Error Estimates for Godunov Mixed Methods for Nonlinear Parabolic Equations

by

Clint Dawson

Technical Report 88-6, May 1988

1A Thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, Rice University.
RICE UNIVERSITY

ERROR ESTIMATES FOR GODUNOV MIXED METHODS
FOR NONLINEAR PARABOLIC EQUATIONS
by
CLINT DAWSON

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE
DOCTOR OF PHILOSOPHY

APPROVED, THESIS COMMITTEE:

Mary F. Wheeler
Prof. Mary F. Wheeler, Mathematical Sciences, Director

William W. Symes
Prof. William W. Symes, Mathematical Sciences

Richard A. Tapia
Prof. Richard A. Tapia, Mathematical Sciences

B. Frank Jones
Prof. B. Frank Jones, Mathematics

Roland Glowinski
Prof. Roland Glowinski, Mathematics,
University of Houston

Houston, Texas
January, 1988
Abstract

Error Estimates for Godunov Mixed Methods for Nonlinear Parabolic Equations

Clint Dawson

Many computational fluids problems are described by nonlinear parabolic partial differential equations. These equations generally involve advection (transport) and a small diffusion term, and in some cases, chemical reactions. In almost all cases they must be solved numerically, which means approximating steep fronts, and handling time-scale effects caused by the advective and reactive processes.

We present a time-splitting algorithm for solving such parabolic problems in one space dimension. This algorithm, referred to as the Godunov-mixed method, involves splitting the differential equation into its advective, diffusive, and reactive components, and solving each piece sequentially. Advection is approximated by a Godunov-type procedure, and diffusion by a mixed finite element method. Reactions split into an ordinary differential equation, which is handled by integration in time. The particular scheme presented here combines the higher-order Godunov MUSCL algorithm with the lowest-order mixed method. This splitting approach is capable of resolving steep fronts and handling the time-scale effects caused by rapid advection and instantaneous reactions.

The scheme as applied to various boundary value problems satisfies maximum principles. The boundary conditions considered include Dirichlet, Neumann and mixed boundary conditions. These maximum principles mimic discretely the classical maximum principles satisfied by the true solution.
The major results of this thesis are discrete $L^\infty(L^2)$ and $L^\infty(L^1)$ error estimates for the method assuming various combinations of the boundary conditions mentioned above. These estimates show that the scheme is essentially first-order in space and time in both norms; however, in the $L^1$ estimates, one sees a much weaker dependence on the lower bound of the diffusion coefficient than is usually derived in standard energy estimates. All of these estimates hold for uniform and non-uniform grid. Error estimates for a lower-order Godunov-mixed method for a fully nonlinear advection-diffusion-reaction problem are also considered. First-order estimates in $L^1$ are derived for this problem.
Acknowledgments

I wish to express heartfelt thanks to Professor Wheeler for her guidance and friendship, and to my parents for their love and support.
Contents

1 Preliminaries 1

1.1 Introduction .................................................. 1
1.2 Notation and definitions ....................................... 5
1.3 Summary of Results .......................................... 9

2 Scheme Development 15

2.1 The basic Godunov-mixed method (GMM) ................. 15
  2.1.1 The higher-order Godunov MUSCL scheme ............... 17
  2.1.2 The lowest-order mixed method ........................ 19
  2.1.3 Slope-limiting ........................................... 21
2.2 Enforcing initial and boundary conditions ............... 22
  2.2.1 Dirichlet boundary conditions .......................... 22
  2.2.2 Neumann boundary conditions ........................... 23
  2.2.3 Mixed boundary condition ............................... 25
  2.2.4 Initial condition ........................................ 25
2.3 Many advection steps per diffusion step .................. 25
2.4 GMM for more general equations ........................... 27
Chapter 1

Preliminaries

1.1 Introduction

Many physical phenomena, in particular many phenomena in fluid dynamics, can be described by nonlinear parabolic partial differential equations or systems of such equations. These equations generally include advection or transport terms, diffusion or dispersion (velocity-dependent diffusion) terms, and in some cases, chemical reaction terms. Moreover, the advection terms are generally larger in magnitude than the dispersion terms; i.e., the problem is advection-dominated. Problems of this type occur, for example, in the fields of reservoir engineering, contaminant transport, aerodynamics, biological flows, and the modelling of semiconductors and catalytic converters.

The difficulties involved in solving these equations numerically have intrigued scientists and engineers for many years. In particular, it is well known that standard finite difference and finite element techniques do not resolve the steep gradients or fronts which are usually associated with these problems. The numerical solutions generated by these schemes often exhibit oscillatory behavior in the neighborhood of the front which is not characteristic of
the true solution. The earliest attempts to control this phenomena led to the development of lower-order schemes such as upwind differencing which overcompensates for the instability by introducing an inordinate amount of numerical diffusion. On coarse grids, this mesh-dependent diffusion is often larger in magnitude than the physical diffusion given in the equation. Thus, while the numerical solutions generated by these methods are stable, and, in fact, satisfy maximum principles, the fronts are still not approximated accurately. In the last several years, however, many schemes have been proposed which combine numerical stability with accurate front resolution. These schemes have generally been successful because they do a better job of approximating the essentially hyperbolic nature of the equation.

Another numerical difficulty for these problems is one of time-scale; i.e., the physical processes of advection, diffusion, and reaction occur on different time-scales and need to be modelled as such. These considerations become particularly acute when reactions are present which occur almost instantaneously.

For problems in multidimensions, a numerical difficulty one can encounter involves sensitivity to grid orientation. A scheme which is sensitive to grid orientation will often exhibit radically different solutions depending on whether the solution is generated on a diagonal grid or a parallel grid. Here the terms diagonal and parallel refer to the orientation of the grid with respect to the direction of flow. Thus, one design criterion for multidimensional algorithms is to inhibit grid orientation effects as much as possible. Lower-order upwinding schemes and spatial operator-splitting schemes, such as alternating direction methods, are especially prone to this phenomenon. For excellent discussions on grid orientation and its effect on algorithm design, see [2, 4, 35].

The methods we wish to investigate (hereafter referred to as Godunov-mixed methods) for solving these types of problems are operator-splitting or, more accurately, time-
splitting techniques based on combining Godunov-type finite difference methods for scalar conservation laws with mixed finite element methods for parabolic equations. (For a general discussion of time-splitting for linear partial differential equations, see LeVeque [29].) The combining of these two schemes is natural because both are finite element methods which utilize discontinuous approximating spaces. These methods are generalizable to many different types of advective flow problems, including multidimensional problems and systems of equations. They also show much promise for parallelization. Versions of this idea have been tested numerically and have been shown to work well for such problems as the one-dimensional Buckley-Leverett equation and the two-dimensional miscible displacement equations [1, 8, 43].

The original Godunov scheme (sometimes referred to as the lower-order or first-order Godunov scheme) [19] for scalar conservation laws has been the basis for many finite difference schemes (see Harten, Lax, and van Leer [22] for an overview of Godunov-type schemes). The algorithm was first extended to a higher-order method by van Leer [39] in his MUSCL scheme for gas dynamics. This work was later refined by Colella [11] and Colella and Woodward [13]. Bell and Shubin [3] were the first to extend these ideas to problems in reservoir simulation. Bell, Dawson, and Shubin [1] later refined their work specifically for two-dimensional problems. Other examples of Godonov-type schemes include the ENO schemes of Chakravarthy, Engquist, Harten, and Osher [7, 20, 23]. The method has also been developed for systems of conservation laws (see, for example, Colella [12]). Furthermore, it has been to extended to a more general finite element setting by Chavent and Jaffre [8], and Chavent and Cockburn [9].

The theory of Godunov-type schemes is concerned mainly with stability properties and convergence to a weak solution satisfying an entropy condition. Harten, Hyman, and Lax
[21] proved stability and convergence to an entropy solution for a class of explicit monotone schemes (which included Godunov's scheme) in one dimension. Crandall and Majda [14] extended these results to problems in n-dimensions. Sanders [36] later generalized this work to nonuniform grids for explicit or implicit 3-point schemes. He also derived an $L^\infty(L^1)$ error estimate, extending work of Kruzkov [24] and Kuznetsov [25]. Error estimates for variations of these schemes have also been derived by LeRoux [28], Lucier [30], and Cockburn [10]. Stability and convergence to entropy solutions for higher-order schemes have been established by Osher and Chakravarthy [32] and by Sweby [37]. Wheeler [41] proved stability and weak convergence for the MUSCL scheme. In this thesis, we are not concerned with convergence to weak solutions, as we are considering only parabolic problems with unique, smooth solutions. To our knowledge, no theoretical results exist for combining Godunov-type schemes with other methods for solving parabolic equations.

Mixed finite element methods have proven to be a very robust tool for solving elliptic and parabolic equations. The scheme was originally developed by Thomas [38] for elliptic equations. Error estimates for elliptic equations which arise out of compressible and incompressible flow problems have been derived for these schemes on rectangular meshes by Brezzi [5], Falk and Osborn [17], Raviart and Thomas [33], and Douglas, Ewing, and Wheeler [16]. Extensions to more general geometry have been developed by Brezzi, Douglas, and Marini [6]. More recently, Glowinski and Wheeler [18] have examined the use of mixed methods in a domain decomposition algorithm for elliptic equations. Extension of these ideas to parabolic equations such as the heat equation is straightforward. In [31] and [40], extensions of the mixed method to more general parabolic equations are given.

In this thesis, we wish to establish conservation of stability (maximum principles) and derive error estimates for Godunov-mixed methods applied to one-dimensional problems of
the form

$$\frac{\partial s}{\partial t} + \frac{\partial f(s)}{\partial x} - \frac{d}{dx} \left( a(x,t,s(x,t)) \frac{\partial s}{\partial x} \right) = r(x,t,s), \quad (x,t) \in Q_T, \quad (1.1)$$

$$Q_T = [0, 1] \times (0, T) = I \times (0, T).$$

with smooth initial data and various types of smooth boundary conditions. Here, by advection-dominated flow we mean that in (1.1), $|f'| >> a$, and by $d/dx$ we are referring to the total derivative with respect to $x$. The Godunov-type scheme we employ is the higher-order MUSCL scheme, which by a simple choice of parameters, reduces to the first-order Godunov scheme. Furthermore, in the dispersion step, we employ the lowest-order Raviart-Thomas approximating spaces and apply special quadrature rules which reduces the mixed method to block-centered finite differences [35, 40]. In our analysis, we assume for simplicity that $f'(s) \geq 0$ and we must have that $0 < a_* \leq a(x,t,s) \leq a^*$ for some positive constants $a_*, a^*$. Throughout this work we assume $s$ and the coefficients are sufficiently smooth so that truncation errors exist.

The rest of this chapter is divided into two sections. In Section 1.2, we give some notation and definitions which we will employ throughout the rest of this thesis. In Section 1.3, we summarize the theoretical results obtained in Chapters 3-4 for the Godunov-mixed method described in Chapter 2. In Appendix D, we discuss the solvability of the differential equations we are considering here, and present conditions which guarantee a classical solution.

### 1.2 Notation and definitions

In this section we give some notation and definitions which we will utilize throughout the rest of the thesis.
The standard notation \( g = O(h^m) \) means there exists a constant \( C \) independent of \( h \) such that
\[
|g(h)| \leq C h^m
\]
as \( h \to 0 \).

Let
\[
\delta_x : 0 = x_0, \ldots, x_{J-\frac{1}{2}} = 1
\]
be a partition of \([0, 1]\) into grid blocks \( B_j \),
\[
B_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], \quad j = 1, \ldots, J - 1.
\]

Let \( x_j \) be the midpoint of \( B_j \), and
\[
\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}},
\]
\[
\Delta x_{j+\frac{1}{2}} = x_{j+1} - x_j, \quad j = 1, \ldots, J - 2,
\]
with \( \Delta x_{\frac{1}{2}} = \Delta x_1 \) and \( \Delta x_{J-\frac{1}{2}} = \Delta x_{J-1} \). Define
\[
\mathcal{M}^k_{-1}(\delta_x) = \{ v | v \text{ is a polynomial of degree } \leq k \text{ on } B_j \},
\]
and set
\[
\mathcal{M}^k_0(\delta_x) = \mathcal{M}^k_{-1}(\delta_x) \cap C^0(I).
\]

Finally, for \( \Delta t^n > 0 \), define
\[
t^n = \sum_{k=0}^{n-1} \Delta t^k,
\]
and let \( N^* \) be such that \( t^{N^*} = T \). In this method, \( \Delta t^n \) is chosen in accordance with a CFL condition as described later.
Let \( a, b \in \mathcal{M}_{-1}^{0}(\delta x) \). We will employ the following discrete difference operators. For \( j = 1, \ldots, J - 2 \), let
\[
    b_{j, x} = \frac{b_{j+1} - b_{j}}{\Delta x_{j+\frac{1}{2}}}.
\]
For \( j = 2, \ldots, J - 1 \), define
\[
    b_{j, x} = \frac{b_{j} - b_{j-1}}{\Delta x_{j-\frac{1}{2}}},
\]
and
\[
    b_{j, \dot{x}} = \frac{b_{j} - b_{j-1}}{\Delta x_{j}}.
\]
For \( b = b^n \) time-dependent, let
\[
    b_{j, t}^{n+1} = \frac{b_{j}^{n+1} - b_{j}^{n}}{\Delta t^{n}}.
\]
Also, for simplicity we will write \( (a_j b_{j, x})_x \) for \( (a_{j+\frac{1}{2}} b_{j, x})_x \). Furthermore, \( b_x \) will refer to the function in \( \mathcal{M}_{-1}^{0}(\delta x) \) with \( b_x = b_{j, x} \) on \( B_j \), \( j = 1, \ldots, J - 2 \), and \( b_x \) will be specified as needed on \( B_{J-1} \); see, for example, \( (2.34) \). Similarly,
\[
    b_x|_{B_j} = b_{j, x}, \quad j = 2, \ldots, J - 1,
\]
and \( b_x \) is defined appropriately on \( B_1 \), e.g. see \( (2.30) \).

Throughout this thesis, we will find it convenient to define extensions of functions in \( \mathcal{M}_{-1}^{0}(\delta x) \) to the intervals \([-\Delta x_1, 0]\) and \([1, 1 + \Delta x_{J-1}]\). For \( b \) an arbitrary function in \( \mathcal{M}_{-1}^{0}(\delta x) \), these extensions are denoted by \( b_0 \) and \( b_J \), respectively, and they are defined by the boundary conditions. By employing this notation, the discrete difference operators given above are valid for all \( j = 1, \ldots, J - 1 \).

We will employ the following discrete inner product and norms. Define
\[
    \langle a, b \rangle = \sum_{j=1}^{J-1} a_j b_j \Delta x_j,
\]
$$||b||_{L^p} = \left( \sum_{j=1}^{J-1} |b_j|^p \Delta x_j \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

and

$$||b||_{L^\infty} = \max_j |b_j|,$$

where

$$\max \equiv \max_{1 \leq j \leq J-1}.$$

When we write $||b||$ we mean $||b||_{L^2}$. Furthermore, we will use the following discrete space-time norms. For $1 \leq p \leq \infty$ and $0 \leq n_1 \leq n_2 \leq N^*$, let

$$||b||_{L^q(n_1,n_2;L^p)} = \left( \sum_{n=n_1}^{n_2} ||b^n||_{L^p}^q \Delta t^n \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty,$$

and

$$||b||_{L^\infty(n_1,n_2;L^p)} = \max_{n_1 \leq n \leq n_2} ||b^n||_{L^p}.$$

Moreover, as shorthand notation we will write $||b||_{L^2(\mathbb{R}^2)}$ for $||b||_{L^2(0,N^*_1;L^2)}$, etc.

We will use the following continuous function spaces with corresponding norms. For $\Gamma$ a measurable domain, let

$$L^p(\Gamma) = \{g \mid \left( \int_\Gamma |g|^p \right)^{\frac{1}{p}} < \infty \}, \quad 1 \leq p < \infty,$$

$$L^\infty(\Gamma) = \{g \mid \sup_\Gamma |g| < \infty \}$$

and let

$$||g||_{L^p(\Gamma)} = \left( \int_\Gamma |g|^p \right)^{\frac{1}{p}},$$

$$||g||_{L^\infty(\Gamma)} = \sup_\Gamma |g|.$$

For $1 \leq p < \infty$ and $g = g(x)$, let

$$\mathcal{H}^p(I) = \{g \mid \frac{d^kg}{dx^k} \in L^2(I), \; 0 \leq k \leq p \},$$
with

$$\|g\|_{\mathcal{H}^p(I)} = \left( \sum_{k=0}^{p} \left\| \frac{d^k g}{d x^k} \right\|_{L^2(I)}^2 \right)^{\frac{1}{2}}.$$  

Also, let

$$\mathcal{W}^q_\infty(\Gamma) = \{ g | \text{all derivates of } g \text{ of order } \leq q \text{ are in } L^\infty \text{ on } \Gamma \},$$

with

$$\|g\|_{\mathcal{W}^q_\infty(\Gamma)} = \max_{1 \leq l \leq q} \|D_x^l g\|_{L^\infty(\Gamma)}.$$  

By $D_x^l g$ we mean any partial derivative of $g$ of order $l$. Here $\Gamma$ could also be time dependent, in which case the variable $x$ would have $t$ as one of its components.

We will also employ continuous space-time norms. Let $0 \leq t_1 \leq t_2 \leq T$, and assume $\Gamma$ is a measurable spatial domain. Define for $1 \leq p < \infty$ and $1 \leq q \leq \infty$,

$$\mathcal{L}^p(t_1, t_2; \mathcal{L}^q(\Gamma)) = \{ g \mid \left( \int_{t_1}^{t_2} \|g\|_{\mathcal{L}^q(\Gamma)}^p dt \right)^{\frac{1}{p}} < \infty \},$$

$$\mathcal{L}^\infty(t_1, t_2; \mathcal{L}^q(\Gamma)) = \{ g \mid \sup_{(t_1, t_2)} \|g\|_{\mathcal{L}^q(\Gamma)} < \infty \},$$

with norms

$$\|g\|_{\mathcal{L}^p(t_1, t_2; \mathcal{L}^q(\Gamma))} = \left( \int_{t_1}^{t_2} \|g\|_{\mathcal{L}^q(\Gamma)}^p dt \right)^{\frac{1}{p}},$$

$$\|g\|_{\mathcal{L}^\infty(t_1, t_2; \mathcal{L}^q(\Gamma))} = \sup_{(t_1, t_2)} \|g\|_{\mathcal{L}^q(\Gamma)}.$$  

For simplicity we will write $\|g\|_{\mathcal{L}^\infty(\Omega_T)}$ for $\|g\|_{\mathcal{L}^\infty(0, T; \mathcal{L}^\infty(I))}$.

### 1.3 Summary of Results

In this section we summarize the major results of this thesis. As indicated earlier, we have restricted our attention to problems of the form

$$\frac{\partial s}{\partial t} + \frac{\partial f(s)}{\partial x} - \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial s}{\partial x} \right) = 0, \quad (x, t) \in I \times (0, T] \equiv Q_T,$$  

(1.2)
or more generally,

$$\frac{\partial s}{\partial t} + \frac{\partial f(s)}{\partial x} - \frac{d}{dx} \left( a(x, t, s(x, t)) \frac{\partial s}{\partial x} \right) = r(x, t, s), \quad (x, t) \in Q_T. \quad (1.3)$$

We assume the initial condition

$$s(x, 0) = s^0(x), \quad x \in I, \quad \quad \quad (1.4)$$

and we consider boundary conditions of the form,

$$s(0, t) = g_0(t), \quad \quad \quad (1.5)$$

$$a(0, t, s) \frac{\partial s}{\partial x}(0, t) = 0, \quad \quad \quad (1.6)$$

or

$$f(s(0, t)) - a(0, t, s) \frac{\partial s}{\partial x}(0, t) = f(g_0(t)) \quad (1.7)$$

at the left endpoint. At the right endpoint we assume either,

$$s(1, t) = g_1(t), \quad \quad \quad (1.8)$$

or

$$a(1, t, s) \frac{\partial s}{\partial x}(1, t) = 0, \quad 0 < t \leq T. \quad \quad \quad (1.9)$$

Throughout we assume $f \in C^1(\mathbb{R})$, $f' \geq 0$, and $0 < a_* \leq a(x, t, s) \leq a^*$.

In Chapter 2, we present a Godunov-mixed finite element procedure for solving problems of the form given above. As we will see, this method is based on splitting (1.2) (similarly (1.3)) into an advection equation,

$$\frac{\partial \tilde{s}}{\partial t} + \frac{\partial f(\tilde{s})}{\partial x} = 0,$$
and a diffusion equation,

\[ \frac{\partial s^*}{\partial t} - \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial s^*}{\partial x} \right) = 0. \]

The result of our splitting method is an approximation \( S^n \in M_{-1}^1(\delta z) \) to \( s(x, t^n) \) at each diffusion time-level \( t^n \) which is of the form

\[ S^n(x)|_{x \in B_j} = S_j^n + (x - x_j) \delta S_j^n. \]  \hspace{1cm} (1.10)

The method we employ to solve the advection equation is the higher-order Godunov MUSCL algorithm of van Leer, while for the diffusion equation we employ a mixed finite element method based on using the lowest-order Raviart-Thomas approximating spaces together with special quadrature rules. Generally, since the MUSCL scheme is explicit and the mixed method implicit in time, one solves the advection equation for many small CFL time-steps for each diffusion step. By CFL time-step we are referring to the well-known Courant-Friedrichs-Lewy time-step restriction which relates the time-step \( \Delta t \) to \( \Delta x \). One must enforce this restriction in the Godunov method to preserve stability of the solution. The necessity for this type of restriction will become clearer when we derive maximum principles for the scheme in Chapter 3.

When reactions are present, one can include them in the diffusion equation, or split them into an ordinary differential equation

\[ \frac{ds}{dt} = r(x, t, s), \]

and solve on a different time-scale than either advection or diffusion.

In Chapter 3 we examine the stability of the scheme derived in Chapter 2. Here we show that \( S^n \) as given by (1.10) satisfies certain maximum principles which mimic in a discrete sense the well-known maximum principles [26] which solutions to (1.2) satisfy; i.e.,
a classical solution \( s \) to (1.2) attains its maximum and minimum on the boundary of \( Q_T \). Thus, we show that when (1.2), (1.4) and Dirichlet boundary conditions (1.5), (1.8) hold, then

\[
m^n \leq S_j^n \leq M^n, \tag{1.11}
\]

where

\[
m^n = \min \left( \min_j S_j^0, \inf_{[0,t^n]} g_0(t), \inf_{[0,t^n]} g_1(t) \right),
\]

and

\[
M^n = \max \left( \max_j S_j^0, \sup_{[0,t^n]} g_0(t), \sup_{[0,t^n]} g_1(t) \right).
\]

When Neumann boundary conditions (1.6), (1.7) hold, we obtain

\[
\min S_j^0 \leq S_j^n \leq \max S_j^0. \tag{1.12}
\]

Finally, when \( s \) satisfies (1.7) at \( x = 0 \) and (1.8) at \( x = 1 \), we obtain (1.11), while a condition combining (1.11) and (1.12) holds when one substitutes Neumann boundary conditions at \( x = 1 \).

