}$ is chi-squared for any $g$-inverse $V^{-}$ of $V$, and given the chi-squaredness of $y'V^{-}y$ we have the chi-squaredness of $\hat{y}'V^{-}\hat{y}$ as the difference $(a - b)$ of two chi-squared variates $a$ and $b$, and its independence of the variate $b$.

2. An ANOVA Result for the General Linear Model

We consider now the quadratic form $\hat{y}'V'\hat{y}$, where $V'$ is a symmetric $g$-inverse of $V$ as given by (1.1). Quadratic forms involving $V'$ play an important role in the analysis of variance (ANOVA) of the linear model $(Y, X\beta, \sigma^2V)$. It is well-known that the total sum of squares $SST = y'V'y$ associated with the linear model $(Y, X\beta, \sigma^2V)$ can be decomposed into independent sums of squares as

\[
SST = y'V'y = \hat{y}'V'\hat{y} + \hat{\epsilon}'V'\hat{\epsilon} = SSM + SSE,
\]

where $\hat{\epsilon} = y - \hat{y}$. The sum of squares for error $SSE = \hat{\epsilon}'V'\hat{\epsilon}$ is distributed chi-squared, but in general the sum of squares for the model $SSM = \hat{y}'V'\hat{y}$ is not, since it does not in general satisfy condition (1.6). Consequently the total sum of squares $SST$ is not in general distributed chi-squared. Dunne (1982) showed that $SSM$ and consequently $SST$ are distributed chi-squared under the linear model $(Y, X\beta, \sigma^2V)$ if and only if $C(X) \subset C(V)$. In that case $V' = V^\varphi$ the Moore-Penrose inverse of $V$, or any $g_1$-inverse of $V$ that is symmetric.

The failure in general of the noncentrality parameter condition (1.6) is due to the almost sure contribution to the sum of squares made by $XUX^\varphi$ being required in $V' = (V + XUX^\varphi)$, which can be seen by writing the total sum of squares as
\[ SST = y'Vy \]
\[ = y'V'(V + XU)'V'y \]
\[ = y'V'XU'V'y + y'VV'y \]
\[ = y'V'XU'V'y + \hat{y}'VV'h + \hat{e}'VV'h \]
\[ = SSS + SSR + SSE \]
\[ = SSM + SSE \]

where we partition the model sum of squares \( SSM = SSS + SSR \) as the sure sum of squares SSS and the regression sum of squares SSR. Note that \( SSS = y'V'XU'V'y \) is invariant (w.p.1) over all observations \( y \) from the linear model \( (Y, X\beta, \sigma^2 V) \), since \( (y_1 - y_2) \in C(V) \), w.p.1 for an arbitrary pair of observations \( y_1 \) and \( y_2 \), and thus

\[ (y_1 - y_2)'V'XU' = \hat{e}'VV'XU' = 0' \]

for every appropriate choice of \( \hat{e} \). Thus we have that the statistic SSS coincides with its expectation, since

\[ SSS = y'V'XU'V'y \]
\[ = \hat{y}'V'XU'V'y \]
\[ = \beta'X'V'XU'V'X\beta, \quad w.p.1 \]

Noting that \( V' = V'VV' \) is a \( g \)-inverse of \( V \) which satisfies (1.2) and (1.3), we have that the class of \( g \)-inverses \( V' \) of \( V \) for which \( \hat{y}'V'y \) is distributed chi-squared, is not empty. By noting that the rank of such \( V' \) is given by \( r(V') = r(V) \) it is clear that they form a proper subset of \( g \)-inverses of \( V \). The quadratic forms SSR and SSE are distributed chi-squared, as well as the adjusted total sum of squares \( SSA = SSR + SSE = SST' - SSS \). The ANOVA-table corresponding to the decomposition (2.1) is given below.

Observe that the df for SSR agree with (1.13). The notion of df implicit here is the dimension of the spaces within which the decomposed parts of \( y \) must lie. If an alternative notion of df as the parameter of a chi-square distribution is adopted, then there is no entry in the df column for SSS and SST. When \( C(X) \subset C(V) \), the table simplifies as SSS is void and SSA is redundant, and \( V'VV' = V' \). However all the quadratic forms admit any \( V' \) for \( V' \) or \( V' \), in that case.
### ANOVA of the Linear Model ($Y, X\beta, \sigma^2 V$)

<table>
<thead>
<tr>
<th>SS</th>
<th>df</th>
<th>Statistic</th>
<th>Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>SSS</td>
<td>$r([X:V]) - r(V)$</td>
<td>$\hat{y}'V'XUX'V'\hat{y}$</td>
<td>$\beta'X'V'XUX'V'X\beta$</td>
</tr>
<tr>
<td>SSR</td>
<td>$r(V) - s$</td>
<td>$\hat{y}'V'VV'\hat{y}$</td>
<td>$\beta'X'V'VV'X\beta + (r(V) - s)\sigma^2$</td>
</tr>
<tr>
<td>SSE</td>
<td>$s = r([X:V]) - r(X)$</td>
<td>$\hat{e}'V'\hat{e}$</td>
<td>$s^2\sigma^2$</td>
</tr>
<tr>
<td>SSA</td>
<td>$r(V)$</td>
<td>$y'V'VV'y$</td>
<td>$\beta'X'V'VV'X\beta + r(V)\sigma^2$</td>
</tr>
<tr>
<td>SST</td>
<td>$r([X:V])$</td>
<td>$y'V'y$</td>
<td>$\beta'X'V'X\beta + r(V)\sigma^2$</td>
</tr>
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REFERENCES