Chapter 4 contains the major results of this thesis. Here we derive error estimates in discrete \( L^\infty(L^2) \) and \( L^\infty(L^1) \) norms for the scheme applied to (1.2), (1.4) with various combinations of the boundary conditions (1.5)-(1.9). The first estimate we derive is for the problem (1.2), (1.4) with Dirichlet boundary conditions (1.5) and (1.8). We show that

\[
||s - S||_{L^\infty(L^2)} \leq C(\Delta t + TE), \tag{1.13}
\]

where \( C \) is a constant which depends on, among other things, \( e^{(\alpha + 1)} \). Here \( s_j^n = s(x_j, t^n) \) and \( TE \) represents the truncation error; i.e., the error which arises when the discretization procedure is applied to the true solution \( s \). In Appendix A we show by Taylor expansion
that for \( s \) and the coefficients and data sufficiently smooth (see Theorem 4.4 for regularity assumptions)

\[
TE \leq C(\Delta x^{3/2} + \Delta t).
\] (1.14)

These results can be extended to include the case \( a = a(x, t, s) \).

For this same problem we also show that

\[
||s - S||_{L^\infty(L^1)} \leq C(\Delta x + \Delta t),
\] (1.15)

where in this case \( C \) depends on \( a \). Thus, we prove the result that if \( a_0 \to 0 \) as \( \Delta x \to 0 \), where \( \alpha < 1/2 \), and if the solutions \( s_{a_0} \) to (1.2) remain smooth as \( a_0 \to 0 \) and converge in \( L^1(0,T;L^1(I)) \) to \( \bar{s} \) satisfying an equation of the form (1.2) with \( a(x, t) \geq 0 \), then our approximation \( S_{a_0} \) to \( s_{a_0} \) also converges to \( \bar{s} \) in this norm. Furthermore, if \( s_{a_0} \) converges to \( \bar{s} \) at some rate, then we can obtain a rate for the convergence of \( S_{a_0} \) to \( \bar{s} \). We note that in some physical applications, the diffusion coefficient can only be assumed to be nonnegative, hence this result may apply to these problems. Finally, for the Dirichlet problem we derive a result similar to (1.13)-(1.14) for the modification to the algorithm whereby one takes several advection steps per diffusion step.

Also in Chapter 4, we derive results for the problem (1.2) with a Dirichlet boundary condition at \( x = 0 \) and a Neumann boundary condition (1.9) at \( x = 1 \), and for the problem (1.2) with a mixed boundary condition (1.7) at \( x = 0 \) and a Dirichlet boundary condition at \( x = 1 \). In the first case we obtain

\[
||s - S||_{L^\infty(L^1)} \leq \frac{C}{a^2}(\Delta x + \Delta t).
\] (1.16)

For the latter case we obtain a result similar to that given by (1.13)-(1.14). Truncation errors for these cases are also given in the appendices.
We claim that all of these results hold when one has nonuniform spatial grid, with the modification that $TE = \mathcal{O}(\Delta x + \Delta t)$. To demonstrate this fact, in Chapter 4 we rederive (1.13) for the case of quasi-uniform spatial grid.

We conclude Chapter 4 by deriving a $\mathcal{O}(\Delta x + \Delta t)$ estimate in $L^\infty(L^1)$ for a first-order Godunov-mixed method applied to the nonlinear equation (1.3) with initial condition (1.4) and Dirichlet boundary conditions. Here we use a different technique which is particular to first-order schemes. Namely we use the fact that the $L^1$ norm of the error at time level $t^{n+1}$ is bounded by the $L^1$ norm of the error at time $t^n$ plus truncation error. This technique can also be used to show that the first-order Godunov procedure for scalar conservation laws is truly globally first-order, when the solution is sufficiently smooth. Truncation error analysis for this procedure is given in Appendix C.
Chapter 2

Scheme Development

In this chapter, we derive Godunov-mixed methods for parabolic equations in one dimension with Dirichlet, Neumann, or mixed boundary conditions. The Godunov scheme we employ is based on van Leer's MUSCL algorithm.

In the first section, we present the basic Godunov-mixed method for a quasi-linear parabolic equation in one space dimension. In Section 2.2, we discuss how we enforce boundary and initial conditions for this scheme. In Section 2.3, a modification of the basic scheme is presented whereby one takes many small advection steps per diffusion step. Finally, in Section 2.4 we derive Godunov-mixed methods for more general parabolic problems.

2.1 The basic Godunov-mixed method (GMM)

Consider the problem

\[
\frac{\partial s}{\partial t} + \frac{\partial f(s)}{\partial x} - \frac{\partial}{\partial x} \left( a(x,t) \frac{\partial s}{\partial x} \right) = 0, \quad (x,t) \in I \times (0,T] \equiv Q_T,
\]

with initial condition

\[
s(x,0) = s^0(x), \quad x \in I.
\]
At $x = 0$, assume $s$ satisfies, for $0 < t \leq T$, either a Dirichlet boundary condition,

$$s(0, t) = g_0(t),$$

(2.3)

a homogeneous Neumann (no-flow) boundary condition,

$$a(0, t) \frac{\partial s}{\partial x}(0, t) = 0,$$

(2.4)

or a mixed boundary condition of the form

$$f(s(0, t)) - a(0, t) \frac{\partial s}{\partial x}(0, t) = f(g_0(t)).$$

(2.5)

At $x = 1$, assume $s$ satisfies either

$$s(1, t) = g_1(t),$$

(2.6)

or

$$a(1, t) \frac{\partial s}{\partial x}(1, t) = 0, \quad 0 < t \leq T.$$ 

(2.7)

We assume that the boundary and initial conditions are compatible; i.e., $s^0(0) = g_0(0)$ when (2.3) holds, etc.; and we also assume $f \in C^1(\mathbb{R})$, $f' \geq 0$, and $a(x, t) > 0$.

Our operator-splitting approach can be outlined as follows. Assume we have an approximation, $S(x, \bar{t})$, to $s(x, \bar{t})$ satisfying (2.1), and we want an approximation to $s(x, t^*)$, where $0 \leq \bar{t} < t^* \leq T$. We first apply the MUSCL scheme to the equation

$$\frac{\partial \bar{s}}{\partial t} + \frac{\partial f(\bar{s})}{\partial x} = 0, \quad (x, t) \in I \times [\bar{t}, t^*],$$

(2.8)

with initial condition

$$\bar{s}(x, \bar{t}) = S(x, \bar{t}), \quad x \in I.$$ 

(2.9)

We then apply the mixed method to the equation

$$\frac{\partial s^*}{\partial t} - \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial s^*}{\partial x} \right) = 0, \quad (x, t) \in I \times [\bar{t}, t^*],$$

(2.10)
with initial condition

\[ s^n(x, t^n) = \bar{s}(x, t^*), \quad x \in I, \]  

(2.11)

where \( \bar{s}(x, t^*) \) is the Godunov approximation to \( \bar{s}(x, t^*) \). The result of these two steps is an approximation, \( s(x, t^*) \), to \( s(x, t^*) \) satisfying (2.1). We then iterate on this process by letting \( s(x, t^*) \) be the initial condition for the next advection step.

We now describe the Godunov MUSCL procedure and the mixed method. We begin with the MUSCL procedure.

2.1.1 The higher-order Godunov MUSCL scheme

In (2.8), let \( \bar{t} = t^n \) and \( t^* = t^{n+1} \). The MUSCL algorithm can be derived by integrating (2.8) over the space-time domain \( B_j \times [t^n, t^{n+1}] \) and applying (2.9). Doing so one obtains

\[
\int_{B_j} (\bar{s}^{n+1}(x) - S^n(x)) \, dx + \int_{t^n}^{t^{n+1}} (f(\bar{s}(x_{j+\frac{1}{2}}, t) - f(\bar{s}(x_{j-\frac{1}{2}}, t)) \, dt = 0,
\]

where \( \bar{s}^{n+1}(x) = \bar{s}(x, t^{n+1}) \), and \( S^n(x) \approx s(x, t^n) \). We approximate the second integral above by the midpoint rule, obtaining, for \( f \) and \( s \) sufficiently smooth,

\[
\int_{B_j} (\bar{s}^{n+1}(x) - S^n(x)) \, dx + \Delta t^n [f(\bar{s}(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}}) - f(\bar{s}(x_{j-\frac{1}{2}}, t^{n+\frac{1}{2}})) + \mathcal{O}(\Delta t^2) = 0,
\]

(2.12)

where \( t^{n+\frac{1}{2}} = (t^n + t^{n+1})/2 \).

Let \( \bar{s}^{n+1} \in \mathcal{M}^0_{-1}(\delta_x) \) be an approximation to \( \bar{s}(x, t^{n+1}) \), and suppose \( S^n \in \mathcal{M}^1_{-1}(\delta_x) \) is an approximation to \( s(x, t^n) \) of the form

\[ S^n(x) = S^n_j + (x - x_j)\delta S^n_j, \quad x \in B_j, \]  

(2.13)

where \( \delta S^n_j \) is determined by a slope-limiting procedure outlined below.
We now must define an approximation to \( f(\bar{s}(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}})) \). For the case \( f' \geq 0 \) we approximate this term by \( f(S^n_{j,L}) \), where \( S^n_{j,L} = \bar{s}(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}}) \) and \( \bar{s} \) satisfies the linear advection problem

\[
\frac{\partial \bar{s}}{\partial t} + f'(S^n_{j}) \frac{\partial \bar{s}}{\partial x} = 0, \quad (x, t) \in I \times (t^n, t^{n+\frac{1}{2}})
\]

with

\[
\bar{s}(x, t^n) = S^n(x), \quad x \in I.
\]

Thus, \( S^n_{j,L} \) is found by evaluating \( S^n(x) \) at the point \( x_{j,L} \), where

\[
x_{j,L} = x_{j+\frac{1}{2}} - \frac{\Delta t^n}{2} f'(S^n_j).
\]

Assuming the CFL condition

\[\sup_j f'(S^n_j) \frac{\Delta t^n}{\Delta x_j} \leq 1, \quad (2.14)\]

we have \( x_{j,L} \in B_j \). Hence, by (2.13),

\[
S^n_{j,L} = S^n(x_{j,L}) = S^n_j + \frac{\Delta x_j}{2} (1 - f'(S^n_j) \frac{\Delta t^n}{\Delta x_j}) \delta S^n_j. \quad (2.15)
\]

This is valid for \( j = 1, \ldots, J - 1; S^n_{0,L} \) is determined by the boundary conditions and is given in the next section.

For more general \( f(s) \), the approximation to \( f(\bar{s}(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}})) \) is determined by calculating left and right states, \( S^n_{j,L} \) and \( S^n_{j,R} \), and solving a Riemann problem. For our immediate purpose, we assume \( f' \geq 0 \).

Given \( S^n_j, j = 1, \ldots, J - 1, \) and \( S^n_{j,L}, j = 0, \ldots, J - 1 \), we obtain \( \bar{s}^{n+1}_j = \bar{s}^{n+1}(x), \quad x \in B_j \), by

\[
\bar{s}^{n+1}_j = S^n_j - \frac{\Delta t^n}{\Delta x_j} [f(S^n_{j,L}) - f(S^n_{j-1,L})]. \quad (2.16)
\]
Equation (2.16) follows from (2.12), (2.13), and the assumption that $\tilde{S}^{n+1} \in M_{-1}^0(\delta_x)$ by observing
\[
\int_{B_j} \tilde{S}^{n+1} dx = \Delta x_j \tilde{S}_j^{n+1},
\]
\[
\int_{B_j} S^n dx = \Delta x_j S_j^n,
\]
and by neglecting the $O(\Delta t^2)$ term.

2.1.2 The lowest-order mixed method

Given $\tilde{S}^{n+1}$ as the initial condition, we derive a mixed method for (2.10) and (2.11) as in [35, 40]. Thus, we use the lowest-order Raviart-Thomas approximating spaces and special quadrature rules.

Let $s^*$ satisfy (2.10). Let
\[
u^{n+1}(x) = -a(x, t^{n+1}) \frac{\partial s^*}{\partial x}(x, t^{n+1}), \quad x \in I,
\]
and let $U^{n+1} \approx u^{n+1}, \quad U^{n+1} \in M_0^1(\delta_x)$. Let $S^{n+1} \approx s^*(x, t^{n+1})$, where $S^{n+1} \in M_{-1}^0(\delta_x)$. The approximations $U^{n+1}$ and $S^{n+1}$ are determined as follows.

First, multiply (2.17) by $v$, where $v$ is a test function in $M_0^1(\delta_x)$, and divide by $v(x, t^{n+1})$, then integrate over $I$ and integrate by parts. Substituting $U^{n+1}$ for $u^{n+1}$ and $S^{n+1}$ for $s^*(x, t^{n+1})$ we obtain
\[
\int_I \frac{1}{a^{n+1}} U^{n+1}(x) v(x) dx - \int_I S^{n+1}(x) v'(x) dx = -sv|_0^I.
\]
(2.18)

Note that when Neumann boundary conditions (2.4) and/or (2.7) hold(s), we choose $v$ above so that $v(0) = 0$ and/or $v(1) = 0$, since in the mixed method, Neumann boundary conditions are the essential boundary conditions; i.e., they are enforced exactly, while Dirichlet boundary conditions are enforced weakly.
Moreover, by applying backward differencing in time in (2.10), replacing \( \phi^*(z, t^{n+1}) \) with \( S^{n+1} \) and \(- \frac{\partial}{\partial z}(a(z, t^{n+1}) \tilde{S}^{n+1}(z, t^{n+1}))\) with \( \frac{d}{dz}(U^{n+1}) \), multiplying by a test function \( w \in \mathcal{M}_{0,1}(\delta z) \) and integrating we find that

\[
\int_I \frac{S^{n+1}(x) - \tilde{S}^{n+1}(x)}{\Delta t^n} w(x) dx + \int_I \frac{d}{dz}(U^{n+1}(x)) w(x) dx = 0. \tag{2.19}
\]

By evaluating the integrals in (2.18) using the trapezoidal rule one has

\[
U^{n+1}(x_{j+\frac{1}{2}}) = -a(x_{j+\frac{1}{2}}, t^{n+1}) \frac{S^{n+1}_{j+1} - S^{n+1}_j}{\Delta x_{j+\frac{1}{2}}}, \quad j = 0, \ldots, J - 1, \tag{2.20}
\]

where for \( j = 1, \ldots, J - 1, S^{n+1}_j = S^{n+1}(x), x \in B_j, \) and \( S^{n+1}_0 \) and \( S^{n+1}_J \) are determined by the boundary conditions. Moreover, choosing \( w \) appropriately and integrating exactly in (2.19), and substituting (2.20), we see that

\[
\frac{S^{n+1}_j - S^{n+1}_j}{\Delta t^n} \left[ a^{n+1}_{j+\frac{1}{2}} \frac{S^{n+1}_{j+1} - S^{n+1}_j}{\Delta x_{j+\frac{1}{2}}} - a^{n+1}_{j-\frac{1}{2}} \frac{S^{n+1}_j - S^{n+1}_{j-1}}{\Delta x_{j-\frac{1}{2}}} \right] = 0, \tag{2.21}
\]

for \( j = 1, \ldots, J - 1 \). Here \( a^{n+1}_{j+\frac{1}{2}} = a(x_{j+\frac{1}{2}}, t^{n+1}). \) Thus, the mixed method with lowest-order approximating spaces and the trapezoidal rule of integration is equivalent to block-centered finite differences applied to (2.10).

We note that when \( a \) is discontinuous at grid block boundaries, as often happens in applications, we can replace \( a^{n+1}_{j+\frac{1}{2}} \) by arithmetically or harmonically averaging the average values of \( a \) over grid blocks \( B_j \) and \( B_{j+1} \). Either of these approximations arises naturally in the context of the mixed formulation as given by (2.18).

Substituting (2.16) into (2.21) we obtain

\[
\frac{S^{n+1}_j - S^n_j}{\Delta t^n} + \frac{f(S^n_{j,L}) - f(S^n_{j-1,L})}{\Delta x_j} \left[ a^{n+1}_{j+\frac{1}{2}} \frac{S^{n+1}_{j+1} - S^n_j}{\Delta x_{j+\frac{1}{2}}} - a^{n+1}_{j-\frac{1}{2}} \frac{S^n_j - S^{n+1}_{j-1}}{\Delta x_{j-\frac{1}{2}}} \right] = 0, \tag{2.22}
\]
for $j=1, \ldots, J-1$. Hence, to determine $S^{n+1}$ we must solve a positive definite, symmetric, tridiagonal system of equations at each time $t^n$.

### 2.1.3 Slope-limiting

The final step in the algorithm is the projection of $S^{n+1}$ into the space $\mathcal{M}^1_{-1}(\delta_x)$. This is done by using (2.13) with limited slope $\delta S^n_j$ determined as follows.

Let

$$
\mu_j = \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}},
$$

and set

$$
\delta S^n_j = \frac{S^n_{j+1} - (1 - \mu_j^2)S^n_j - \mu_j^2 S^n_{j-1}}{\mu_j(\Delta x_{j+\frac{1}{2}} + \Delta x_{j-\frac{1}{2}})},
$$

for $j = 2, \ldots, J - 2$. The limited slope $\delta S^n_j$ is given by

$$
\delta S^n_j = \min(\delta_{\text{lim}} S^n_j, |\delta S^n_j|) \cdot \text{sgn}(\delta S^n_j),
$$

where

$$
\delta_{\text{lim}} S^n_j = \begin{cases} 
\alpha_{l,j} \min(\{|S^n_{j,x}|, |S^n_{j,x-1}|\}, & \text{if } (S^n_{j,x}) \cdot (S^n_{j,x-1}) > 0, \\
0, & \text{otherwise}.
\end{cases}
$$

Here $\alpha_{l,j}$ is a parameter satisfying

$$
0 \leq \alpha_{l,j} \leq 2 \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}, \quad j = 1, \ldots, J - 2,
$$

and

$$
0 \leq \alpha_{l,J-1} \leq 2.
$$

Limited slopes $\delta S^n_1$ and $\delta S^n_{J-1}$ incorporating the boundary conditions are given in the next section.
2.2 Enforcing initial and boundary conditions

2.2.1 Dirichlet boundary conditions

With Dirichlet boundary conditions a limited slope can be calculated for the first interval by setting

\[
\delta S_1^n = \min(\delta_{lim} S_1^n, |\delta S_1^n|) \cdot \text{sgn}(\delta S_1^n),
\]

where

\[
\delta S_1^n = \frac{S_1^n - (1 - 4\mu_0^2)S_1^n - 4\mu_1^2 g_0^n}{\mu_1 (\Delta x_1 + 2\Delta x_{\frac{1}{2}})},
\]

and

\[
\delta_{lim} S_1^n = \begin{cases} 
\alpha_{i,1} \min(|S_{1,x}^n|, |S_{1,z}^n|), & \text{if } (S_{1,x}^n) \cdot (S_{1,z}^n) > 0, \\
0, & \text{otherwise.} 
\end{cases}
\]

Here \( \alpha_{i,1} \) satisfies (2.26),

\[
S_{1,z}^n = \frac{S_1^n - g_0^n}{\Delta x_1},
\]

and

\[
\mu_1 = \frac{\Delta x_{\frac{1}{2}}}{\Delta x_1}.
\]

A similar formula for the term \( \delta S_{J-1}^n \) can be derived with \( \alpha_{i,J-1} \) satisfying (2.27); i.e.,

\[
\delta S_{J-1}^n = \min(\delta_{lim} S_{J-1}^n, |\delta S_{J-1}^n|) \cdot \text{sgn}(\delta S_{J-1}^n),
\]

where

\[
\delta S_{J-1}^n = \frac{4\mu_{J-1}^2 g_0^n - (4\mu_{J-1}^2 - 1)S_{J-1}^n - S_{J-2}^n}{\mu_{J-1} (\Delta x_{J-1} + 2\Delta x_{J-\frac{1}{2}})},
\]

and

\[
\delta_{lim} S_{J-1}^n = \begin{cases} 
\alpha_{i,J-1} \min(|S_{J-1,x}^n|, |S_{J-1,z}^n|), & \text{if } (S_{J-1,x}^n) \cdot (S_{J-1,z}^n) > 0, \\
0, & \text{otherwise.} 
\end{cases}
\]
Here,

\[
S_{j-1,x}^n = \frac{g_1^n - S_j^n}{\Delta x_{j-1}},
\]

and

\[
\mu_{j-1} = \frac{\Delta x_{\frac{3}{2}}}{\Delta x_1}.
\]

Dirichlet boundary conditions (2.3) and (2.6) are further enforced by setting

\[
S_{0,L}^n = g_0(t^{n+\frac{1}{2}}),
\]

(2.36)

and defining the extensions \(S_0^{n+1}\) and \(S_j^{n+1}\) by

\[
S_0^{n+1} = 2g_0^{n+1} - S_1^{n+1},
\]

(2.37)

\[
S_j^{n+1} = 2g_j^{n+1} - S_{j-1}^{n+1}.
\]

(2.38)

Equation (2.36) is derived from the approximation

\[
f(S_{0,L}^n) \approx f(t^{n+\frac{1}{2}}) = f(g_0^{n+\frac{1}{2}}).
\]

Conditions (2.37) and (2.38) are derived from the mixed method formulation; i.e. (2.18) and (2.20).

Instead of (2.36), we could define

\[
S_{0,L}^n = g_0^n.
\]

(2.39)

This change results in a \(O(\Delta t)\) error in the analysis but greatly simplifies maximum principle arguments, as we will see in Chapter 3.

2.2.2 Neumann boundary conditions

In the case of Neumann boundary conditions, (2.4) and (2.7), we set

\[
\delta S_1^n = \delta S_{j-1}^n = 0,
\]

(2.40)
and we define the extensions

\[ S_0^{n+1} = S_1^{n+1}, \]  

(2.41)

and

\[ S_J^{n+1} = S_{J-1}^{n+1}. \]  

(2.42)

Moreover, we set

\[ S_{0,L}^n = S_1^n. \]  

(2.43)

Condition (2.40) follows from (2.4) and the approximation

\[ \delta S_1^n \approx \frac{\partial s}{\partial x}(x_1, t^n) = O(\Delta x). \]

Equations (2.41) and (2.42) are derived from (2.20) by forcing \( U_{J-1}^{n+1} = U_{J-\frac{1}{2}}^{n+1} = 0. \) Finally, (2.43) is derived by "reflecting" the solution across \( x = 0; \) thus, we apply (2.15) with \( S_0^n = S_1^n \) and \( \delta S_0^n = 0. \)

As we will see in our analysis in Chapter 4, we must be careful in defining an appropriate \( S_{J-1,L}^n. \) Following the derivation of the MUSCL algorithm as given above, it would seem

\[ S_{J-1,L}^n = S_{J-1}^n. \]  

(2.44)

However, by setting

\[ S_{J-1,L}^n = S_{J-1}^{n+1}, \]  

(2.45)

we are able to avoid technical difficulties associated with the boundary conditions and obtain a better \( L^1 \) error estimate. From a computational point of view clearly (2.44) is preferable since it is explicit; however, the modification (2.45) is well-defined, even though we must now solve a nonlinear equation. Either definition works from a truncation error point of view.
2.2.3 Mixed boundary condition

For the mixed boundary condition (2.5), we enforce this condition in (2.22) for $j = 1$ by setting
\[
f(S^n_{0,L}) - a^n_{\frac{1}{2}} \frac{S^{n+1}_1 - S^{n+1}_0}{\Delta x_1} = \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} f(g_0(t))dt.
\] (2.46)

We also set
\[
\delta S^n_1 = 0.
\] (2.47)

Thus we approximate the solution by a constant in the first interval. This automatically introduces a $O(\Delta x)$ term into the error analysis.

2.2.4 Initial condition

Calculating $\delta S^n_j$ by (2.25), $j = 2, \ldots, J - 2$, and calculating $\delta S^n_0$ and $\delta S^n_{J-1}$ by the proper formulas as given above, we approximate the initial condition (2.2) by setting
\[
S^0(x) = S^0_j + (x - x_j) \delta S^0_j, \quad x \in B_j,
\] (2.48)

where
\[
S^0_j = \frac{1}{\Delta x_j} \int_{B_j} s^0(x)dx.
\] (2.49)

2.3 Many advection steps per diffusion step

In this section, we present a modification [43] of the scheme given in Section 2.1 whereby one takes several advection steps per diffusion step. This approach is very practical for problems where CFL constraints such as (2.14) require $\Delta t^n$ to be small to preserve stability of the Godunov scheme.
Recall from Section 2.1 that diffusion is added implicitly; thus, in the mixed method, there is no stability constraint on $\Delta t$ but at each diffusion time step a system of equations must be solved. Advection, on the other hand, is explicit and very local. Hence one advection step is much less expensive than one diffusion step. By solving for diffusion at every few advection steps instead of at every advection step, we reduce the expense of the overall computation.

Due to the nature of the mixed boundary condition, this case is not considered here. We do, however, formulate the scheme for Dirichlet or Neumann boundary conditions.

For simplicity, assume $K \geq 1$ is fixed. Then the diffusion time-step $\Delta t^n$ is given by

$$\Delta t^n = \sum_{k=0}^{K-1} \Delta t_{s}^{k,n},$$

where $\Delta t_{s}^{k,n}$ satisfies

$$\sup_{j} f'(S_j^{k,n}) \frac{\Delta t_{s}^{k,n}}{\Delta x_j} \leq 1, \quad j = 1, \ldots, J - 1,$$  \hspace{1cm} (2.50)

with $S_j^{k,n}$ is defined below.

Given $S^n \in \mathcal{M}^{-1}_1(\delta x)$, where $S^n$ is of the form (2.13), let

$$S_{j}^{0,n} = S_j^n, \quad j = 1, \ldots, J - 1.$$  \hspace{1cm} (2.51)

Define, for $k = 1, \ldots, K - 1$,

$$S_j^{k,n} = S_j^{k-1,n} - \frac{\Delta t_{s}^{k,n}}{\Delta x_j} [f(S_j^{k-1,n}) - f(S_{j-1}^{k-1,n})],$$  \hspace{1cm} (2.52)

where,

$$S_{j,L}^{k-1,n} = S_j^{k-1,n} + \frac{\Delta x_j}{2} (1 - f'(S_j^{k-1,n}) \frac{\Delta t_{s}^{k,n}}{\Delta x_j}) \delta S_j^{k-1,n},$$  \hspace{1cm} (2.53)

and $\delta S_j^{k-1,n}$ is calculated by slope-limiting as in (2.25). The terms $S_{0,L}^{k-1,n}$, $\delta S_1^{k-1,n}$, and $\delta S_{J-1}^{k-1,n}$ are determined by the boundary conditions; see Section 2.2.
Thus, in this case, for \( j = 1, \ldots, J - 1 \),
\[
S_j^{n+1} = S_j^n + \sum_{k=0}^{K-1} \frac{\Delta t_{k,n}^{j}}{\Delta x_j} [f(S_{j,L}^k) - f(S_{j-1,L}^k)].
\] (2.54)

Combining (2.54) with (2.21), we obtain, for \( j = 1, \ldots, J - 1 \),
\[
\frac{S_j^{n+1} - S_j^n}{\Delta t^n} + \sum_{k=0}^{K-1} \frac{\Delta t_{k,n}^{j}}{\Delta x_j} [f(S_{j,L}^k) - f(S_{j-1,L}^k)]
- \frac{1}{\Delta x_j} [a_{j+\frac{1}{2}}^{n+1} S_{j+1}^{n+1} - a_{j}^{n+1} S_{j}^{n+1}] - a_{j-\frac{1}{2}}^{n+1} S_{j-1}^{n+1} - a_{j-\frac{1}{2}}^{n} S_{j-1}^{n} = 0.
\] (2.55)

Thus to determine \( S^{n+1} \) we must solve a tridiagonal system of equations at each time \( t^n = \sum_{i=0}^{n-1} \Delta t^i \).

As in Section 2.1, we complete this step in the algorithm by projecting \( S^{n+1} \) into the space \( M_{-1}(\delta_x) \) using the slope-limiting procedure described in Sections 2.1 and 2.2 and applying (2.13); \( S^{n+1} \) then becomes the initial condition for the next advection step.

### 2.4 GMM for more general equations

We now extend the scheme presented in Sections 2.1-2.3 for the nonlinear problem
\[
\frac{\partial s}{\partial t} + \frac{\partial f(s)}{\partial x} - \frac{d}{dx} \left( a(x,t,s(x,t)) \frac{\partial s}{\partial x} \right) = r(x,t,s), \quad (x,t) \in Q_T,
\] (2.56)

with initial condition
\[
s(x,0) = s^0(x), \quad x \in I,
\] (2.57)

and boundary conditions
\[
s(0,t) = g_0(t),
\] (2.58)
\[
a(0,t,s(0,t)) \frac{\partial s}{\partial x}(0,t) = 0,
\] (2.59)
or
\[ f(s(0,t)) - a(0,t,s(0,t)) \frac{\partial s}{\partial x}(0,t) = 0, \quad (2.60) \]
and
\[ s(1,t) = g_1(t), \quad (2.61) \]
or
\[ a(1,t,s(1,t)) \frac{\partial s}{\partial x}(1,t) = 0, \quad t \in (0,T]. \quad (2.62) \]

Here we have three physical processes occurring simultaneously: advection, diffusion and "reaction" corresponding with \( f, a, \) and \( r, \) respectively. Thus, we could perform time-splitting at three different levels by extending the ideas of Section 2.3.

Suppose we have an approximation \( S^n(x) \) to \( s(x,t^n) \) satisfying (2.56), and we want to calculate \( S^{n+1}. \) Assume the reaction term is varying on a smaller time-scale than advection and diffusion. We could then time-split our algorithm by the following procedure, which is based on an algorithm given in [42].

Let \( \Delta t_R \) be the time-step for reactions, \( \Delta t_A \) and \( \Delta t_D \) the time-steps for advection and diffusion, respectively. Assume for positive integers \( K \) and \( M, \)
\[ \Delta t_A = K \Delta t_R, \quad (2.63) \]
\[ \Delta t_D = M \Delta t_A, \quad (2.64) \]

and \( \Delta t_A \) is determined by a CFL restriction such as (2.50).

The first step in the splitting is to solve
\[ \frac{d\hat{s}}{dt} = r(x,t,\hat{s}), \quad (x,t) \in I \times (t^n,t^n + \Delta t_A], \quad (2.65) \]
with initial condition
\[ \hat{s}(x,t^n) = S^n(x), \quad x \in I. \quad (2.66) \]
Let \( \bar{S}^{K,1,n} \approx \bar{s}(x, t^n + \Delta t_A) \) by applying some ordinary differential equation solver to (2.65)-(2.66) for \( K \) time-steps of size \( \Delta t_R \).

The next step is to solve

\[
\frac{\partial \bar{s}}{\partial t} + \frac{\partial f(\bar{s})}{\partial x} = 0, \quad (x, t) \in I \times (t^n, t^n + \Delta t_A),
\]

with initial condition

\[
\bar{s}(x, t^n) = \bar{S}^{K,1,n}(x), \quad x \in I.
\]

Let \( \bar{S}^{1,n} \) be the Godunov approximation to \( \bar{s}(x, t^n + \Delta t_A) \). Assuming \( M > 1 \) in (2.64), we would return to (2.65), and solve this equation with initial condition \( \bar{S}^{1,n} \).

After \( M \) steps of the above procedure, we now solve

\[
\frac{\partial s^*}{\partial t} - \frac{d}{dx} \left( a(x, t, s^*) \frac{\partial s^*}{\partial x} \right) = 0, \quad (x, t) \in I \times (t^n, t^{n+1}],
\]

with initial condition

\[
s^*(x, t^n) = \bar{S}^{M,n}(x), \quad x \in I,
\]

using the mixed method. Here \( \bar{S}^{M,n} \) is the Godunov approximation after \( M \) steps of iterating between (2.65) and (2.67).

To be more precise, suppose we solve (2.65)-(2.66) by the explicit Euler method at each point \( x_j \) for \( K \) steps of size \( \Delta t_R \), then we obtain

\[
\bar{S}_j^{K,m,n} = \bar{s}_j^{m-1,n} + \sum_{k=0}^{K-1} \Delta t_R r(x_j, t^{k,m,n}, \bar{s}_j^{k,m,n}),
\]

where

\[
\bar{s}_j^{0,n} = S_j^n,
\]

and for \( m > 1 \), \( \bar{s}_j^{m-1,n} \) is defined below. Here, for \( k=1, \ldots, K \),

\[
\bar{s}_j^{k,m,n} = \bar{s}_j^{k-1,m,n} + \Delta t_R r(x_j, t^{k-1,m,n}, \bar{s}_j^{k-1,m,n}),
\]

\[
\bar{s}_j^{K,m,n} = \bar{s}_j^{K-1,m,n} + \Delta t_R r(x_j, t^{K-1,m,n}, \bar{s}_j^{K-1,m,n}),
\]

\[
\bar{s}_j^{K,m,n} = \bar{s}_j^{K-1,m,n} + \Delta t_R r(x_j, t^{K-1,m,n}, \bar{s}_j^{K-1,m,n}),
\]
where

$$\tilde{S}_j^{0,m,n} = \tilde{S}_j^{m-1,n},$$  \hspace{1cm} (2.74)

and

$$t^{k,m,n} = t^n + (m-1)\Delta t_A + k\Delta t_R.$$  \hspace{1cm} (2.75)

The next step is to project $\tilde{S}^{K,m,n}$ into the space $M_{-1}^1(\delta_x)$ by calculating limited slopes and applying (2.13). Then, the Godunov approximation $\tilde{S}^{m,n} \in M_{0}^0(\delta_x)$ to $\bar{s}(x,t^n + \Delta t_A)$ satisfying (2.67) with initial condition $\tilde{S}^{K,m,n}$ is given by

$$\tilde{S}_j^{m,n} = \tilde{S}_j^{K,m,n} + \frac{\Delta t_A}{\Delta x_j}[f(\tilde{S}_{j,L}^{K,m,n}) - f(\tilde{S}_{j-1,L}^{K,m,n})].$$  \hspace{1cm} (2.76)

We now return to (2.71) and iterate the above procedure for $M$ steps, obtaining finally $\bar{S}^{M,n}$.

The step is completed by computing $S^{n+1}$ via the mixed method applied to (2.69). Using the lowest order Raviart-Thomas spaces and the trapezoidal rule as in Section 2.1 we obtain

$$\frac{S_j^{n+1} - \tilde{S}_j^{M,n+1}}{\Delta t^n} = \frac{1}{\Delta x_j} \left[ a_{j+\frac{1}{2}}^{n+1} \frac{S_{j+1}^{n+1} - S_j^{n+1}}{\Delta x_{j+\frac{1}{2}}} - a_{j-\frac{1}{2}}^{n+1} \frac{S_j^{n+1} - S_{j-1}^{n+1}}{\Delta x_{j-\frac{1}{2}}} \right] = 0,$$  \hspace{1cm} (2.77)

for $j = 1, \ldots, J - 1$. Here $a_{j+\frac{1}{2}}^{n+1} = a(x_{j+\frac{1}{2}}, t^{n+1}, S_{j+\frac{1}{2}}^{n+1})$, $j = 1, \ldots, J - 1$, where for $j = 1, \ldots, J - 2$,

$$S_{j+\frac{1}{2}}^{n+1} = \frac{1}{2} (S_j^{n+1} + S_{j+1}^{n+1}),$$  \hspace{1cm} (2.78)

and $S_{\frac{1}{2}}^{n+1}$ and $S_{\frac{3}{2}}^{n+1}$ are again defined by the boundary conditions.

Thus, to determine $S^{n+1}$, we must use Newton’s method or some other nonlinear equation solver on (2.77). For problems where nonlinear solvers aren’t efficient or don’t converge,
we could modify the algorithm above by defining \( a^{n+1}_{j+\frac{1}{2}} \) by extrapolation in time. This modification does not change our analysis considerably, and the approximation \( S^{n+1} \) is then determined by simply solving a positive definite, symmetric, tridiagonal system of equations, as in the linear case.

The above procedure is but one splitting that could be performed on (2.56). In Chapter 4, we will analyze a splitting based on applying the first-order Godunov scheme and the mixed method to this equation.

Assume reaction, advection, and diffusion occur on roughly the same time scale, so that \( \Delta t_R = \Delta t_A = \Delta t_D \). Then, we can derive a splitting procedure whereby we apply first-order Godunov to (2.8), and then apply the mixed method to the problem

\[
\frac{\partial s^*}{\partial t} - \frac{d}{dx} \left( a(x, t, s^*) \frac{\partial s^*}{\partial x} \right) = r(x, t, s^*), \quad (x, t) \in I \times (t^n, t^{n+1}],
\]

with initial condition

\[
s^*(x, t^n) = \bar{s}^{n+1}(x), \quad x \in I.
\]

Here \( \bar{s}^{n+1} \) is the first-order Godunov approximation to \( \bar{s}(x, t^{n+1}) \) satisfying (2.8).

The first-order Godunov scheme is basically the higher-order scheme derived in Section 2.1 with \( \delta S_j^n = 0 \). Thus, \( S_j^n = \tilde{S}_j^n \), and

\[
\tilde{S}_j^{n+1} = S_j^n - \frac{\Delta t^n}{\Delta x_j} [f(S_j^n) - f(S_{j-1}^n)].
\]

The above formula is valid for \( j = 2, \ldots, J - 1 \). For \( j = 1 \),

\[
\tilde{S}_1^{n+1} = S_1^n - \frac{\Delta t}{\Delta x_1} [f(S_1^n) - f(S_{0,L}^n)],
\]

where \( S_{0,L}^n \) is determined by boundary conditions as in Section 2.2, with the modification that in the case (2.3) holds, we set

\[
S_{0,L}^n = g_0^n,
\]
instead of $g_0^{n+\frac{1}{2}}$.

Finally, applying the mixed method to (2.79) and using the midpoint rule to calculate $s_B^{n+1}$ we obtain

$$
\frac{S_j^{n+1} - \tilde{S}_j^{n+1}}{\Delta t} = \frac{1}{\Delta x_j} \left[ a_{j+\frac{1}{2}}^{n+1} \frac{S_{j+1}^{n+1} - S_j^{n+1}}{\Delta x_{j+\frac{1}{2}}} - a_{j-\frac{1}{2}}^{n+1} \frac{S_j^{n+1} - S_{j-1}^{n+1}}{\Delta x_{j-\frac{1}{2}}} \right] = r(x_j, t^{n+1}, S_j^{n+1}),
$$

(2.84)

with $\tilde{S}^{n+1}$ given by (2.81)-(2.83) and $a_{j+\frac{1}{2}}^{n+1}$ defined as above with $S_{j+\frac{1}{2}}^{n+1}$ given by (2.78).

Thus, finding $S^{n+1}$ again involves solving a nonlinear system of equations. Earlier remarks about using extrapolation of coefficients in (2.84) also hold here.
Chapter 3

Maximum Principles

In this chapter, we extend earlier results of Wheeler [41] and demonstrate that, with certain assumptions on the time-step and the slope-limiting parameter \( \alpha_{i,j} \), the Godunov-mixed method (GMM) approximation described in Chapter 2 satisfies discrete maximum principles. Thus, the scheme is stable.

The results of this chapter can be summarized as follows. For \( S \) the GMM approximation to \( s \) satisfying (2.1), (2.2), with Dirichlet boundary conditions (2.3) and (2.6), we have

\[
m^n \leq S^n_j \leq M^n
\]  

(3.1)

for \( j = 1, \ldots, J - 1 \) and \( n = 0, \ldots, N^* \); where

\[
m^n = \min \left( \min_j S^0_j, \inf_{[0,t^n]} g_0(t), \inf_{[0,t^n]} g_1(t) \right)
\]

and

\[
M^n = \max \left( \max_j S^0_j, \sup_{[0,t^n]} g_0(t), \sup_{[0,t^n]} g_1(t) \right).
\]

For the problem with Neumann boundary conditions (2.4) and (2.7), \( S \) satisfies

\[
\min_j S^0_j \leq S^n_j \leq \max_j S^0_j.
\]  

(3.2)
Furthermore, when \( s \) satisfies a mixed boundary condition (i.e. (2.5)) at \( x = 0 \) and (2.6) at \( x = 1 \), \( S \) satisfies (3.1), and when (2.7) holds at \( x = 1 \), we have

\[
\min\left(\min_j S_j^0, \inf_{[0,t^n]} g_0(t)\right) \leq S_j^p \leq \max\left(\max_j S_j^0, \sup_{[0,t^n]} g_0(t)\right). \tag{3.3}
\]

The above inequalities are proved for the basic scheme presented in Chapter 2, Sections 2.1 and 2.2. The results are easily extendable to the extension of the scheme given in Section 2.3, and to the nonlinear scheme of Section 2.4 when \( r \equiv 0 \).

Inequality (3.1) is proved in Section 3.1, and (3.2) is derived in Section 3.2. In Section 3.3, (3.1) is shown to hold for the problem with a mixed boundary condition at \( x = 0 \) and a Dirichlet boundary condition at \( x = 1 \). Maximum principles for any combination of the boundary conditions are easily derived by combining elements of the proofs given in the following sections.

### 3.1 Dirichlet boundary conditions

In this section, we derive (3.1) for the GMM approximation, \( S \), to \( s \) satisfying (2.1)-(2.3) and (2.6). Thus \( S \) satisfies (2.22) with boundary data given in Section 2.2.1. We also prove that (3.1) holds with the modification to the scheme whereby

\[
S_{0,L}^n = g_0(t^n). \tag{3.4}
\]

The results of this section are given in the following lemmas.

**Lemma 3.1** Let \( S \) satisfy (2.22) and (2.28)-(2.38). Let the slope-limiting parameters \( \alpha_{i,j} \) satisfy

\[
0 \leq \alpha_{i,j} \leq 2\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}, \quad j = 1, \ldots, J - 2, \tag{3.5}
\]
and

$$0 \leq \alpha_{i,j-1} \leq 2.$$  (3.6)

Let

$$g_{0,*} = \sup_{\{t \mid g_0'(t) < 0\}} |g_0'(t)|,$$

and

$$g_0^{*} = \sup_{\{t \mid g_0'(t) > 0\}} g_0'(t),$$

and let

$$\Delta x = \min_j \Delta x_j,$$

$$\alpha_l = \max_j \frac{\alpha_{i,j} \Delta x_j}{\Delta x_{j-l}}.$$  (3.7)

Assume $\Delta t^n$ satisfies the following constraints for each $n$:

$$\frac{\Delta t^n}{\Delta x_n} \sup_{S^n} f'(s) \leq \frac{1}{1 + \alpha_{l,l}},$$  (3.8)

and when $\alpha_{l,1}$ and $S^n_1 - g^n_0 \neq 0$,

$$\frac{\Delta t^n}{\Delta x_1} \sup_{S^n} f'(s)[1 + \frac{\alpha_{l,1}}{2} + \frac{g^n_{0,*} \Delta t^n}{2(S^n_1 - g^n_0)}] \leq 1, \text{ if } S^n_1 - g^n_0 > 0,$$  (3.9)

or

$$\frac{\Delta t^n}{\Delta x_1} \sup_{S^n} f'(s)[1 + \frac{\alpha_{l,1}}{2} - \frac{g^n_{0,*} \Delta t^n}{2(S^n_1 - g^n_0)}] \leq 1, \text{ if } S^n_1 - g^n_0 < 0;$$  (3.10)

and

$$\Delta t^n g^n_{0,*} \leq 2(S^n_1 - g^n_0), \text{ if } S^n_1 - g^n_0 > 0,$$  (3.11)

or

$$\Delta t^n g^n_{0,*} \leq -2(S^n_1 - g^n_0), \text{ if } S^n_1 - g^n_0 < 0.$$  (3.12)
Here $S^n$ and $\tilde{S}^n$ are closed intervals in $\mathbb{R}$ defined as follows:

$$S^n = \left[ \min \left( \inf_{[0,T]} g_0(t), g_1^n, \min_{j} S^n_j \right), \max \left( \sup_{[0,T]} g_0(t), g_1^n, \max_{j} S^n_j \right) \right],$$

and

$$\tilde{S}^n = \left[ \min \left( \inf_{[0,T]} g_0(t), S^n_1, S^n_2 \right), \max \left( \sup_{[0,T]} g_0(t), S^n_1, S^n_2 \right) \right].$$

If the above assumptions on $\Delta t^n$ and $\alpha_{1,j}$ are satisfied, then $S$ satisfies (3.1).

**Lemma 3.2** Let $S$ and $\alpha_{1,j}$ satisfy the hypothesis of Lemma (3.1) with the exception that $S^n_{0,L}$ is given by (3.4). Let $\Delta t^n$ satisfy

$$\frac{\Delta t^n}{\Delta x} \sup_{S^n} f'(s) \leq \frac{1}{1 + \frac{\alpha_1}{\Delta}},$$

(3.13)

where

$$\alpha_l = \max_{j} \frac{\alpha_{1,j} \Delta x_j}{\Delta x_{j-\frac{1}{2}}},$$

$$\Delta x = \min_{j} \Delta x_j,$$

and

$$\tilde{S}^n = \left[ \min (g_0^n, g_1^n, \min_{j} S^n_j), \max (g_0^n, g_1^n, \max_{j} S^n_j) \right].$$

Then $S$ satisfies (3.1).

**Proof** (Lemma 3.1). We first define some notation. Let

$$\delta S^n_j = S^n_j - S^n_{j-1}, \quad j = 2, \ldots, J - 1,$$

$$\delta S^n_1 = S^n_1 - g^n_0,$$

and let

$$h_j(s) = f'(s) \frac{\Delta t^n}{\Delta x_j}.$$

(3.14)
Recall from Chapter 2

\[ S_{j+1}^n = S_j^n - \frac{\Delta t^n}{\Delta x_j} [f(S_{j+1}^n) - f(S_j^n - S_{j-1}^n)], \quad (3.15) \]

where

\[ S_{0,L}^n = \theta_0^{n+\frac{1}{2}}, \quad (3.16) \]

and for \( j = 1, \ldots, J - 1, \)

\[ S_{j,L}^n = S_j^n + \frac{\Delta x_j}{2} (1 - h_j(S_j^n)) \delta S_j^n, \quad (3.17) \]

with \( \delta S_j^n \) given by (2.25), (2.28), or (2.32).

Case 1: \( j = 2, \ldots, J - 1. \) Applying the Mean Value Theorem to \( f \) in (3.15) we find

\[ S_{j+1}^n = S_j^n - h_j(\beta_j)(S_{j,L}^n - S_{j-1,L}^n) \]

\[ = S_j^n - D_{j-\frac{1}{2}} \delta S_j^n, \quad (3.18) \]

where

\[ D_{j-\frac{1}{2}} = \left\{ \begin{array}{ll}
   h_j(\beta_j) \delta S_j^n + \frac{\Delta x_j}{2} (1 - h_j(S_j^n)) \delta S_j^n, & \text{if } \delta S_j^n \neq 0, \\
   -\frac{\Delta x_j}{2} (1 - h_{j-1}(S_{j-1}^n)) \delta S_{j-1}^n / \delta S_j^n, & \text{otherwise},
\end{array} \right. \quad (3.19) \]

and \( \beta_j \) is some point between \( S_{j,L}^n \) and \( S_{j-1,L}^n. \) We now follow the argument given by Wheeler to show \( 0 \leq D_{j-\frac{1}{2}} \leq 1. \)

First, note that if \( \delta S_j^n > 0 \) for \( 1 \leq j \leq J - 2, \) then since \( h_j(S_j^n) \leq 1 \) by (3.8) and

\[ \Delta x_j \delta S_j^n \leq \frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} \alpha_{l,j}(S_{j+1}^n - S_j^n) \leq 2(S_{j+1}^n - S_{j}^n) \]

by (2.25) and (3.5), we have by (3.17),

\[ S_j^n < S_{j,L}^n \leq S_{j+1}^n. \]
Similarly, if $\delta S^n_j \leq 0$, then

$$S^n_{j+1} < S^n_{j,L} \leq S^n_j.$$ 

Thus

$$\min(S^n_j, S^n_{j+1}) \leq S^n_{j,L} \leq \max(S^n_j, S^n_{j+1}), \quad (3.20)$$

for $1 \leq j \leq J - 2$. For $j = J - 1$, if $\delta S^n_{J-1} > 0$, then

$$\Delta x_{J-1} \delta S^n_{J-1} \leq \alpha_{J-1}(g^n_1 - S^n_{J-1})$$

$$\leq 2(g^n_1 - S^n_{J-1})$$

by (2.32) and (3.6). Thus

$$S^n_{J-1} < S^n_{J-1,L} \leq g^n_1.$$ 

Similarly, if $\delta S^n_{J-1} \leq 0$,

$$g^n_1 \leq S^n_{J-1,L} < S^n_{J-1}.$$ 

Hence

$$\min(S^n_{J-1}, g^n_1) \leq S^n_{J-1,L} \leq \max(S^n_{J-1}, g^n_1). \quad (3.21)$$

Thus, since $\beta_j$ is between $S^n_{j,L}$ and $S^n_{j-1,L}$, then by (3.20), (2.36), and (3.21), $\beta_j \in S^n$ for $1 \leq j \leq J - 1$, hence

$$h_j(\beta_j) = \frac{\Delta t^n}{\Delta x_j} f'(\beta_j)$$

$$\leq \frac{\Delta t^n}{\Delta x_j} \sup_{S^n} f'(s)$$

$$\leq \frac{1}{1 + \frac{\alpha}{2}}, \quad (3.22)$$
by (3.8).

We note that by the slope-limiting procedure, $\delta\tilde{S}^n_j$ and $\delta\tilde{S}^n_{j-1}$ are zero or have the same
sign as $\delta\tilde{S}^n_j$. Hence by (3.19) and (3.5),

$$
D_{j-\frac{1}{2}} \geq h_j(\beta_j)(1 - \frac{\Delta x_{j-1}\alpha_{l,j-1}}{2\Delta x_{j-\frac{1}{2}}}) \\
\geq 0.
$$

(3.23)

Moreover, by (3.19), (3.22), (3.7), and (3.8),

$$
D_{j-\frac{1}{2}} \leq h_j(\beta_j)(1 + \frac{\alpha_{l,j}\Delta x_j}{2\Delta x_{j-\frac{1}{2}}}) \\
\leq \max_j h_j(\beta_j)(1 + \frac{\alpha_{l,j}}{2}) \\
\leq 1.
$$

(3.24)

By (3.18), and (3.23)-(3.24), if $\delta\tilde{S}^n_j \geq 0$, then

$$
S^n_{j-1} \leq \tilde{S}^n_{j+1} \leq S^n_j.
$$

(3.25)

Similarly, if $\delta\tilde{S}^n_j \leq 0$, then

$$
S^n_j \leq \tilde{S}^n_{j+1} \leq S^n_{j+1}.
$$

(3.26)

Thus, by (3.25) and (3.26), for $2 \leq j \leq J - 1$,

$$
\min_{1 \leq j \leq J-1} S^n_j \leq \tilde{S}^n_{j+1} \leq \max_{1 \leq j \leq J-1} S^n_j.
$$

(3.27)

Case 2: $j=1$. For $j=1$, recall

$$
\tilde{S}^{n+1}_1 = S^n_1 - \frac{\Delta t^n}{\Delta x_1} [f(S^n_{1,L}) - f(g_0^{n+\frac{1}{2}})] \\
= S^n_1 - h_1(\beta_1)(S^n_{1,L} - g_0^{n+\frac{1}{2}}) \\
= \begin{cases} 
S^n_1 - D_{\frac{1}{2}} \delta\tilde{S}^n_1, & \text{if } \delta\tilde{S}^n_1 \text{ and } \alpha_{l,1} \neq 0, \\
S^n_1 - h_1(\beta_1)(S^n_{1} - g_0^{n+\frac{1}{2}}), & \text{otherwise}.
\end{cases}
$$

(3.28)
Here $\beta_1$ is some point between $S^n_1$ and $g_0^{n+\frac{1}{2}}$, and

$$D_{\frac{1}{2}} = h_1(\beta_1)(\eta_0 + \frac{\Delta x_1}{2}(1 - h_1(S^n_1))\frac{\delta S^n_1}{\delta n_1}),$$

(3.29)

where

$$\eta_0 = (S^n_1 - g_0^{n+\frac{1}{2}})/\delta S^n_1.$$  

(3.30)

First, assume $\delta S^n_1$ and $\alpha_{1,1} \neq 0$. Then $\delta S^n_1$ is zero or has the same sign as $\delta S^n_1$. Thus, by (3.29) and (3.30),

$$D_{\frac{1}{2}} \geq h_1(\beta_1)\eta_0$$

$$= h_1(\beta_1)(1 - \frac{\Delta t_n g^n_0(i)}{2 \delta S^n_1}),$$

for some $i \in (t^n, t^{n+\frac{1}{2}})$ by the Mean Value Theorem applied to $\eta_0$. Hence,

$$D_{\frac{1}{2}} \geq \begin{cases} 
  h_1(\beta_1)(1 - \frac{\Delta t_n g^n_0(i)}{2 \delta S^n_1}), & \text{if } \delta S^n_1 > 0, \\
  h_1(\beta_1)(1 + \frac{\Delta t_n g^n_0(i)}{2 \delta S^n_1}), & \text{if } \delta S^n_1 < 0, \\
  0, & \text{otherwise} 
\end{cases}$$

(3.31)

by (3.11) and (3.12).

Furthermore, since $\beta_1$ is between $S^n_{1,L}$ and $g_0^{n+\frac{1}{2}}$ and $S^n_{1,L}$ is between $S^n_1$ and $S^n_2$ (see (3.20)), we have $\beta_1 \in S^n$. Thus

$$D_{\frac{1}{2}} \leq h_1(\beta_1)(\eta_0 + \frac{\alpha_{1,1}}{2})$$

$$= h_1(\beta_1)(1 - \frac{\Delta t_n g^n_0(i)}{2 \delta S^n_1} + \frac{\alpha_{1,1}}{2})$$

$$\leq \begin{cases} 
  \frac{\Delta t_n}{\Delta x_1} \sup_{S^n} f'(s)(1 + \frac{\Delta t_n g^n_0(i)}{2 \delta S^n_1} + \frac{\alpha_{1,1}}{2}), & \text{if } \delta S^n_1 > 0, \\
  \frac{\Delta t_n}{\Delta x_1} \sup_{S^n} f'(s)(1 - \frac{\Delta t_n g^n_0(i)}{2 \delta S^n_1} + \frac{\alpha_{1,1}}{2}), & \text{if } \delta S^n_1 < 0, \\
  1, & \text{otherwise} 
\end{cases}$$

(3.32)

by (3.29), (3.30), (3.9) and (3.10).
Applying (3.31) and (3.32) in (3.28) one obtains

\[ \min(S_1^n, g_0^n) \leq \bar{S}_1^{n+1} \leq \max(S_1^n, g_0^n) \]  

(3.33)

when \( \bar{S}_1^n \) and \( \alpha_{1,1} \neq 0 \).

Now assume \( \bar{S}_1^n \) or \( \alpha_{1,1} = 0 \), then since \( 0 \leq h_1(\beta_1) \leq 1 \) we obtain from (3.28),

\[ \min(S_1^n, g_0^{n+\frac{1}{2}}) \leq \bar{S}_1^{n+1} \leq \max(S_1^n, g_0^{n+\frac{1}{2}}). \]  

(3.34)

Combining (3.27), (3.33), and (3.34) we find

\[ \min(\min_j S_j^n, \inf_{[0,t^{n+\frac{1}{2}}]} g_0(t)) \leq \bar{S}_j^{n+1} \leq \max(\max_j S_j^n, \sup_{[0,t^{n+\frac{1}{2}}]} g_0(t)). \]  

(3.35)

Now, recall

\[ S_j^{n+1} = \frac{\Delta t^n \left[ a_{j+\frac{1}{2}} \frac{S_{j+1}^{n+1} - S_j^{n+1}}{\Delta x_{j+\frac{1}{2}}} + a_{j-\frac{1}{2}} \frac{S_j^{n+1} - S_{j-1}^{n+1}}{\Delta x_{j-\frac{1}{2}}} \right]}{\Delta x_j}. \]  

(3.36)

Suppose

\[ \bar{S}_j^{n+1} = \max_j S_j^{n+1} \]

for some \( j \) between 2 and \( J - 2 \). Let

\[ \sigma_j^+ = \frac{\Delta t^n a_{j+\frac{1}{2}}^{n+1}}{\Delta x_j \Delta x_{j+\frac{1}{2}}}, \]

\[ \sigma_j^- = \frac{\Delta t^n a_{j-\frac{1}{2}}^{n+1}}{\Delta x_j \Delta x_{j-\frac{1}{2}}}. \]

Then \( \sigma_j^+ \) and \( \sigma_j^- \) are greater than zero for all \( j \), and

\[ S_j^{n+1} \leq -\sigma_j^- S_j^{n+1} + (1 + \sigma_j^- + \sigma_j^+) S_j^{n+1} - \sigma_j^+ S_{j+1}^{n+1} \]

\[ = \bar{S}_j^{n+1}. \]

Thus

\[ \bar{S}_j^{n+1} \leq \max_j \bar{S}_j^{n+1}. \]  

(3.37)
Now, suppose
\[ S_1^{n+1} = \max_j S_j^{n+1}. \]

Recall \( S_0^{n+1} = S_1^{n+1} - 2g_0^{n+1} \), hence
\[ -2\sigma_1^- g_0^{n+1} + (1 + 2\sigma_1^- + \sigma_1^+) S_1^{n+1} - \sigma_1^+ S_2^{n+1} = S_1^{n+1}. \tag{3.38} \]

We have two cases. Either \( S_1^{n+1} \leq g_0^{n+1} \) or \( S_1^{n+1} \geq g_0^{n+1} \). In the latter case, by the argument given above applied to (3.38) we obtain
\[ S_1^{n+1} \leq \bar{S}_1^{n+1}. \]

Thus, in either case,
\[ S_1^{n+1} \leq \max(\bar{S}_1^{n+1}, g_0^{n+1}). \tag{3.39} \]

Similarly, if
\[ S_{j-1}^{n+1} = \max_j S_j^{n+1}, \]
we obtain
\[ S_{j-1}^{n+1} \leq \max(\bar{S}_{j-1}^{n+1}, g_1^{n+1}). \tag{3.40} \]

Thus, combining (3.37), (3.39), and (3.40) with (3.35) we find
\[ \max_j S_j^{n+1} \leq \max_j (\max S_j^n, \sup_{[0,t^{n+1}]} g_0(t), \sup_{[0,t^{n+1}]} g_1(t)). \]

By an argument similar to the one above we can also show
\[ \min_j S_j^{n+1} \geq \min_j (\min S_j^n, \inf_{[0,t^{n+1}]} g_0(t), \inf_{[0,t^{n+1}]} g_1(t)). \]

The lemma now follows by induction on \( n \).
Proof (Lemma 3.2). The proof of Lemma 3.2 is identical to the proof of Lemma 3.1 except for the bound for $\bar{S}_1^{n+1}$. In this case

$$\bar{S}_1^{n+1} = S_1^n - \frac{\Delta t^n}{\Delta x_1} [f(S_1^n, L) - f(g^n_0)]$$

$$= S_1^n - h_1(\beta_1)(S_1^n, L - g^n_0)$$

$$= S_1^n - D_1 \hat{\delta} S_1^n,$$ \hspace{1cm} (3.41)

where

$$D_1 = \begin{cases} 
  h_1(\beta_1)[\hat{\delta} S_1^n + \frac{\Delta t^n}{\Delta x_1} \{1 - h_1(S_1^n)\} \hat{\delta} S_1^n]/\hat{\delta} S_1^n, & \text{if } \hat{\delta} S_1^n \neq 0, \\
  0, & \text{otherwise.} 
\end{cases} \hspace{1cm} (3.42)$$

Here $\beta_1$ is some point between $S_1^n, L$ and $g^n_0$.

Thus

$$D_1 \geq h_1(\beta_1) \geq 0$$ \hspace{1cm} (3.43)

since $\hat{\delta} S_1^n$ is zero or has the same sign as $\hat{\delta} S_1^n$. Furthermore, by (3.20), $\beta_1 \in \hat{S}^n$, thus by (3.42) and (3.13),

$$D_1 \leq h_1(\beta_1)(1 + \frac{\alpha_{1,1}}{2})$$

$$\leq \frac{\Delta t^n}{\Delta x_1} \sup_{\hat{\delta} S_1^n} f'(s)(1 + \frac{\alpha_{1,1}}{2})$$

$$\leq 1.$$ \hspace{1cm} (3.44)

Hence by (3.41), (3.43) and (3.44), we obtain

$$\min(S_1^n, g_0^n) \leq \bar{S}_1^{n+1} \leq \max(S_1^n, g_0^n).$$ \hspace{1cm} (3.45)

The Lemma follows by combining (3.45) with (3.27), (3.37), (3.39), and (3.40), and using induction on $n$. 
3.2 Neumann boundary conditions

In this section, we prove that (3.2) holds for the GMM approximation to \( s \) satisfying (2.1), (2.2), (2.4), and (2.7). We prove this result for both definitions of \( S^n_{j-1,L} \), (2.44) and (2.45). The result is given in the following lemma.

**Lemma 3.3.** Let \( S \) satisfy (2.22), (2.48)-(2.49), and (2.40)-(2.43), with \( S^n_{j-1,L} \) given by either (2.44) or (2.45). Let \( \alpha_{i,j} \) satisfy

\[
0 \leq \alpha_{i,j} \leq 2 \frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}}
\]

for \( j = 1, \ldots, J - 2 \), and

\[
0 \leq \alpha_{i,J-1} \leq 2.
\]

Let \( \Delta t^n \) satisfy

\[
\frac{\Delta t^n}{\Delta x} \sup_{S^n} f'(s) \leq \frac{1}{1 + \frac{\alpha_i}{2}}, \tag{3.46}
\]

where

\[
\Delta x = \min_j \Delta x_j,
\]

\[
\alpha_i = \max_j \frac{\alpha_{i,j} \Delta x_j}{\Delta x_{j-\frac{1}{2}}},
\]

and

\[
S^n = [\min_j S^n_j, \max_j S^n_j].
\]

Then \( S \) satisfies (3.2).

**Proof.** First, assume (2.44) holds. Then, for \( j = 2, \ldots, J - 2 \), we have, by the same argument given in the proof of Lemma 3.1, that

\[
\min_{1 \leq j \leq J-1} S^n_j \leq \bar{S}^{n+1} \leq \max_{1 \leq j \leq J-1} S^n_j, \tag{3.47}
\]
where again $\bar{S}_{n+1}$ is given by (3.15).

For $j = 1$, since $S_{0,J}^n = S_1^n$, we have

$$\bar{S}_{1}^{n+1} = S_1^n - \frac{\Delta t^n}{\Delta x_1} [f(S_1^n) - f(S_1^n)] = S_1^n. \tag{3.48}$$

Furthermore,

$$\bar{S}_{J-1}^{n+1} = S_{J-1}^n - \frac{\Delta t^n}{\Delta x_{J-1}} [f(S_{J-1}^n) - f(S_{J-2,L}^n)]$$

since $\delta S_{J-1}^n = 0$. By the same argument used to derive (3.47) one can show

$$\min(S_{J-1}^n, S_{J-2}^n) \leq \bar{S}_{J-1}^{n+1} \leq \max(S_{J-1}^n, S_{J-2}^n). \tag{3.49}$$

Combining (3.47)-(3.49), we have

$$\min_j S_j^n \leq \bar{S}_j^{n+1} \leq \max_j S_j^n \tag{3.50}$$

for all $j = 1, \ldots, J - 1$.

Now assume

$$\bar{S}_j^{n+1} = \max_j S_j^{n+1}$$

for some $2 \leq j \leq J - 2$, where $S_j^{n+1}$ satisfies (3.36). By the same argument given in the proof of Lemma 3.1 we obtain

$$S_j^{n+1} \leq \bar{S}_j^{n+1}. \tag{3.51}$$

Assume

$$S_1^{n+1} = \max_j S_j^{n+1}.$$

Then by (3.36) and the fact that $S_0^{n+1} = S_1^{n+1}$ we have

$$(1 + \sigma_1^+) S_1^{n+1} - \sigma_1^+ S_2^{n+1} = \bar{S}_1^{n+1}.$$
where again
\[ \sigma^+_1 = \frac{\Delta t^n a^{n+1}}{\Delta x_1 \Delta x_{\frac{1}{2}}} > 0. \]

Thus
\[ S^{n+1}_1 \leq (1 + \sigma^+_1) S^{n+1}_1 - \sigma^+_1 S^{n+1}_2, \]
\[ = \tilde{S}^{n+1}_1. \] (3.52)

Similarly, if
\[ S^{n+1}_j = \max_j S^{n+1}_j, \]
then
\[ S^{n+1}_{j-1} \leq \tilde{S}^{n+1}_{j-1}. \] (3.53)

Thus, by (3.51)-(3.53), and (3.50) we obtain
\[ \max_j S^{n+1}_j \leq \max_j S^n_j. \] (3.54)

Furthermore, by a similar argument, we find that
\[ \min_j S^{n+1}_j \geq \min_j S^n_j. \] (3.55)

Next, assume (2.45) holds. Then (3.54) and (3.55) hold if \( S^{n+1}_j \) is the maximum or minimum of \( S^{n+1} \) for some \( 1 \leq j \leq J - 2 \). Moreover, we have that \( S^{n+1}_{j-1} \) satisfies
\[ S^{n+1}_{j-1} + \frac{\Delta t^n}{\Delta x_{j-1}} [f(S^{n+1}_{j-1}) - f(S^n_{j-2, L})] \]
\[ + \sigma^-_{j-1} (S^{n+1}_{j-1} - S^{n+1}_{j-2}) = S^n_{j-1}, \]
where
\[ \sigma^-_{j-1} = \frac{\Delta t^n a^{n+1}}{\Delta x_{j-1} \Delta x_{j-\frac{1}{2}}} > 0. \]
Thus,

\[(1 + h_{J-1}(\hat{S}) + \sigma_{J-1}^+) S_{J-1}^{n+1} - \sigma_{J-1}^- S_{J-2}^{n+1}\]
\[= S_{J-1}^n + h_{J-1}(\hat{S}) S_{J-2,L}^n,\]

where \(\hat{S} \in (\min(S_{J-1}^{n+1}, S_{J-2,L}^n), \max(S_J^{n+1}, S_{J-2,L}^n))\), and where \(h_j(s)\) is again defined by (3.14). Suppose \(S_{J-1}^{n+1} = \max S_j^{n+1}\), then

\[(1 + h_{J-1}(\hat{S})) S_{J-1}^{n+1} \leq S_{J-1}^n + h_{J-1}(\hat{S}) S_{J-2,L}^n\]
\[\leq (1 + h_{J-1}(\hat{S})) \max_{1 \leq j \leq J-1} S_j^n,\]

where we have used (3.20). Thus, we have

\[S_{J-1}^{n+1} \leq \max_{1 \leq j \leq J-1} S_j^n. \tag{3.56}\]

Similarly, if \(S_{J-1}^{n+1} = \min_j S_j^{n+1}\), then

\[S_{J-1}^{n+1} \geq \min_{1 \leq j \leq J-1} S_j^n. \tag{3.57}\]

Thus, (3.54) and (3.55) hold, and the lemma follows by induction on \(n\).

3.3 Mixed boundary condition

We conclude this chapter by demonstrating discrete maximum principles for the GMM approximation to \(s\) satisfying (2.1)-(2.2), (2.5), and either (2.6) or (2.7).

The results of this section are given in the following lemmas. We prove the first lemma, Lemma 3.4. The proof of the second lemma is obtained trivially by combining elements of the proofs of the Lemma 3.4 and Lemma 3.3.

**Lemma 3.4** Let \(\alpha_{l,j}, j = 2, \ldots, J-1\) satisfy the hypothesis of Lemma (3.1). Let \(S\) be the GMM approximation to \(s\) satisfying (2.1)-(2.2), (2.5), with a Dirichlet boundary condition...
at $x = 1$, i.e. (2.6). Let $\Delta t^n$ satisfy

$$\frac{\Delta t^n}{\Delta x} \sup_{S^n} f'(s) \leq \frac{1}{1 + \frac{\alpha_t}{2}},$$

(3.58)

where

$$\Delta x = \min_j \Delta x_j,$$

$$\alpha_t = \max_{2 \leq j \leq J - 1} \frac{\alpha_{t,j} \Delta x_j}{\Delta x_{j - \frac{1}{2}}},$$

and

$$\hat{S}^n = [\min_j (\min_j S_j^n, \inf_{[0,T]} g_0(t), g_1^n), \max_j (\max_j S_j^n, \sup_{[0,T]} g_0(t), g_1^n)].$$

Then $S$ satisfies (3.1).

**Lemma 3.5** Let $\alpha_{t,j}, j = 2, \ldots, J - 2, \Delta x$, and $\alpha_t$ be given as above. Let $\Delta t^n$ satisfy

$$\frac{\Delta t^n}{\Delta x} \sup_{S^n} f'(s) \leq \frac{1}{1 + \frac{\alpha_t}{2}},$$

(3.59)

where

$$\hat{S}^n = [\min_j (\min_j S_j^n, \inf_{[0,T]} g_0(t)), \max_j (\max_j S_j^n, \sup_{[0,T]} g_0(t))].$$

Let $S$ be the GMM approximation to $s$ satisfying (2.1)-(2.2), (2.5), and (2.7). Then $S$ satisfies (3.4).

**Proof** (Lemma 3.4). For $2 \leq j \leq J - 1$, previous arguments given in the proof of Lemma 3.1 can be used to show

$$\min_{1 \leq j \leq J - 1} S_j^n \leq \hat{S}^{n+1}_j \leq \max_{1 \leq j \leq J - 1} S_j^n,$$

and if

$$S_j^{n+1} = \max_j S_j^{n+1}$$
for some $2 \leq j \leq J - 1$, then

$$S_j^{n+1} \leq \max(S_j^{n+1}, g_1^{n+1})$$

$$\leq \max_j (\max S_j^n, g_1^{n+1}).$$

(3.60)

Moreover, if

$$S_j^{n+1} = \min_j S_j^{n+1}$$

for some $2 \leq j \leq J - 1$, then

$$S_j^{n+1} \geq \min(S_j^{n+1}, g_1^{n+1})$$

$$\geq \min_j (\min S_j^n, g_1^{n+1}).$$

(3.61)

For $j = 1$, we have by (2.22) and (2.46)-(2.47),

$$(1 + \sigma_1^+)S_1^{n+1} - \sigma_1^+ S_2^{n+1}$$

$$= S_1^n - \frac{\Delta t^n}{\Delta x_1} \left[f(S_1^n) - \frac{1}{\Delta x_1} \int_{t^n}^{t^{n+1}} f(g_0(t))dt\right].$$

Thus, if $S_1^{n+1} > S_2^{n+1}$, then

$$S_1^{n+1} < (1 + \sigma_1^+)S_1^{n+1} - \sigma_1^+ S_2^{n+1}$$

$$= S_1^n - \frac{1}{\Delta x_1} \int_{t^n}^{t^{n+1}} [f(S_1^n) - f(g_0(t))]dt$$

$$= S_1^n - \frac{1}{\Delta x_1} \int_{t^n}^{t^{n+1}} f'(\xi(t))(S_1^n - g_0(t))dt$$

$$= \left(1 - \frac{1}{\Delta x_1} \int_{t^n}^{t^{n+1}} f'(\xi(t))dt\right) S_1^n + \frac{1}{\Delta x_1} \int_{t^n}^{t^{n+1}} f'(\xi(t))g_0(t)dt,$$

where $\xi(t)$ is some point between $S_1^n$ and $g_0(t)$. Hence,

$$\frac{1}{\Delta x_1} \int_{t^n}^{t^{n+1}} f'(\xi(t))dt \leq \frac{\Delta t^n}{\Delta x_1} \sup_{[t^n, t^{n+1}]} f'(\xi(t))$$

$$\leq \frac{\Delta t^n}{\Delta x_1} \sup S^n$$

$$\leq 1,$$
by (3.58). Moreover, by the monotonocity assumption on $f$, i.e. $f' \geq 0$, we have

$$\frac{1}{\Delta x_1} \int_{t^n}^{t^{n+1}} f'(\xi(t)) dt \geq 0.$$ 

Thus

$$S_1^{n+1} < \left(1 - \frac{1}{\Delta x_1} \int_{t^n}^{t^{n+1}} f'(\xi(t)) dt\right) S_1^n + \frac{1}{\Delta x_1} \int_{t^n}^{t^{n+1}} f'(\xi(t)) g_0(t) dt$$

$$\leq \max(S_1^n, \sup_{[t^n, t^{n+1}]} g_0(t)). \tag{3.62}$$

Similarly, if $S_1^{n+1} < S_2^{n+1}$, then

$$S_1^{n+1} > \min(S_1^n, \inf_{[t^n, t^{n+1}]} g_0(t)). \tag{3.63}$$

Thus, by (3.60)-(3.63),

$$\min(\min_{j} S_j^n, \inf_{[t^n, t^{n+1}]} g_0(t), g_1^{n+1}) \leq S_2^{n+1} \leq \max(\max_{j} S_j^n, \sup_{[t^n, t^{n+1}]} g_0(t), g_1^{n+1}).$$

The lemma now follows by induction on $n$. 

Chapter 4

Error Estimates

In this chapter we prove the major results of this thesis. Here we derive error estimates in discrete $L^\infty(L^2)$ and $L^\infty(L^1)$ norms for the scheme applied to (2.1), (2.2) with various combinations of the boundary conditions (2.3)-(2.7).

In Sections 4.1-4.3, we derive estimates for the problem (2.1), (2.2) with Dirichlet boundary conditions (2.3), (2.6). $L^\infty(L^2)$ estimates are given in Section 4.1, and an $L^\infty(L^1)$ estimate is given in Section 4.2. Moreover, in Section 4.3 we derive a result for this problem for the modification to the algorithm whereby one takes several advection steps per dispersion step.

In Section 4.4, we derive $L^\infty(L^1)$ results for the problem (2.1)-(2.2) with a Dirichlet boundary condition at $x = 0$ and a Neumann boundary condition at $x = 1$. In Section 4.5, we derive $L^\infty(L^2)$ results for the problem (2.1)-(2.2) with a mixed boundary condition at $x = 0$ and a Dirichlet boundary condition at $x = 1$.

We claim that the above results can be extended to nonuniform spatial grid and to general, positive $a = a(x, t, s)$. To demonstrate the former claim, in Section 4.6 we rederive the estimates of Section 4.1 for the case of quasi-uniform spatial grid.
In Section 4.7, we conclude this chapter by deriving an \( L^\infty(L^1) \) estimate for a first-order Godunov-mixed method applied to the nonlinear equation (2.56) with initial condition (2.57) and Dirichlet boundary conditions.

Truncation error analyses for the above problems are described in the Appendices A, B, and C.

Throughout this chapter, \( \Delta x = \max_j \Delta x_j, \Delta t = \max_n \Delta t^n \), and \( C \) represents a generic constant, independent of \( \Delta x \) and \( \Delta t \). Furthermore, the notation

\[
C = C(\phi)
\]

means \( C \) is a constant which depends on \( \phi \).

### 4.1 \( L^\infty(L^2) \) estimate-Dirichlet problem

In this section we derive a discrete \( L^\infty(L^2) \) error estimate for the scheme presented in Section 2.1. We assume Dirichlet boundary conditions hold at both \( x = 0 \) and \( x = 1 \).

The main result of this section can be stated as follows. When \( s \) satisfying (2.1)-(2.3) and (2.6), the coefficients, and initial and boundary data are sufficiently smooth, \( f' \geq 0 \), and \( a(x,t) \geq a_* > 0 \), then the Godunov-mixed method (GMM) approximation \( S \) to \( s \) satisfies

\[
||s - S||_{L^\infty(L^2)} = O(\Delta x + \Delta t).
\]  

(4.1)

Here \( s^n_j = s(x_j,t^n) \), and \( \Delta t^n \) is chosen to satisfy the hypothesis of Lemma 3.1 if \( S^n_{0,L} = g^n_0 + \frac{1}{2} \) or the hypothesis of Lemma 3.2 if \( S^n_{0,L} = g^n_0 \). Assume for the moment that \( S^n_{0,L} = g^n_0 + \frac{1}{2} \).

We will remark at the end of this section about the changes in the estimate when the latter choice is made. In this section, we also assume the spatial mesh is uniform.

Let \( s \) satisfy (2.1)-(2.3), (2.6). Integrating (2.1) over \( B_j \times [t^n,t^{n+1}] \), we find that, for
\[ j = 1, \ldots, J - 1, \]
\[
\frac{s_{j}^{n+1} - s_{j}^{n}}{\Delta t} + \frac{f(s_{j,L}^{n}) - f(s_{j-1,L}^{n})}{\Delta x} \]
\[- \frac{1}{\Delta x} \left[ a_{j+\frac{1}{2}}^{n+1} s_{j+\frac{1}{2}}^{n+1} - a_{j-\frac{1}{2}}^{n+1} s_{j-\frac{1}{2}}^{n+1} \right] = E_{j}^{n}. \] (4.2)

Here \( E_{j}^{n} \) represents truncation error, and

\[
E_{j}^{n} = E_{T,j}^{n} + (E_{A,j}^{n})_{x} + (E_{B,j}^{n})_{x}, \quad (4.3)
\]

where

\[
E_{T,j}^{n} = \frac{s_{j}^{n+1} - s_{j}^{n}}{\Delta t} - \frac{1}{\Delta t} \left[ \frac{1}{\Delta x} \int_{B_{j}} s_{j}^{n+1} dx - \frac{1}{\Delta x} \int_{B_{j}} s_{j}^{n} dx \right], \quad (4.4)
\]

\[
E_{A,j}^{n} = f(s_{j,L}^{n}) - \frac{1}{\Delta t} \int_{t_{n}}^{t_{n+1}} f(s(x_{j+\frac{1}{2}}, t)) dt, \quad (4.5)
\]

and

\[
E_{B,j}^{n} = a_{j+\frac{1}{2}}^{n+1} \frac{s_{j+\frac{1}{2}}^{n+1} - s_{j-\frac{1}{2}}^{n+1}}{\Delta x} - \frac{1}{\Delta t} \int_{t_{n}}^{t_{n+1}} (a \frac{\partial s}{\partial x})(x_{j+\frac{1}{2}}, t) dt. \quad (4.6)
\]

In (4.2)-(4.6), for \( n = 0, \ldots, N^{*} \),

\[
s_{j}^{n} = s(x_{j}, t^{n}), \quad j = 1, \ldots, J - 1; \quad (4.7)
\]

and, as in Chapter 2, we define extensions \( s_{0}^{n} \) and \( s_{J}^{n} \) of \( s^{n} \) to the intervals \([-\Delta x, 0]\) and \([1, 1 + \Delta x]\), respectively, which incorporate boundary data. These extensions are given by

\[
s_{0}^{n} = 2g_{0}^{n} - s_{1}^{n}, \quad (4.8)
\]

\[
s_{J}^{n} = 2g_{1}^{n} - s_{J-1}^{n}. \quad (4.9)
\]

Moreover, we define

\[
s_{0,L}^{n} = g_{0}^{n+\frac{1}{2}}, \quad (4.10)
\]
\begin{align}
  s_{j,L}^n &= s_j^n + \frac{\Delta x}{2} (1 - h(s_j^n)) \delta s_j^n, \quad j = 1, \ldots, J - 1, \\
  \delta s_j^n &= \frac{s_{j+1}^n - s_j^{n-1}}{2\Delta x}, \quad j = 2, \ldots, J - 2, \\
  \delta s_1^n &= \frac{s_2^n + 3s_1^n - 4s_0^n}{3\Delta x}, \\
  \delta s_{j-1}^n &= \frac{4s_j^n - 3s_{j-1}^n - s_{j-2}^n}{3\Delta x}.  
\end{align}

Multiplying (4.2) by an arbitrary constant \( w_j \Delta x \), and summing on \( j \) we obtain

\begin{equation}
\langle s_t^{n+1} + f(s_L^n)_x - (a s_{xi}^{n+1})_x, w \rangle = \langle E^n, w \rangle, \quad w \in M_{-1}^{0}(\delta x).  
\end{equation}

Moreover, multiplying (2.22) by \( w_j \Delta x \) and summing on \( j \), we find that the GMM approximation \( S \) satisfies

\begin{equation}
\langle S_t^{n+1} + f(S_L^n)_x - (a S_{xi}^{n+1})_x, w \rangle = 0, \quad w \in M_{-1}^{0}(\delta x).  
\end{equation}

Let

\begin{align}
  \xi_j^n &= s_j^n - S_j^n, \\
  \xi_{j,L}^n &= s_{j,L}^n - S_{j,L}^n.  
\end{align}

Then, subtracting (4.17) from (4.16) we obtain

\begin{equation}
\langle \xi_t^{n+1} + (f(s_L^n) - f(S_L^n))_x - (a \xi_{xi}^{n+1})_x, w \rangle = \langle E^n, w \rangle.  
\end{equation}
Note that by (4.10), (4.8)-(4.9), and (2.36)-(2.38), we have

\[ \xi^0_{0,L} = 0, \quad (4.19) \]

and the extensions \( \xi^{n+1}_0 \) and \( \xi^{n+1}_j \) of \( \xi^{n+1} \) satisfy

\[ \xi^{n+1}_0 = -\xi^{n+1}_1, \quad (4.20) \]

and

\[ \xi^{n+1}_j = -\xi^{n+1}_{j-1}, \quad n = 0, \ldots, N^* - 1. \quad (4.21) \]

The procedure we follow to derive our estimate is based on an idea given in Rose [34]. In this procedure, one sums by parts to place the discrete differences on the test function \( w \). One then substitutes for \( w \) a time-dependent test function which we denote by \( Z^n \). Summing by parts gives rise to an adjoint linear parabolic problem which \( Z^n \) is chosen to satisfy, with boundary conditions determined by the boundary conditions of the original problem, and initial condition determined by the type of estimate one wants to derive; e.g., \( L^\infty(L^2) \), \( L^\infty(L^1) \), etc.

Thus, consider \( \langle (f(s^n_L) - f(S^n_L))_x, w \rangle \). By summing by parts and using (4.19) we have

\[
\begin{align*}
\langle (f(s^n_L) - f(S^n_L))_x, w \rangle &= \sum_{j=1}^{J-1} (f(s^n_{j,L}) - f(S^n_{j,L}))_x w_j \Delta x \\
&= - \sum_{j=1}^{J-1} (f(s^n_{j,L}) - f(S^n_{j,L})) w_{j,x} \Delta x \\
&\quad + (f(s^n_{j-1,L}) - f(S^n_{j-1,L})) w_j \\
&= - \sum_{j=1}^{J-2} (f(s^n_{j,L}) - f(S^n_{j,L})) w_{j,x} \Delta x \\
&\quad + (f(s^n_{j-1,L}) - f(S^n_{j-1,L})) w_{j-1}.
\end{align*}
\]

Thus we obtain
\begin{align}
\langle (f(s_L^n) - f(S_L^n))_\varepsilon, w \rangle &= -\langle (f(s_L^n) - f(S_L^n)), \kappa w_x \rangle \\
&+ (f(s_{j-1,L}^n) - f(S_{j-1,L}^n))(w_J + w_{J-1})/2 \\
&= -\langle A^n \xi_L^n, \kappa w_x \rangle + B_A^n(w), 
\end{align}

where

\begin{align}
B_A^n(w) &= \kappa_{J-1} A_{j-1}^n \xi_{j-1,L}^n (w_J + w_{J-1}) 
\end{align}

with \( \kappa_j \) and \( A_j^n \) defined by

\begin{align}
\kappa_j &= \begin{cases} 
1, & \text{for } j = 1, \ldots, J - 2, \\
\frac{1}{2}, & \text{for } j = J - 1,
\end{cases}
\end{align}

and for \( j = 0, \ldots, J - 1, \)

\begin{align}
A_j^n &= \begin{cases} 
(f(s_{j,L}^n) - f(S_{j,L}^n))/\xi_{j,L}^n, & \text{if } \xi_{j,L}^n \neq 0, \\
0, & \text{otherwise.}
\end{cases}
\end{align}

Recalling from Section 1.2 the definition of \((a_j^{n+1} \xi_j^{n+1})_\varepsilon\) and summing by parts twice we find that

\begin{align}
\langle (a^{n+1} \xi^{n+1})_\varepsilon, w \rangle &= \langle \xi^{n+1}, (a^{n+1} w_\varepsilon)_\varepsilon \rangle + B_D^n(w), 
\end{align}

where

\begin{align}
B_D^n(w) &= \frac{1}{\Delta x} \left[ a_{j-\frac{1}{2}}^{n+1} (\xi_{j-1}^{n+1} - \xi_{j-1}^{n+1}) w_{J-1} - a_{j-\frac{1}{2}}^{n+1} \xi_{j-1}^{n+1} (w_J - w_{J-1}) \right] \\
&- \frac{1}{\Delta x} \left[ a_{\frac{1}{2}}^{n+1} (\xi_0^{n+1} - \xi_{0}^{n+1}) w_0 - a_{\frac{1}{2}}^{n+1} \xi_0^{n+1} (w_1 - w_0) \right] \\
&= -\frac{1}{\Delta x} \left[ a_{j-\frac{1}{2}}^{n+1} \xi_{j-1}^{n+1} (w_J + w_{J-1}) - a_{\frac{1}{2}}^{n+1} \xi_{\frac{1}{2}}^{n+1} (w_1 + w_0) \right]
\end{align}

by (4.20) and (4.21).

Let \( w = Z^n \in \mathcal{M}_{-1}(\delta_x) \), where \( Z^n \) will be defined shortly. Then, substituting (4.22) and (4.26) into (4.18) we obtain

\begin{align}
\langle \xi_1^{n+1}, Z^n \rangle - \langle \xi_0^n, \kappa A^n Z_\varepsilon^n \rangle - \langle \xi^{n+1}, (a^{n+1} Z_\varepsilon^n)_\varepsilon \rangle &= \langle E^n, Z^n \rangle + B^n(Z^n)
\end{align}
where

\[ B^n(Z^n) = B^n_D(Z^n) - B^n_A(Z^n). \]  \hspace{1cm} (4.28)

Multiplying above by \( \Delta t^n \) and summing on \( n, n = 0, 1, \ldots, N \), where \( N \leq N^* \) is arbitrary, and summing by parts on \( n \) we obtain

\[ \langle \xi^N, Z^N \rangle = \sum_{n=0}^{N-1} \langle \xi^{n+1}, Z_t^{n+1} + (a^{n+1}Z_x^n)_x \rangle \Delta t^n \]
\[ + \sum_{n=0}^{N-1} \langle \xi^n, \kappa A^n Z^n_x \rangle \Delta t^n \]
\[ + \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} B^n(Z^n) \Delta t^n \]
\[ + \langle \xi^0, Z^0 \rangle \equiv I_1 + I_2 + I_3 + I_4 + I_5. \]  \hspace{1cm} (4.29)

**Choice of \( Z^n \).** We choose \( Z^n \) to be the block-centered finite difference (i.e. lowest-order mixed method with trapezoidal rule) approximation to \( z(x, t^n) \), where \( z \) satisfies the "backward" parabolic equation

\[ \frac{\partial z}{\partial t} + \frac{\partial}{\partial x} \left( \bar{a}(x, t) \frac{\partial z}{\partial x} \right) = 0, \quad (x, t) \in I \times [0, t^N), \]  \hspace{1cm} (4.30)

\[ z(x, t^N) = \xi^N_j, \quad x \in B_j, \]  \hspace{1cm} (4.31)

\[ z(0, t) = z(1, t) = 0, \quad 0 \leq t < t^N. \]  \hspace{1cm} (4.32)

Here \( \bar{a}(x, t) = a(x, t + \Delta t^n) \), \( t^n \leq t < t^{n+1} \).

Thus, from Sections 2.1 and 2.2, we know that \( Z^n \) satisfies for \( n = 0, \ldots, N - 1 \),

\[ \langle Z_t^{n+1} + (a^{n+1}Z^n_x)_x, v \rangle = 0, \quad v \in \mathcal{M}^0(\delta_x), \]  \hspace{1cm} (4.33)

\[ Z_j^N = \xi^N_j, \quad j = 1, \ldots, J - 1, \]  \hspace{1cm} (4.34)

and the extensions \( Z^0_0 \) and \( Z^0_j \) are given by

\[ Z^0_0 = -Z^0_1, \quad Z^0_j = -Z^0_{j-1}. \]  \hspace{1cm} (4.35)
We now derive bounds energy-type bounds for $Z$ which we will utilize in the estimates of $I_1 - I_5$. These bounds are given in the following lemma.

**Lemma 4.1** Let $Z$ satisfy (4.33)-(4.35). Let $0 < a_* \leq a(x,t)$. Then

$$
\|Z\|_{L^\infty(0,N;L^2)} \leq \|\xi^N\|, \quad (4.36)
$$

and

$$
\|Z_x\|_{L^2(0,N-1;L^2)} + \|Z_t\|_{L^2(0,N-1;L^2)} \leq \frac{C}{a_*} \|\xi^N\|. \quad (4.37)
$$

**Proof.** Let $Y_j^n = Z_j^{N-n}$, $j = 1, \ldots, J-1$, $\bar{a}^n = \bar{a}^{N-n}$, and $\Delta t^n = \Delta t^{N-n}$. Then $Y$ satisfies

$$
\langle Y_j^{n+1} - (\bar{a}^{n+1} Y_{x}^{n+1})_x, v \rangle = 0, \quad v \in M_{-1}^{0}(\delta_x), \quad (4.38)
$$

$$
Y_j^0 = \xi^N, \quad j = 1, \ldots, J-1, \quad (4.39)
$$

$$
Y_0^n = -Y_1^n, \quad Y_j^n = -Y_{j-1}^n. \quad (4.40)
$$

Let $v = Y^{n+1}$ in (4.38). Then summing the second term in (4.38) by parts, utilizing (4.40) and applying the inequality $b(b-c) \geq \frac{1}{2}(b^2 - c^2)$ to the first term we obtain

$$
\frac{\|Y^{n+1}\|^2 - \|Y^n\|^2}{2\Delta t^{n+1}} + \frac{a_*}{2} \left[ \|Y_x^{n+1}\|^2 + \|Y_{x}^{n+1}\|^2 \right] \leq 0. \quad (4.41)
$$

Multiplying (4.41) by $\Delta t^{n+1}$ and summing on $n$, $n = 0, 1, \ldots, M-1$, where $M \leq N$ is arbitrary, we obtain

$$
\|Y^M\|^2 + \frac{a_*}{2} \sum_{n=0}^{M-1} \left( \|Y_x^{n+1}\|^2 + \|Y_{x}^{n+1}\|^2 \right) \Delta t^{n+1} \leq \|Y^0\|^2.
$$

The lemma now follows by recalling the definition of $Y_j^n$. 

Returning to (4.29), we see that by (4.33)-(4.35), (4.23), (4.27), and (4.28), \( I_1 = I_4 = 0 \), and

\[
||\xi^N||^2 = I_2 + I_3 + I_5. \tag{4.42}
\]

**Estimate of \( I_2 \).** Consider

\[
I_2 = \sum_{n=0}^{N-1} (\xi^n, \kappa A^n Z^n_x) \Delta t^n
\leq ||\kappa A||_{L^\infty(L^\infty)} ||\xi||_{L^2(0,N-1;L^2)} ||Z_x||_{L^2(0,N-1;L^2)}
\leq \frac{C ||A||_{L^\infty(L^\infty)}^2}{a_*} ||\xi||_{L^2(0,N-1;L^2)}^2 + \frac{1}{8} ||\xi^N||^2 \tag{4.43}
\]

where we have applied the Cauchy-Schwarz inequality, (4.37) of Lemma 4.1, and the well-known inequality

\[
ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2, \quad \epsilon > 0. \tag{4.44}
\]

Here we have chosen \( \epsilon = 1/4 \).

We now estimate \( ||\xi||_{L^2(0,N-1;L^2)} \). Recall

\[
\xi^n_{j,L} = s^n_{j,L} - S^n_{j,L}
= \xi^n_j + \frac{\Delta x}{2} (1 - h(s^n_j)) \delta s^n_j - \frac{\Delta x}{2} (1 - h(S^n_j)) \delta S^n_j. \tag{4.45}
\]

By the slope-limiting procedure (2.25), (2.28), and (2.32) there exist \( \omega^n_j \) with \( 0 \leq \omega^n_j \leq 1 \) such that

\[
\delta S^n_j = \omega^n_j \delta S^n_j, \tag{4.46}
\]

where \( \delta S^n_j \) is given by (2.24), (2.29), or (2.33). Hence

\[
\delta s^n_j - \delta S^n_j = \omega^n_j (\delta s^n_j - \delta S^n_j) + (1 - \omega^n_j) \delta s^n_j
\]

so that

\[
\Delta x |\delta s^n_j - \delta S^n_j| \leq \Delta x \omega^n_j |\delta s^n_j - \delta S^n_j| + \Delta x (1 - \omega^n_j) |\delta s^n_j|. \tag{4.47}
\]
Furthermore, by (4.13)-(4.15) and (2.24), (2.29), and (2.33) we have

\[
\Delta x |\delta s_{j}^n - \delta S_{j}^n| \leq \begin{cases} 
|\xi_{j+1}^n| + |\xi_{j}^n|, & \text{for } j = 2, \ldots, J - 2, \\
|\xi_{j}^n| + |\xi_{j-1}^n|, & \text{for } j = 1, \\
|\xi_{j-1}^n| + |\xi_{j-2}^n|, & \text{for } j = J - 1.
\end{cases} 
\]  

(4.48)

Thus, by (4.45)-(4.48), and recalling that by (4.20) and (4.21), $|\xi_{0}^n| = |\xi_{1}^n|$ and $|\xi_{J}^n| = |\xi_{J-1}^n|$, we see that for all $j = 1, \ldots, J - 1$,

\[
|\xi_{j,L}^n| \leq |\xi_{j}^n| + \frac{\Delta x}{2} |1 - h(S_{j}^n)| |\delta s_{j}^n - \delta S_{j}^n| + \frac{\Delta x}{2} |h(S_{j}^n) - h(S_{j,n})| |\delta s_{j}^n| \\
\leq C(|\xi_{j}^n| + |\xi_{j+1}^n| + |\xi_{j-1}^n|) \\
+ C\Delta x \left( |1 - \omega_{j}^n| + |h(s_{j}^n) - h(S_{j}^n)| \right) |\delta s_{j}^n|.
\]  

(4.49)

Squaring both sides in (4.49), applying (4.44) with $\epsilon = 1$, multiplying by $\Delta x \Delta t^n$, and summing on $j$ and $n$ we obtain

\[
\|\xi_{L}^n\|_{L^2(0,N-1;L^2)}^2 \leq C\|\xi_{L}^n\|_{L^2(0,N-1;L^2)}^2 \\
+ C(\|1 - \omega\|_{L^{\infty}(L^\infty)}, \|h(s) - h(S)\|_{L^{\infty}(L^\infty)}) \Delta x^2 \|\delta s_{L}^n\|_{L^2(L^2)}^2.
\]  

(4.50)

Assuming $s$ and the initial condition $s^0$ are sufficiently smooth (e.g., $\frac{\partial s}{\partial x}$ and $\frac{\partial s^0}{\partial x}$ exist and are bounded), we have

\[
\|\delta s_{L}^n\|_{L^2(L^2)} \leq C \equiv C_{2,1}.
\]  

(4.51)

Also, by maximum principles for $s$ and the time-step restrictions given in Lemma 3.1.,

\[
|h(s_{j}^n) - h(S_{j}^n)| = \frac{\Delta t^n}{\Delta x} |f'(s_{j}^n) - f'(S_{j}^n)| \\
\leq C \equiv C_{2,2}.
\]  

(4.52)

Furthermore, by the definition of $A_{j}^n$, i.e. (4.25), and the Mean Value Theorem, $A_{j}^n = f'(\bar{x}_{j}^n)$, where $\bar{x}_{j}^n$ is a point between $s_{j,L}^n$ and $S_{j,L}^n$. By maximum principles for $s$ and $S$, we
have that $\tilde{\xi}_j^n$ is contained in a compact subset of $\mathbb{R}$ for all $j$ and $n$, and this set is independent of $\Delta x$ and $\Delta t$. Thus since $f \in C^1(\mathbb{R})$, $A_j^n$ is bounded; i.e.,

$$
||A||_{L^\infty(L^\infty)} \leq C \equiv C_{2,3}.
$$

(4.53)

Hence, by (4.43), (4.50), and (4.51)-(4.53), we have

$$
I_2 \leq (C_2)^2(||\xi||_{L^2(0,N-1; L^2)}^2 + \Delta x^2) + \frac{1}{8}||\xi^N||^2,
$$

(4.54)

where

$$
C_2 = a_{k}^{-\frac{1}{3}} C(C_{2,1}, C_{2,2}, C_{2,3}).
$$

(4.55)

**Remark 3.1.** For $f'$ Lipschitz continuous, we can refine our argument above somewhat; i.e., we have

$$
|h(s_j^n) - h(S_j^n)| = \frac{\Delta t^n}{\Delta x} |f'(s_j^n) - f'(S_j^n)| \leq C(L_{f'}) |\xi_j^n|,
$$

where $L_{f'}$ is a Lipschitz constant for $f'$. Thus,

$$
||\xi_{j,L}||_{L^2(0,N-1; L^2)} \leq C(||\xi_j^n|| + ||\xi_{j-1}^n|| + ||\xi_{j+1}^n||) + \Delta x C (||1 - \omega_j^n||) ||\delta s_j^n||,
$$

and

$$
||\xi||_{L^2(0,N-1; L^2)} \leq C||\xi||_{L^2(0,N-1; L^2)} + C(1 - \omega ||\xi||_{L^\infty(L^\infty)} \Delta x^2).
$$

Hence

$$
I_2 \leq (C'_2)^2(||\xi||_{L^2(0,N-1; L^2)}^2 + ||1 - \omega||_{L^\infty(L^\infty)} \Delta x^2) + \frac{1}{8}||\xi^N||^2,
$$

(4.56)

where

$$
C'_2 = a_k^{-\frac{1}{3}} C(C_{2,1}, C_{2,3}, L_{f'}).
$$

(4.57)
Hence, if the problem is sufficiently dispersive so that one can set \( \delta S^n_j = \delta S^n_j \) for all \( j \) and \( n \); i.e. \( \omega^n_j \equiv 1 \), then the \( \mathcal{O}(\Delta x^2) \) term above disappears. Numerical testing [15] indicates that when \( a_* \) is sufficiently large, slope-limiting is not necessary to preserve stability. Thus, in these situations the dominant error is the truncation error. However, the question of when to turn the slope-limiter on and when to turn it off has not been answered theoretically.

**Remark 3.2.** Assume \( f' \) is Lipschitz continuous and the slope-limiting parameter \( \alpha, j \equiv 1 \). As we will see in the appendix, without changing our global truncation error estimate we could modify our definition of \( s^n_{j,L} \) by setting

\[
 s^n_{j,L} = s^n_j + \frac{\Delta x}{2} (1 - h(s^n_j)) \delta \delta^n_j,
\]

where

\[
 \delta \delta^n_j = \begin{cases} 
 s^n_{j,x}, & \text{if } \delta S^n_j = S^n_{j,x}, \\
 s^n_{j,x}, & \text{if } \delta S^n_j = S^n_{j,x}, \\
 \delta s^n_j, & \text{if } \delta S^n_j = 0 \text{ or } \delta S^n_j = \delta S^n_j.
\end{cases}
\]

Now one can argue heuristically that the number of times \( \delta S^n_j \) is set to zero when \( \frac{\partial s(S^n)}{\partial x} \neq 0 \) is bounded for each \( n \), and furthermore this bound, which we will call \( L^* \), is independent of \( \Delta x \) and \( \Delta t \). In other words, we argue that, except in regions where \( S \) and \( s \) are constant, \( S \) has at most a fixed, finite number of local maxima and/or minima independent of the mesh spacing. Under this assumption,

\[
 |\xi^n_{j,L}| \leq |\xi^n_j| + \frac{\Delta x}{2} |1 - h(s^n_j)| |\delta s^n_j - \delta S^n_j| + \frac{\Delta x}{2} |h(s^n_j) - h(S^n_j)| |\delta \delta^n_j|
\]

and

\[
 ||\xi||L^2(0,N-1; L^2) \leq C||\xi||L^2(0,N-1; L^2) + C \Delta x^2 \sum_{n=0}^{N-1} \sum_{j \in T^n} |\delta \delta^n_j|^2 \Delta x \Delta t^n,
\]
where

\[ \mathcal{J}^n = \{ j \mid \delta s_j^n = 0 \} . \]

Since the cardinality of \( \mathcal{J}^n \) is bounded by \( L^* < \infty \) for each \( n \) independent of \( \Delta x \) and \( \Delta t \), we have

\[ \| \xi \|_{L^2(0, N-1; L^2)}^2 \leq C \| \xi \|_{L^2(0, N-1; L^2)}^2 + C(L^*) \Delta x^3. \]

Hence

\[ I_2 \leq (C_2')^2 (\| \xi \|_{L^2(0, N-1; L^2)}^2 + \Delta x^3) + \frac{1}{8} \| \xi' \|_{L^2}^2, \quad (4.58) \]

where

\[ C_2' = a_x^{-\frac{1}{2}} C(C_{2,1}, C_{2,3}, L_f', L^*). \quad (4.59) \]

**Estimate of \( I_3 \).** Next, consider

\[ I_3 = \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n \]

\[ = \sum_{n=0}^{N-1} \langle E^n_T, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} \langle (E^n_A)_z, Z^n \rangle \Delta t^n \]

\[ + \sum_{n=0}^{N-1} \langle (E^n_D)_z, Z^n \rangle \Delta t^n \]

\[ \equiv I_3' + I_3'' + I_3''', \quad (4.60) \]

where \( E_T, E_A, \) and \( E_D \) are given by (4.4)-(4.6).

Consider the first term above. By Cauchy-Schwarz, (4.36) and (4.44) we have

\[ I_3' = \sum_{n=0}^{N-1} \langle E^n_T, Z^n \rangle \Delta t^n \]

\[ \leq C \| E_T \|_{L^2(L^2)}^2 + \frac{1}{24} \| \xi' \|_{L^2}^2. \quad (4.61) \]
Next, recall

\[
I_3'' = \sum_{n=0}^{N-1} \langle \langle E''_n \rangle_z, Z^n \rangle \Delta t^n
\]

\[
= \sum_{n=0}^{N-1} \sum_{j=1}^{J-1} \left[ \left( f(s^n_{j+\frac{1}{2}}, - \frac{1}{\Delta t^n} \int_t^{t+n+1} f(s(x_{j+\frac{1}{2}}, t)) dt \right)_x Z^n_j \right] \Delta x \Delta t^n
\]

\[
= \sum_{n=0}^{N-1} \sum_{j=1}^{J-1} \left[ \left( f(s^n_{j+\frac{1}{2}}, f(s(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}})) - \text{avg}_{[n, n+1]} f(s(x_{j+\frac{1}{2}}, t)) \right)_x Z^n_j \Delta x \right] \Delta t^n
\]

\[
+ \sum_{n=0}^{N-1} \sum_{j=1}^{J-1} \left[ \left( f(s(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}})) - \text{avg}_{[n, n+1]} f(s(x_{j+\frac{1}{2}}, t)) \right)_x Z^n_j \Delta x \right] \Delta t^n
\]

\[
\equiv \sum_{n=0}^{N-1} \langle \langle E''_n \rangle_z, Z^n \rangle \Delta t^n + \langle \langle E''_n \rangle_x, Z^n \rangle \Delta t^n, \tag{4.62}
\]

where

\[
\text{avg}_G g(\sigma) = \frac{1}{m(T)} \int_T g(\sigma) d\sigma. \tag{4.63}
\]

Summing by parts, and using the identities \( Z^n_j = -Z^n_{j-1} \), and \( s^n_{0,L} = s^{n+\frac{1}{2}} = s^{n+\frac{1}{2}} \), we have

\[
\sum_{n=0}^{N-1} \langle \langle E''_n \rangle_x, Z^n \rangle \Delta t^n = -\sum_{n=0}^{N-1} \langle E''_n, \kappa Z^n \rangle \Delta t^n
\]

\[
\leq \frac{C}{\Delta x} \| E''_n \|^2_{L^2(L^2)} + \frac{1}{48} \| \kappa^n \|^2,
\tag{4.64}
\]

by Cauchy-Schwarz, (4.37), and (4.44).

Summing by parts again, and using the identities \( Z^n_j = -Z^n_{j-1} \), and \( Z^n_0 = -Z^n_1 \) we obtain

\[
\sum_{n=0}^{N-1} \langle \langle E''_n \rangle_z, Z^n \rangle \Delta t^n
\]

\[
= -\frac{1}{2} \sum_{n=0}^{N-1} \left[ \sum_{j=1}^{J-1} \left( f(s_{j+\frac{1}{2}}, f(s(x_{j+\frac{1}{2}}, t))) Z^n_{j, x} \Delta x \right) \Delta t^n
\]

\[
- \frac{1}{2} \sum_{n=0}^{N-1} \left[ \sum_{j=1}^{J-1} \left( f(s_{j-\frac{1}{2}}, f(s(x_{j-\frac{1}{2}}, t))) Z^n_{j-1, x} \Delta x \right) \Delta t^n
\]

\[
\equiv -\frac{1}{2} \sum_{n=0}^{N-1} \left( \langle E''_{AR}, Z^n \rangle + \langle E''_{AL}, Z^n \rangle \right) \Delta t^n
\]
\[ \leq \frac{C}{a_*} \left[ \|E_{AR}''\|^2_{L^2(L^2)} + \|E_{AL}''\|^2_{L^2(L^2)} + \frac{1}{48} \|\xi''\|^2 \right]. \tag{4.65} \]

Thus, combining (4.62), (4.64)-(4.65) we have

\[ I_3'' \leq \frac{C}{a_*} \left[ \|E_{AR}'\|^2_{L^2(L^2)} + \|E_{AR}''\|^2_{L^2(L^2)} + \|E_{AL}''\|^2_{L^2(L^2)} + \frac{1}{24} \|\xi''\|^2 \right]. \tag{4.66} \]

Now consider \( I_3'''. \) We have

\[ I_3''' = \sum_{n=0}^{N-1} (\langle (E^n_D)_{x_1}, Z^n \rangle) \Delta t^n \]

\[ = \sum_{n=0}^{N-1} \left[ \sum_{j=1}^{J-1} \left( a_{n+1, n+1, j}^{n+1} s_{n+1, j}^{n+1} - \text{avg}_{x_1, t} (a \frac{\partial \xi}{\partial x})(x_{j+\frac{1}{2}}, t) \right) Z_j^n \Delta x \right] \Delta t^n \]

\[ = -\frac{1}{2} \sum_{n=0}^{N-1} \left[ \sum_{j=1}^{J-1} \left( a_{n+1, n+1, j}^{n+1} s_{n+1, j}^{n+1} - \text{avg}_{x_1, t} (a \frac{\partial \xi}{\partial x})(x_{j+\frac{1}{2}}, t) \right) Z_j^n \Delta x \right] \Delta t^n \]

\[ -\frac{1}{2} \sum_{n=0}^{N-1} \left[ \sum_{j=1}^{J-1} \left( a_{n+1, n+1, j}^{n+1} s_{n+1, j}^{n+1} - \text{avg}_{x_1, t} (a \frac{\partial \xi}{\partial x})(x_{j-\frac{1}{2}}, t) \right) Z_j^n \Delta x \right] \Delta t^n \]

\[ = -\frac{1}{2} \sum_{n=0}^{N-1} \left[ (E^n_{DR}, Z^n_x) + (E^n_{DL}, Z^n_x) \right] \Delta t^n \tag{4.67} \]

by summation by parts. Thus

\[ I_3''' \leq \frac{C}{a_*} \left[ \|E_{DR}''\|^2_{L^2(L^2)} + \|E_{DL}''\|^2_{L^2(L^2)} + \frac{1}{24} \|\xi''\|^2 \right] \tag{4.68} \]

by Cauchy-Schwarz, (4.37) and (4.44).

Combining (4.61), (4.66), and (4.68) with (4.60), we obtain

\[ I_3 \leq \frac{C}{a_*} \left[ \|E_{AR}'\|^2_{L^2(L^2)} + \|E_{AR}''\|^2_{L^2(L^2)} + \|E_{AL}''\|^2_{L^2(L^2)} \right. \]

\[ + \|E_{DL}'\|^2_{L^2(L^2)} + \|E_{DL}''\|^2_{L^2(L^2)} + C\|E_T''\|^2_{L^2(L^2)} + \frac{1}{8} \|\xi''\|^2 \]

\[ \equiv (C_3)^2 (TE)^2 + \frac{1}{8} \|\xi''\|^2, \tag{4.69} \]

where

\[ C_3 = C(a_*^{-\frac{1}{2}}). \tag{4.70} \]
Estimate of \( I_5 \). Finally, recall

\[
I_5 = \langle \xi^0, \mathcal{Z}^0 \rangle.
\]

By Taylor expansion with integral remainder, we have

\[
|\xi^0_j| = |s^0_j - \text{avg}_{B_j} s^0(x)| 
\leq C \Delta x^{\frac{3}{2}} \| \frac{d^2 s^0}{dx^2} \|_{C^2(B_j)}. 
\]

Thus

\[
I_5 \leq C \| \xi^0 \|^2 + \frac{1}{8} \| \xi^N \|^2 
\leq (C_5)^2 \Delta x^4 + \frac{1}{8} \| \xi^N \|^2, \tag{4.71}
\]

where

\[
C_5 = C(\| s^0 \|_{H^2(I)}). \tag{4.72}
\]

Combining (4.54), (4.69), and (4.71) with (4.42) and hiding the \( \frac{1}{8} \| \xi^N \|^2 \) terms on the left, we have

\[
\| \xi^N \|^2 \leq (C_2)^2 \left( \| \xi \|^2_{L^2(0,N-1; L^2)} + \Delta x^2 \right) + (C_3)^2 (TE)^2 + (C_5)^2 \Delta x^4. \tag{4.73}
\]

By applying a discrete version of Gronwall’s Lemma [27] we complete the proof of the following theorem.

**Theorem 4.1** Let \( S \) be the GMM approximation to \( s \) satisfying (2.1)-(2.3), (2.6) as given in Sections 2.1 and 2.2. Assume \( f, a, s, s^0, g_0 \) and \( g_1 \) satisfy the following:

(i) \( f \in C^1(\mathbb{R}), f' \geq 0, \)

(ii) \( a(x,t) \geq a_* > 0, \)
(iii) \( \frac{\partial^2}{\partial x^2} \) and \( \frac{\partial^2}{\partial x \partial t} \) exist and are bounded in \( Q_T \),

(iv) \( s^0 \in \mathcal{H}^2(I) \),

(v) \( g_0, g_1 \in C^1(0, T) \).

Assume uniform spatial mesh and assume \( \Delta t^n \) is chosen according to Lemma 3.1. Then

\[
\| s - S \|_{L^\infty(L^2)} \leq C_\xi (\Delta x + TE),
\]

where \( TE \) is the truncation error given by (4.69), and

\[
C_\xi = C(\sqrt{\varepsilon (C_3)^2}, C_3, C_5).
\]

Recall from Remark 3.1 and the estimate of \( I_2 \) that if \( f' \) is Lipschitz continuous, and if \( \delta S_j^n = \delta S_j^n \) for every \( j \) and \( n \), then the \( \mathcal{O}(\Delta x) \) term disappears in (4.54). Thus, instead of (4.73) we have

\[
\| \xi^N \|^2 \leq (C_2')^2 \| \xi \|^2_{L^2(0, N-1; L^2)} + (C_3)^2 \Delta x^4 + (C_3)^2 (TE)^2.
\]

Hence we obtain the following.

**Theorem 4.2** Let the hypothesis of Theorem 4.1 hold. Assume \( f' \) is Lipschitz continuous and the limited slope \( \delta S_j^n = \delta S_j^n \) for every \( j \) and \( n \). Then

\[
\| s - S \|_{L^\infty(L^2)} \leq C_\xi' (\Delta x^2 + TE),
\]

where

\[
C_\xi' = C(\sqrt{\varepsilon (C_4')^2}, C_3, C_5).
\]
If the assumptions given in Remark 3.2 hold, then, instead of (4.73) we have
\[ \|\xi^N\|^2 \leq (C_2^\alpha)^2 \left( \|\xi\|^2_{L^2(0,N-1;L^2)} + \Delta x^3 \right) + (C_3)^2(TE)^2 + (C_5)^2\Delta x^4. \]
Hence we obtain the following.

**Theorem 4.3** Let the hypothesis of Theorem 4.1 hold. Assume \( f' \) is Lipschitz continuous, \( \alpha_{1,j} \equiv 1 \), and the limited slope \( \delta S_j^n = 0 \) for only a finite number of \( j \)'s where \( \frac{\partial a(x,t^n)}{\partial x} \neq 0 \), and furthermore this number is bounded for all \( n \) independent of \( \Delta x \) and \( \Delta t \). Then
\[ \|s - S\|_{L^\infty(L^2)} \leq C_\xi''(\Delta x^{\frac{1}{2}} + TE), \] (4.78)
where
\[ C_\xi'' = C(\sqrt{e(C_2^\alpha)^2}, C_3, C_5). \] (4.79)

In Appendix A of the thesis, we derive a bound for \( TE \). We now summarize the result.

**Theorem 4.4** Let \( TE \) be the truncation error given by (4.69). Let \( f, a, s^0, g_0, g_1 \), and \( s \) satisfy the following smoothness assumptions.

(i) \( f \in C^2(\mathbb{R}) \),

(ii) \( a \frac{\partial a}{\partial x} \in C^1(Q_T) \),

(iii) \( s^0 \in C^2(I) \cap H^3(I) \),

(iv) \( g_0, g_1 \in C^2(0,T) \),

(v) \( s \) and all first and second partials of \( s \) with respect to \( z \) and \( t \) exist everywhere in \( Q_T \) and are bounded, \( \frac{\partial^3 s}{\partial x^3} \) and \( \frac{\partial^3 s}{\partial x^2} \) exist and are in \( L^2(0,T;L^\infty(I)) \) and \( L^\infty(0,T;L^2(I)) \), respectively.
Assume $\Delta t^n = \mathcal{O}(\Delta x)$. Then

$$TE \leq C_{TE}(\Delta x^{\frac{3}{2}} + \Delta t),$$

(4.80)

where

$$C_{TE} = C(T, C'_A, C''_A, C''_{AL}, C_{DR}, C_{DL}),$$

(4.81)

with $C_T, C'_A, C''_A, C''_{AL}, C_{DR},$ and $C_{DL}$ constants given by (A.10), (A.27), (A.28), (A.31), (A.37), and (A.43), respectively.

Note that, since we only obtain a $\mathcal{O}(\Delta x + \Delta t)$ estimate for the error anyway, we could have reduced our smoothness assumptions on $s$ and obtained a $\mathcal{O}(\Delta x + \Delta t)$ truncation error estimate. To obtain this result we need only assume $s \in \mathcal{W}^2_\infty(Q_T)$.

The Corollaries below follow immediately from Theorems 4.1-4.4.

**Corollary 4.1** Let the hypotheses of Theorems 4.1 and 4.4 hold. Then

$$||s - S||_{L^\infty(L^2)} \leq C(C_T, C_{TE})(\Delta x + \Delta t).$$

(4.82)

**Corollary 4.2** Let the hypotheses of Theorems 4.2 and 4.4 hold. Then

$$||s - S||_{L^\infty(L^2)} \leq C(C'_T, C_{TE})(\Delta x^{\frac{3}{2}} + \Delta t).$$

(4.83)

**Corollary 4.3** Let the hypotheses of Theorems 4.3 and 4.4 hold. Then

$$||s - S||_{L^\infty(L^2)} \leq C(C''_T, C_{TE})(\Delta x^{\frac{3}{2}} + \Delta t).$$

(4.84)
Assume we enforce the left boundary condition (2.3) by setting $S_{0,L}^n = g_0^n$ instead of $g_0^{n+\frac{1}{2}}$. Then, by defining $s_{0,L}^n = g_0^n$ the only change in our argument above is the addition of a term involving $f(g_0^n) - f(g_0^{n+\frac{1}{2}})$ in $TE$. Since this term is $O(\Delta t)$, the global estimate for $TE$ given by (4.80) remains unchanged. Thus, we have the following result.

**Theorem 4.5** Let the hypotheses of Theorems 4.1 and 4.4 hold with the exception that $\Delta t^n$ is chosen as in Lemma 3.2. Define $S_{0,L}^n = g_0^n$ in the Godunov-mixed method. Then

$$||s - S||_{L^\infty(L^2)} \leq C(C_{\xi}, C_{TE})(\Delta x + \Delta t).$$

(4.85)

We also obtain estimates similar to those given in Corollaries 4.2 and 4.3 for this case when the proper assumptions on the slope-limiting terms are made.

Finally, we remark that the above estimates are easily extendable to positive $a = a(x,t,s)$. For this case we modify the $I_1$ and $I_2$ terms in (4.29) to be

$$I_1 = \sum_{n=0}^{N-1} \langle \xi^{n+1}, Z_t^{n+1} + (a^{n+1}(S^{n+1})Z_{\xi}^n) \rangle \Delta t^n$$

and

$$I_2 = \sum_{n=0}^{N-1} \sum_{j=1}^{J-1} [\xi_{j,L}^{n+1} A_j^n - (a^{n+1}(S^n) - a^{n+1}(S^{n+1})) s_{j+\frac{1}{2}}^{n+1} Z_j x \Delta x \Delta t^n,$$

where

$$s_{j+\frac{1}{2}}^{n+1} = \frac{s_{j}^{n+1} + s_{j+1}^{n+1}}{2}$$

and

$$s_{j+\frac{1}{2}}^{n+1} = \frac{s_{j}^{n+1} + s_{j+1}^{n+1}}{2}.$$
Moreover, we have an additional truncation error term involving
\[ a_{j+\frac{1}{2}}^{n+1}(\tilde{s}_{j+\frac{1}{2}}^{n+1}) - a_{j+\frac{1}{2}}^{n+1}(s(x_{j+\frac{1}{2}}, t^{n+1})). \]

Assuming \( a \) is Lipschitz in \( s \), we have that the above term is \( O(\Delta x^2) \), and we also obtain
\[
I_2 \leq C(\|\xi\|_{L^2(0,N;L^2)}^2 + \Delta x^2) + \frac{1}{8}\|\xi^N\|^2,\]

The remaining truncation error and initial error estimates are essentially unchanged. Thus, we have
\[
\|\xi^N\|^2 \leq C\|\xi\|_{L^2(0,N;L^2)}^2 + C(\Delta x^2 + \Delta t^2),
\]

and a \( O(\Delta x + \Delta t) \) \( L^2 \) estimate again follows by applying the discrete Gronwall Lemma.

### 4.2 \( L^\infty(L^1) \) estimate-Dirichlet problem

By modifying the argument given in Section 4.1, one can obtain an \( L^\infty(L^1) \) error estimate for the Dirichlet problem. While this estimate does not give a better rate of convergence, it does give a better constant. Here \( C \) depends on \( a_{\ast}^{-2} \) instead of \( e^{a_{\ast}^{-1}} \) as given in Theorem 4.1, where \( a_{\ast} \) is again the lower bound on \( a(x,t) \). Specifically, in this section we show
\[
\|S - s\|_{L^\infty(L^1)} \leq \frac{C}{a_{\ast}^2}(\Delta x + \Delta t).
\]

Thus, if the coefficients and boundary and initial data of the problem are such that \( s \) remains smooth as \( a_{\ast} \to 0 \), we may allow \( a_{\ast} \to 0 \) as \( \Delta x^\epsilon \), where \( 0 \leq \epsilon < \frac{1}{2} \).

In this estimate, we need the time-step to be quasi-uniform, in particular, we must have
\[
C^\ast \geq \frac{\Delta t^n}{\Delta t^{n+1}} \geq C_{\ast}.
\]  
(4.86)
One way of ensuring that (4.86) holds is to make one of the following choices. Either 
\[ S^{n+\frac{1}{2}}_{0,L} = g_0^{n+\frac{1}{2}} \] and \( \alpha_{L,1} = 0 \), or \( S^{n}_{0,L} = g_0^n \). Under either of these choices, we are assured by

Lemmas 3.1 and 3.2 that \( \Delta t^n \) can be chosen to be proportional to \( \Delta x \); i.e.

\[ \Delta t^n = C^n \Delta x, \]

where \( C^n \) is bounded below and above for all \( n \) independent of \( \Delta x \).

Recall from Section 4.1, that the error \( \xi = s - S \) satisfies

\[ \langle \xi^N, Z^N \rangle = \sum_{n=0}^{N-1} \langle \xi^{n+1}, Z_t^{n+1} + (a^{n+1} Z^n_x) \rangle \Delta t^n \]

\[ + \sum_{n=0}^{N-1} \langle \xi^n, \kappa A^n Z^n_x \rangle \Delta t^n \]

\[ + \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} B^n(Z^n) \Delta t^n \]

\[ + \langle \xi^0, Z^0 \rangle \]

(4.87)

where \( A^n \) is given by (4.25), \( \kappa \) by (4.24), \( B^n \) by (4.28), \( E^n \) by (4.3), and \( Z^n \in M_{0,1}^2(\delta_x) \) remains to be chosen.

Also, recall

\[ \xi^n_{j,L} = \xi^n_j + \frac{\Delta x}{2} (1 - h(s^n_j)) d s^n_j - \frac{\Delta x}{2} (1 - h(S^n_j)) d S^n_j \]

\[ \equiv \xi^n_j + \Delta x (\gamma(s^n_j) - \tilde{\gamma}(S^n_j)). \]

(4.88)

Thus, substituting (4.88) into (4.87) and manipulating we find that

\[ \langle \xi^N, Z^N \rangle = \sum_{n=0}^{N-1} \langle \xi^{n+1}, Z_t^{n+1} + \kappa A^n Z^n_x + (a^{n+1} Z^n_x) \rangle \Delta t^n \]

\[ + \sum_{n=0}^{N-1} \langle \xi^n - \xi^{n+1} + \Delta x (\gamma(s^n) - \tilde{\gamma}(S^n)), \kappa A^n Z^n_x \rangle \Delta t^n \]

\[ + \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} B^n(Z^n) \Delta t^n \]

\[ + \langle \xi^0, Z^0 \rangle \]

\[ \equiv I_1 + I_2 + I_3 + I_4 + I_5. \]

(4.89)
Choice of $Z^n$. In this case we choose $Z$ to be an implicit upwinding-mixed method approximation to $z(x,t)$ satisfying

$$
\frac{\partial z}{\partial t} + p(x,t) \frac{\partial z}{\partial x} + \frac{\partial}{\partial x}(\bar{a}(x,t) \frac{\partial z}{\partial x}) = 0, \quad (x,t) \in I \times [0,t^N),
$$

(4.90)

$$
z(x,t^N) = \text{sgn } \xi_j^N, \quad x \in B_j,
$$

(4.91)

$$
z(0,t) = z(1,t) = 0, \quad 0 \leq t < t^N,
$$

(4.92)

where $p$ is a function satisfying $p(x_{j+\frac{1}{2}},t^n) = \kappa_j A_j^n$, $j = 1, \ldots, J - 1$, and $\bar{a}(x,t)$ satisfies $\bar{a}(x,t) = a(x,t + \Delta t^n)$, $t^n \leq t < t^{n+1}$.

Hence $Z$ satisfies

$$
(Z_t^{n+1} + \kappa A^n Z_x^n + (a^{n+1} Z_x^n)_x, v) = 0, \quad v \in \mathcal{M}_{-1}^0(\delta_x),
$$

(4.93)

$$
Z_j^N = \text{sgn } \xi_j^N, \quad j = 1, \ldots, J - 1,
$$

(4.94)

$$
Z_0^n = -Z_1^n, \quad Z_j^n = -Z_{j-1}^n, \quad n = 0, \ldots, N.
$$

(4.95)

We now have two lemmas concerning $Z$.

**Lemma 4.2** Let $Z$ satisfy (4.93)-(4.95), then

$$
||Z||_{L^\infty(0,N; L^\infty)} \leq 1.
$$

(4.96)

**Proof.** As we did in the proof of Lemma 4.1, let $Y^n = Z^{N-n}$, $\bar{a}^n = \bar{a}^{N-n}$, $\Delta t^n = \Delta t^{N-n}$, and let $\bar{A}^n = \kappa A^{N-n}$. Then $Y$ satisfies

$$
(Y_t^{n+1} - \bar{A}^{n+1} Y_x^{n+1} - (\bar{a}^{n+1} Y_x^{n+1})_x, v) = 0, \quad v \in \mathcal{M}_{-1}^0(\delta_x),
$$

(4.97)

$$
Y_j^0 = \text{sgn } \xi_j^N, \quad j = 1, \ldots, J - 1,
$$

(4.98)

$$
Y_0^n = -Y_1^n, \quad Y_j^n = -Y_{j-1}^n, \quad n = 0, \ldots, N.
$$

(4.99)
Choosing \( v = 1 \) on \( B_j \) and 0 elsewhere we have, for \( j = 1, \ldots, J - 1, \)

\[
Y^{n+1}_{j+\frac{1}{2}} - \bar{A}^{n+1}_{j+\frac{1}{2}} Y^{n+1}_{j+\frac{1}{2}} - (\bar{a}^{n+1}_{j+\frac{1}{2}} Y^{n+1}_{j+\frac{1}{2}})_x = 0.
\]

Let \( \beta_j = \bar{A}^{n+1}_{j+\frac{1}{2}} \Delta_{x_{j+\frac{1}{2}}}, \sigma_j = \bar{a}^{n+1}_{j+\frac{1}{2}} \Delta_{x_{j+\frac{1}{2}}}, \) then \( \beta_j \geq 0, \sigma_j > 0, \) and

\[
(1 + \beta_j + \sigma_j + \sigma_{j-1}) Y^{n+1}_j - (\beta_j + \sigma_j) Y^{n+1}_{j+1} - \sigma_{j-1} Y^{n+1}_{j-1} = Y^n_j.
\]

Assume

\[
|Y^{n+1}_j| = \max_{1 \leq j \leq J-1} |Y^{n+1}_j|,
\]

for some \( 1 \leq j \leq J - 1. \) Then, since \( |Y^{n+1}_0| = |Y^{n+1}_1| \) and \( |Y^{n+1}_J| = |Y^{n+1}_{J-1}| \) by (4.99),

\[
|Y^{n+1}_j| = \max_{0 \leq j \leq J} |Y^{n+1}_j|,
\]

and

\[
|Y^{n+1}_j| \leq |(\beta_j + \sigma_j) Y^{n+1}_{j+1} + (1 + \beta_j + \sigma_j + \sigma_{j-1}) Y^{n+1}_j - \sigma_{j-1} Y^{n+1}_{j-1}|
\]

\[
\leq |(\beta_j + \sigma_j) Y^{n+1}_{j+1} + (1 + \beta_j + \sigma_j + \sigma_{j-1}) Y^{n+1}_j - \sigma_{j-1} Y^{n+1}_{j-1}|
\]

\[
= |Y^n_j|.
\]

Here we have used the elementary inequality \( ||a| - |b| - |c|| \leq |a - b - c| \).

Thus

\[
||Y^{n+1}||_{L^\infty} = |Y^{n+1}_j| \leq |Y^n_j| \leq ||Y^n||_{L^\infty}.
\]

The lemma now follows by applying induction on \( n \) and using (4.98).

**Lemma 4.3** Let \( Z \) satisfy (4.93)-(4.95), then

\[
||Z||_{L^2(0,N-1; L^2)} + ||Z||_{L^2(0,N-1; L^2)} \leq \frac{CC_{2,3}}{a_*},
\]

(4.100)

where \( C_{2,3} \) is a bound for \( ||A||_{L^\infty(L^\infty)} \), see (4.53).
Proof. Let $Y^n$, $A^n$, $\bar{a}^n$, and $\Delta t^n$ be defined as in the proof of Lemma 4.2. Let $v = Y^{n+1}$ in (4.97). Then summing by parts, using (4.99), and applying the inequality $b(b - c) \geq \frac{1}{2}(b^2 - c^2)$ we have

$$
\frac{||Y^{n+1}||^2 - ||Y^n||^2}{2\Delta t^{n+1}} + \frac{a_*}{2} \left[ ||Y_x^{n+1}||^2 + ||Y_x^{n+1}||^2 \right]
$$

$$
\leq \langle A^{n+1} Y_x^{n+1} , Y^{n+1} \rangle
$$

$$
\leq \frac{C}{a_*} ||A^{n+1}||^2 ||Y^{n+1}||^2 + \frac{a_*}{4} ||Y_x^{n+1}||^2
$$

$$
\leq \frac{CC_{2,3}^2}{a_*} + \frac{a_*}{4} ||Y_x^{n+1}||^2,
$$

where we have used Lemma 4.2. Hiding the $\frac{a_*}{4} ||Y_x^{n+1}||^2$ term, multiplying above by $\Delta t^{n+1}$ and summing on $n$ we obtain

$$
||Y^M||^2 + \frac{a_*}{2} \sum_{n=0}^{M-1} \left( ||Y_x^{n+1}||^2 + ||Y_x^{n+1}||^2 \right) \Delta t^{n+1}
$$

$$
\leq ||Y^0||^2 + \frac{CC_{2,3}^2}{a_*}
$$

$$
\leq ||Y^0||^2 + \frac{CC_{2,3}}{a_*}
$$

$$
\leq \frac{CC_{2,3}^2}{a_*} + 1
$$

where $M \leq N$ is arbitrary. Recalling the definition of $Y^n$, the proof is complete.

Returning now to (4.89), we have, by (4.93)-(4.95),

$$
||\xi^N||_{L^1} = I_2 + I_3 + I_5.
$$

(4.101)

Before estimating $I_2$, we derive crude bounds for $||\xi^{n+1}_x||_{L^2(L^2)}$, $||\xi_x||_{L^2(L^2)}$, and $||\xi_x||_{L^2(L^2)}$. These bounds are summarized in the following lemma.

**Lemma 4.4** Let the regularity assumptions on $s$ and the data given in Theorem 4.4 hold. Assume $0 < a_* \leq a(x,t) \leq a^*$, $a$ is Lipschitz with respect to $t$ with Lipschitz constant $L_t$, and assume (4.86) holds. Then

$$
||\xi_x||_{L^2(L^2)} + ||\xi_x||_{L^2(L^2)} \leq \frac{C_{\xi_x}}{a_*},
$$

(4.102)
and
\[ ||\xi_t^{n+1}||_{L^2(L^2)} \leq \frac{C_\xi_t}{a_*}, \tag{4.103} \]
where
\[ C_\xi = C(||s_0||_{L^\infty(I)}, ||g_0||_{L^\infty(0,T)}, ||g_1||_{L^\infty(0,T)}, C_{TE}, C_{2,3}), \tag{4.104} \]
and
\[ C_\xi_t = C(C_\xi, a_*, ||\frac{\partial s}{\partial x}||_{L^\infty(Q_T)}, ||g_0||_{L^\infty(0,T)}, ||\frac{ds_0}{dx}||_{L^\infty(I)}, t), \tag{4.105} \]
with \( C_{TE} \) given in Theorem 4.4.

**Proof.** Recall that for \( w \in \mathcal{M}_{-1}^0(\delta_x) \), the error \( \xi \) satisfies
\[ \langle \xi_t^{n+1} + (f(s_L^n) - f(S_L^n))_\delta - (a\xi_t^{n+1})_\delta, w \rangle = \langle E^n, w \rangle, \tag{4.106} \]
and
\[ \xi^0_{0,L} = 0, \]
\[ \xi^{n+1}_0 = -\xi^{n+1}_1, \]
\[ \xi^{n+1}_j = -\xi^{n+1}_{j-1}. \]

Let \( w = \xi^{n+1} \) above, then summing by parts and using the inequality \( b(b-c) \geq \frac{1}{2}(b^2-c^2) \)
we have
\[ \frac{||\xi^{n+1}||^2 - ||\xi^n||^2}{2\Delta t^n} + \frac{a_*}{2} (||\xi^{n+1}||^2 + ||\xi^{n+1}_\delta||^2) \]
\[ \leq -\langle (f(s_L^n) - f(S_L^n))_\delta, \xi^{n+1} \rangle + \langle E^n, \xi^{n+1} \rangle. \tag{4.107} \]

Summing by parts as in (4.22) we find
\[ -\langle (f(s_L^n) - f(S_L^n))_\delta, \xi^{n+1} \rangle = \langle \kappa A^n \xi^n_L, \xi^{n+1} \rangle \]
\[ \leq \frac{CC_{2,3}^2}{a_*} ||\xi^n_L||^2_{L^\infty} + \frac{a_*}{4} ||\xi^{n+1}||^2. \tag{4.108} \]
By Lemma 3.1, (3.20)-(3.21), (2.49), (4.10)-(4.11), (4.13)-(4.15), and maximum principles for $s$, we have

$$
|\xi_{i,j}^n| \leq |S_{i,j}^n| + |s^n_{i,j}| \\
\leq \|S^n\|_{L^\infty} + \|s^n\|_{L^\infty} + C\Delta x |s^n_j| \\
\leq C \left( \|s^0\|_{L^\infty(0,T)} + \|g_0\|_{L^\infty(0,T)} + \|g_1\|_{L^\infty(0,T)} \right) \\
\equiv C_\infty. \quad (4.109)
$$

Thus, by (4.108) and (4.109),

$$
-(f(s^n_x) - f(s_x^n))_x, \xi^{n+1} \leq \frac{CC_2^2 C_\infty^2}{a_*} + \frac{a_*}{4} \|\xi^{n+1}\|^2. \quad (4.110)
$$

Next, recall from Theorem 4.4 that the truncation error term $TE$ given by (4.69) satisfies

$$
\|TE\|_{L^2(L^2)} \leq C_{TE}(\Delta x^{\frac{3}{2}} + \Delta t).
$$

Also, recall from (4.3) that

$$
E^n_T = E^n_{T,j} + (E^n_{A,j})_x + (E^n_{D,j})_x,
$$

where $E_T$, $E_A$, and $E_D$ are given by (4.4), (4.5), and (4.6), respectively. Hence for any $M \leq N^*$ we have

$$
\sum_{n=0}^M \langle E^n, \xi^{n+1} \rangle \Delta t^n \leq \|E\|_{L^1(0,M;L^1)} \|\xi\|_{L^\infty(L^\infty)} \\
\leq C_\infty \|E\|_{L^2(L^2)} \\
\leq C_\infty \left( \|E_T\|_{L^2(L^2)} + \|(E_A)_x\|_{L^2(L^2)} + \|(E_D)_x\|_{L^2(L^2)} \right) \\
\leq \frac{C_\infty}{\Delta x} \|TE\|_{L^2(L^2)} \\
\leq CC_\infty C_{TE} \\
\equiv C_E, \quad (4.111)
$$
since $\Delta t = O(\Delta x)$.

Applying (4.110) in (4.107), hiding the $a_n^\epsilon ||\xi^n_{t+1}||^2$ term on the left, multiplying by $\Delta t^n$, summing on $n$, and applying (4.111) we obtain

$$
||\xi^M||^2 + a_* \sum_{n=0}^{M-1} \left[ ||\xi^{n+1}||^2 + ||\xi^{n+1}||^2 \right] \Delta t^n
\leq \frac{CC_2^2C_\epsilon^2}{\epsilon^2} + C_E + ||\xi^0||^2.
$$

(4.112)

By the quasi-uniformity assumption on $\Delta t^n$, (4.86), we have

$$
\sum_{n=0}^{M-1} \left[ ||\xi^{n+1}||^2 + ||\xi_{t+1}||^2 \right] \Delta t^n = \sum_{n=0}^{M-1} \left[ ||\xi^{n+1}||^2 + ||\xi_{t+1}||^2 \right] \frac{\Delta t^n}{\Delta t^{n+1}} \Delta t^{n+1}
\geq C \left[ ||\xi||_{L^2(1,M;L^2)} + ||\xi||_{L^2(1,M;L^2)} \right].
$$

Furthermore, by the regularity of $s^0$ and the compatibility between $s^0$ and $g_0$ and $g_1$, we have

$$
||\xi^0|| + ||\xi^0|| \leq C_{TE}\Delta x.
$$

Thus from (4.112) we obtain

$$
||\xi||_{L^2(0,M;L^2)}^2 + ||\xi||_{L^2(0,M;L^2)}^2 \leq \frac{C_{\xi}}{a_*^2}.
$$

Since $M$ is arbitrary, this proves (4.102).

To derive (4.103), let $w = \xi^{n+1}_{t}$ in (4.106), then again summing by parts and using the inequality $b(b - c) \geq \frac{1}{2}(b^2 - c^2)$ we obtain

$$
||\xi^{n+1}_{t}||^2 + \frac{1}{4} \left[ \sum_{j=1}^{J-1} \left( a_{j+1/2}^{n+1} ||\xi^{n+1}_{j,x}||^2 \right) \Delta x + \sum_{j=1}^{J-1} \left( a_{j-1/2}^{n+1} ||\xi^{n+1}_{j,x}||^2 \right) \Delta x \right]
\leq -\langle (f(s^n_{L}))_{x} , \xi^{n+1}_{t} \rangle + \langle E^n , \xi^{n+1}_{t} \rangle
+ \frac{1}{4} \left[ \sum_{j=1}^{J-1} [a_{j+1/2}^{n+1} ||\xi^n_{j,x}||^2 + a_{j-1/2}^{n+1} ||\xi^n_{j,x}||^2] \Delta x \right]
\leq C||f(s^n_{L}) - f(s^n_{L})||^2 + C||E^n||^2 + \frac{1}{2} ||\xi^{n+1}_{t}||^2
+C(L,t)(||\xi^n_{t,x}||^2 + ||\xi^n_{t,x}||^2).
$$

(4.113)
Consider

\[ \| (f(s^n_L) - f(S^n_L)) x \| \leq C \left( \| (f(s^0_L)) x \| ^2 + \| (f(S^n_L)) x \| ^2 \right). \]  

(4.114)

Assume for the moment that \( S^n_{0,L} = g_0^{n+\frac{1}{2}} \), then,

\[ \frac{f(s^n_{j,L}) - f(s^n_{j-1,L})}{\Delta x} = f'(\tilde{s}^n_j) \frac{s^n_{j,L} - s^n_{j-1,L}}{\Delta x} \]

\[ = \begin{cases} 
  f'(\tilde{s}^n_j) \left( s^n_{j,a} + \frac{1}{2} (1 - h(s^n_j)) \delta s^n_j \right) \\
  \quad - \frac{1}{2} (1 - h(s^n_{j-1})) \delta s^n_{j-1}, \quad j = 2, \ldots, J - 1, \\
  f'(\tilde{s}^n_1) \left( (s^n_1 - g_0^{n+\frac{1}{2}}) / \Delta x \right) \\
  \quad + \frac{1}{2} (1 - h(s^n_j)) \delta s^n_1, \quad j = 1,
\end{cases} \]

where \( \tilde{s}^n_j \) is some point between \( s^n_{j,L} \) and \( s^n_{j-1,L} \). By the regularity assumptions on \( s \) and \( s^0 \), we have for \( j = 2, \ldots, J - 1 \) and \( n \geq 0 \),

\[ \| (f(s^n_{j,L})) x \| \leq C C_{2,3} \left( \| \frac{\partial s}{\partial x} \|_{L^\infty(Q_T)} + \| \frac{d s^0}{d x} \|_{L^\infty(I)} \right) \]

\[ \leq C. \]

Furthermore, by the regularity assumptions on \( s, s^0, \) and \( g_0, \) and the fact that \( \Delta t^n = O(\Delta x) \), we have

\[ \| (f(s^n_{1,L})) x \| \leq C C_{2,3} \left( \| \frac{\partial s}{\partial x} \|_{L^\infty(Q_T)} + \| g_0 \|_{L^\infty(0,T)} + \| \frac{d s^0}{d x} \|_{L^\infty(I)} \right) \]

\[ \leq C. \]

Thus

\[ \| (f(s^n_L)) x \| ^2 \leq C C_{2,3}^2 \left( \| \frac{\partial s}{\partial x} \|_{L^\infty(Q_T)}^2 + \| g_0 \|_{L^\infty(0,T)}^2 + \| \frac{d s^0}{d x} \|_{L^\infty(I)}^2 \right) \]

\[ \equiv C^2_{s^t}. \]  

(4.115)

Next, consider

\[ \frac{f(S^n_{j,L}) - f(S^n_{j-1,L})}{\Delta x} = f'(\tilde{s}^n_j) \frac{S^n_{j,L} - S^n_{j-1,L}}{\Delta x} \]
\[
\begin{align*}
\{ f'(\tilde{S}_j^n)(S_{j,x}^n + \frac{1}{2}(1 - h(S_j^n))\delta S_j^n & \notag \\
- \frac{1}{2}(1 - h(S_{j-1}^n))\delta S_{j-1}^n), & \quad j = 2, \ldots, J - 1, \\
\frac{f'(\tilde{S}_1^n)((S_1^n - g_0^{n+\frac{1}{2}})/\Delta x & 
otag \\
+ \frac{1}{2}(1 - h(S_1^n))\delta S_1^n), & \quad j = 1, 
\}
\end{align*}
\]

where \( \tilde{S}_j^n \) is a point between \( S_{j,L}^n \) and \( S_{j-1,L}^n \). For \( j = 2, \ldots, J - 1, \) and \( n \geq 1 \) we have by the slope-limiting procedure (2.25) and (2.33),

\[
(f(S_{j,L}^n))_x = f'(\tilde{S}_j^n)(S_{j,x}^n + \frac{\tilde{\omega}_j^n}{2}(1 - h(S_j^n))S_{j,x}^n \\
- \frac{\tilde{\omega}_{j-1}^n}{2}(1 - h(S_{j-1}^n))S_{j,x}^n),
\]

where \( 0 \leq \tilde{\omega}_j^n, \tilde{\omega}_{j-1}^n \leq 2 \) satisfy

\[
\delta S_j^n = \tilde{\omega}_j^n S_{j,x}^n, \\
\delta S_{j-1}^n = \tilde{\omega}_{j-1}^n S_{j,x}^n.
\]

Thus

\[
|f(S_{j,L}^n))_x| \leq CC_{2,3} \left| S_{j,x}^n \right| \leq CC_{2,3} \left( \left| s_{j,x}^n \right| + \left| \xi_j^n \right| \right). \tag{4.116}
\]

For \( j = 1 \), we have

\[
(f(S_{1,L}^n))_x = f'(\tilde{S}_1^n) \left( \frac{S_1^n - g_0^{n+\frac{1}{2}}}{\Delta x} + \frac{\tilde{\omega}_1^n}{2}(1 - h(S_1^n))S_{2,x}^n \right) \\
= f'(\tilde{S}_1^n) \left( \frac{s_0^n - g_0^{n+\frac{1}{2}} - \xi_1^n}{\Delta x} + \frac{\tilde{\omega}_1^n}{2}(1 - h(S_1^n))S_{2,x}^n \right) \\
= f'(\tilde{S}_1^n) \left( \frac{s_0^n - g_0^{n+\frac{1}{2}}}{\Delta x} + \frac{\Delta t^n g_0^n - g_0^{n+\frac{1}{2}}}{\Delta x} - \frac{1}{2} \frac{\xi_1^n - \xi_0^n}{\Delta x} \\
+ \frac{\tilde{\omega}_1^n}{2}(1 - h(S_1^n))(s_{2,x}^n - \xi_{2,x}^n) \right),
\]

where in the third term above we used the fact that \( \xi_0^n = -\xi_1^n \). Thus

\[
|f(S_{1,L}^n))_x| \leq CC_{2,3} \left( \left| \frac{\partial s}{\partial x} \right|_{L^\infty(Q_T)} + \left| g_0' \right|_{L^\infty(0,T)} + \left| \xi_{1,x}^n \right| + \left| \xi_{2,x}^n \right| \right). \tag{4.117}
\]
by the regularity assumption on \( s \) and \( g_0 \), and since \( \Delta t^n = O(\Delta x) \). Hence, by (4.116) and (4.117), for \( n \geq 1 \),

\[
\| (f(S^n_x))^B \|_2^2 \leq C C_{2,3}^2 \left( \| \frac{\partial s}{\partial x} \|_{C^\infty(Q_T)}^2 + \| \xi^n_x \|^2 + \| g_0 \|_{L^\infty(0,T)}^2 \right). \tag{4.118}
\]

For \( n = 0 \), following the argument given above, we have, by the smoothness and consistency assumptions on \( s^0 \) and \( g_0 \),

\[
\| (f(S^n_x))^B \|_2^2 \leq C C_{2,3}^2 \left( \| \frac{d s^0}{d x} \|_{L^\infty(I)}^2 + \| g_0 \|_{L^\infty(0,T)}^2 \right) \leq C. \tag{4.119}
\]

The bounds (4.115) and (4.118)-(4.119) are valid for \( S_{0,L}^n = g_0^{n+\frac{1}{2}} \). They are even easier to obtain when \( S_{0,L}^n = g_0^{n} \). Thus, they hold in either case.

Now multiplying (4.113) by \( \Delta t^n \) and summing on \( n \) we obtain

\[
\sum_{n=0}^{M-1} \| \xi^{n+1}_x \|^2 \Delta t^n + a_* \left( \| \xi^n_x \|^2 + \| \xi^n_M \|^2 \right)
\]
\[
\leq C \sum_{n=0}^{M-1} \left( \| (f(s^n_x))^B \|_2^2 + \| (f(S^n_x))^B \|_2^2 \right) \Delta t^n + C \| E \|_{L^2(L^2)}^2
\]
\[
+ a^* \left( \| \xi^n_2 \|^2 + \| \xi^n_3 \|^2 \right) + C(L) \left( \| \xi^n_x \|_{L^2(L^2)}^2 + \| \xi^n_2 \|_{L^2(L^2)}^2 \right)
\]
\[
\leq C \left( \| \xi^n_2 \|_{L^2(L^2)}^2 + \| \xi^n_3 \|_{L^2(L^2)}^2 \right) + C_{2*}^2 + C_{TE}^2 + C_{TE} a^* \Delta x^2
\]
\[
\leq \frac{C \xi_{\perp}}{a^*}
\]

by (4.114)-(4.115), (4.118)-(4.119), (4.111), and (4.102). Since \( M \) above is arbitrary, this proves (4.103).

**Estimate of \( I_2 \).** With the above bounds, we can now estimate \( I_2 \). Recall

\[
I_2 = \sum_{n=0}^{N-1} (\xi^n - \xi^{n+1} + \Delta x \left( \gamma(s^n) - \hat{\gamma}(S^n) \right), \kappa A^a Z^a_x) \Delta t^n
\]
\[
\leq C_{2,3} \left( \Delta t \| \xi^{n+1}_x \|_{L^2(L^2)} + \Delta x \| \gamma(s) - \hat{\gamma}(S) \|_{L^2(L^2)} \right)
\]
\[
\times \| Z_x \|_{L^2(L^2)}. \tag{4.120}
\]
Consider the second term above. We have

\[
\gamma(s^n_j) - \gamma(S^n_j) = \frac{1}{2}(1 - h(s^n_j))\delta s^n_j - \frac{1}{2}(1 - h(S^n_j))\delta S^n_j \\
= \frac{1}{2} \omega^n_j (1 - h(S^n_j)) (\delta s^n_j - \delta S^n_j) + \frac{1}{2} \left[ h(S^n_j) - 1 \right] \omega^n_j + (1 - h(s^n_j)) \delta s^n_j ,
\]

(4.121)

where \( \omega^n_j \) is given by (4.46). By the definition of \( \delta s^n_j \), i.e. (4.13)-(4.15), and (2.24), (2.29), and (2.33), we have

\[
|\delta s^n_j - \delta S^n_j| \leq C \left( |\xi^n_{j,x}| + |\xi^n_{j,x}| \right) .
\]

Thus,

\[
||\gamma(s) - \gamma(S)||_{L^2(L^2)} \leq C \left( ||\xi_x||_{L^2(L^2)} + ||\xi_x||_{L^2(L^2)} + ||\delta s||_{L^2(L^2)} \right) \\
\leq \frac{C_\xi}{a^*_e} + C_{2,1} ,
\]

(4.122)

by (4.102), where \( C_{2,1} \) is given by (4.51).

Hence, by (4.120), (4.122), (4.103), and Lemmas 4.3 and 4.4,

\[
I_2 \leq C_2 (\Delta t + \Delta x) .
\]

(4.123)

where

\[
C_2 = a^*_e C(C_{\xi_x}, C_{\xi_t}, C_{2,1}, C_{2,3}) .
\]

(4.124)

Estimate of \( I_3 \). For the estimate of \( I_3 \), we can do one of two things. Recall

\[
I_3 = \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n .
\]

One approach is given in Section 4.1; by this approach we sum by parts to obtain
\[ I_3 \leq \frac{C}{a_*} \left[ \| E_A' \|_{L^2(L^2)} + \| E_{AR}' \|_{L^2(L^2)} + \| E_{AR}'' \|_{L^2(L^2)} \\
+ \| E_{DL} \|_{L^2(L^2)} + \| E_{DR} \|_{L^2(L^2)} \right] + \| E_T \|_{L^2(L^2)} \\
+ C \sum_{n=0}^{N-1} |f(s_{0,L}^n) - f(s_{\frac{n+\frac{1}{2}}{2}}^n)| \Delta t^n \\
= a_*^{-1} C_{TE} T E \\
\equiv C_3 T E, \quad (4.125) \]

where \( E_A', E_{AR}', E_{DL}, \) and \( E_{DR}, \) and \( E_T \) are given by \((4.62), (4.65), (4.67), \) and \((4.4)\), respectively. The term

\[ |f(s_{0,L}^n) - f(s_{\frac{n+\frac{1}{2}}{2}}^n)| = \begin{cases} 
0, & \text{if } S_{0,L}^n = g_0^{n+\frac{1}{2}}, \\
O(\Delta t), & \text{if } S_{0,L}^n = g_0^n.
\end{cases} \]

Thus the estimate of \( TE \) given in Theorem 4.4 is valid.

The other approach is to apply Cauchy-Schwarz and Lemma 4.2 to obtain

\[ I_3 \leq \| E \|_{L^1(L^1)}. \quad (4.126) \]

If \( a_* = O(1) \), both approaches result in a \( O(\Delta x^{\frac{3}{2}} + \Delta t) \) estimate for the truncation error assuming enough smoothness; however the estimate \((4.125)\) requires one less degree of smoothness on \( s \) than \((4.126)\). If \( a_* = O(\Delta x^\epsilon) \), where \( \epsilon > 0 \), we lose a \( \Delta x^{-\epsilon} \) in \((4.125)\) but we will lose a \( \Delta x^{-2\epsilon} \) in the estimate of \( I_2 \) anyway, so the global convergence rate is unchanged. Thus, for these reasons and for simplicity, we choose the former approach over the latter.

**Estimate of \( I_5 \).** Finally, consider

\[ I_5 = \langle x^0, Z^0 \rangle \\
\leq \| x^0 \|_{L^1} \\
\leq C_5 \Delta x^2, \quad (4.127) \]
where
\[ C_5 = C \left( \| d^2 s^0 \|_{L^1(I)} \right). \] (4.128)

Combining (4.123), (4.125), (4.127) with (4.101) we obtain
\[ \| \xi^N \|_{L^1} \leq C_2(\Delta x + \Delta t) + C_3 T E + C_5 \Delta x^2. \] (4.129)

Thus we have the following theorems.

**Theorem 4.6** Let the hypotheses of Theorems 4.1 and 4.4 hold. Assume (4.86) holds and a is Lipschitz in t. Then
\[ \| s - S \|_{L^\infty(L^1)} \leq C_\xi (\Delta x + \Delta t), \] (4.130)
where
\[ C_\xi = C(C_2, C_3, C_5, C_{TE}). \] (4.131)

Thus \( \| s - S \|_{L^\infty(L^1)} = O(a_*^{-2}(\Delta x + \Delta t)). \)

**Theorem 4.7** Let the hypothesis of Theorem 4.6 hold. Assume \( a_* = O(\Delta x^\epsilon), \) where \( 0 \leq \epsilon < \frac{1}{2}. \) For each sufficiently small \( \Delta x, \) let \( s_{a_*} \) be the solution to (2.1)-(2.3), (2.6) with \( a_* \leq a(x, t). \) Assume \( s_{a_*} \to \bar{s} \) in \( L^1(0,T; L^1(I)) \) as \( \Delta x \to 0, \) where \( \bar{s} \) satisfies an equation of the form (2.1) with \( a(x, t) \geq 0. \) Moreover, assume for each \( \Delta x \) that \( s_{a_*} \) satisfies the smoothness assumptions of Theorem 4.4. Then the GMM approximation, \( S_{a_*} \) to \( s_{a_*} \) satisfies
\[ \| s_{a_*} - S_{a_*} \|_{L^\infty(L^1)} \leq C \left( \Delta x^{1-2\epsilon} + \frac{\Delta t}{\Delta x^{2\epsilon}} \right). \] (4.132)

Furthermore, \( \| S_{a_*} - \bar{s} \|_{L^1(0,T; L^1(Q_T))} \to 0 \) as \( \Delta x \to 0. \)
Proof. Estimate (4.132) follows immediately from (4.130) with $a_* = O(\Delta x^\epsilon)$. Also

$$
\| S_{a_*} - \bar{s} \|_{L^1(0,T; L^1(I))} = \int_0^T \int_I | S_{a_*} - \bar{s} | \, dx \, dt
$$

$$
= \sum_{n=0}^{N^*} \sum_{j=1}^{J-1} \int_{t^n}^{t^{n+1}} \int_{B_j} | S^n_{a_*}(x) - \bar{s}(x,t) | \, dx \, dt
$$

$$
\leq \sum_{n=0}^{N^*} \sum_{j=1}^{J-1} | S^n_{a_*}^{n,j} - s^n_{a_*}^{n,j} | \Delta x \Delta t^n
$$

$$
+ \sum_{n=0}^{N^*} \sum_{j=1}^{J-1} \int_{B_j} | (x - x_j) \delta S^n_j | \, dx \, \Delta t^n
$$

$$
+ \sum_{n=0}^{N^*} \sum_{j=1}^{J-1} \int_{t^n}^{t^{n+1}} \int_{B_j} | s^n_{a_*}^{n,j} - s_{a_*}(x,t) | \, dx \, dt
$$

$$
+ \| s_{a_*} - \bar{s} \|_{L^1(0,T; L^1(I))}
$$

$$
\leq C \Delta x^{1-2\epsilon} + C \Delta x \| \delta S \|_{L^2(L^2)}
$$

$$
+ C(\Delta x + \Delta t) + \| s_{a_*} - \bar{s} \|_{L^1(0,T; L^1(I))},
$$

since $\Delta t = O(\Delta x)$. Recalling from the proof of Lemma 4.4 that

$$
\| \delta S \|_{L^2(L^2)} \leq C + \| \xi \|_{L^2(L^2)} \leq \frac{C}{a_*},
$$

we have

$$
\| S_{a_*} - \bar{s} \|_{L^1(0,T; L^1(I))} \leq C \Delta x^{1-2\epsilon} + \| s_{a_*} - \bar{s} \|_{L^1(0,T; L^1(I))}.
$$

Hence

$$
\lim_{\Delta x \to 0} \| S_{a_*} - \bar{s} \|_{L^1(0,T; L^1(I))} = 0.
$$

4.3 Many advection steps per dispersion step

We now extend the methods of Section 4.1 to derive a $O(\Delta x + \Delta t)$ error estimate in $L^\infty(L^2)$ for the modification of the scheme given in Section 2.3 whereby one takes $K$ advection steps per dispersion step, where $K$ is bounded above independent of $\Delta t$. 
We prove this result for the case where $s$ satisfies (2.1) and (2.2) with Dirichlet boundary conditions at $x = 0$ and $x = 1$.

Assume the mesh is uniform, $K$ is fixed, and $\Delta t^{k,n}_s$ satisfies the hypothesis of Lemma 3.1. Thus the dispersion time-step $\Delta t^n$ is determined by

$$\Delta t^n = \sum_{k=0}^{K-1} \Delta t^{k,n}_s .$$

Integrating (2.1) over $B_j \times [t^n, t^{n+1}]$, multiplying by a constant $w_j \Delta x$, and summing on $j$ we obtain

$$(s_t^{n+1} + \sum_{k=0}^{K-1} \frac{\Delta t^{k,n}_s}{\Delta t^n} (f(s^{k,n}_L))_x - (a^{n+1} s^{n+1}_s)_x, w) = (E^n, w), \quad w \in \mathcal{M}^0_{-1}(\delta_x).$$  \hspace{1cm} (4.133)

In this case

$$E^n_j = E^n_{T,j} + \sum_{k=0}^{K-1} \frac{\Delta t^{k,n}_s}{\Delta t^n} (E^{k,n}_{A,j})_x + (E^n_{D,j})_x,$$ \hspace{1cm} (4.134)

where $E^n_{T,j}$ is given by (4.4), $E^n_{D,j}$ is given by (4.5), and

$$E^{k,n}_{A,j} = f(s^{k,n}_{j,L}) - \frac{1}{\Delta t^{k,n}_s} \int_{t^{k,n}}^{t^{k+1,n}} f(s(x_{j+\frac{1}{2}}, t))dt.$$ \hspace{1cm} (4.135)

Here we are using the following notation; for $n = 0, \ldots, N^*$, and $k = 0, \ldots, K - 1$,

$$s^{k,n}_j = s(x_j, t^{k,n}),$$ \hspace{1cm} (4.136)

$$s^n_j = s^0_j, \quad j = 1, \ldots, J - 1,$$ \hspace{1cm} (4.137)

where

$$t^{k,n} = t^n + \sum_{l=0}^{k-1} \Delta t^{l,n}_s .$$

We define

$$s^n_0 = 2g^n_0 - s^n_1, \quad s^n_j = 2g^n_j - s^n_{j-1} .$$ \hspace{1cm} (4.138)
Furthermore,

\[ s_{0,L}^{k,n} = g_0^{k+\frac{1}{2},n}, \]  

\[ s_{j,L}^{k,n} = s_{j,n}^{k,n} + \Delta x \frac{1}{2} (1 - h(s_{j,n}^{k,n})) \delta s_{j,n}^{k,n}, \quad j = 1, \ldots, J - 1, \]  

where

\[ h(s) = \frac{\Delta t_{\sigma,n}^{k,n}}{\Delta x} f'(s) \]  

and \( \delta s_{j,n}^{k,n} \) is given by (4.13)-(4.15).

Multiplying (2.55) by \( w_j \Delta x \) and summing on \( j \) we have that the GMM approximation, \( S \), to \( s \) satisfies

\[ \langle S_t^{n+1} + \sum_{k=0}^{K-1} \frac{\Delta t_{s,n}^{k,n}}{\Delta t^n} (f(S_{L}^{k,n}))- (a^{n+1}S_{\xi}^{n+1}) \rangle , w \rangle = 0, \]  

\[ w \in M_{-1}(\delta_{\xi}). \]  

Let

\[ \xi_{j,n}^{k,n} = s_{j,n}^{k,n} - S_{j,n}^{k,n}, \quad k = 0, \ldots, K - 1, \]  

\[ \xi_{j,n}^{n} = \xi_{j,0,n}, \]  

and

\[ \xi_{j,L}^{k,n} = s_{j,L}^{k,n} - S_{j,L}^{k,n}, \quad k = 0, \ldots, K - 1. \]  

Then, subtracting (4.142) from (4.133) we obtain

\[ \langle \xi_t^{n+1} + \sum_{k=0}^{K-1} \frac{\Delta t_{s,n}^{k,n}}{\Delta t^n} (f(s_{L}^{k,n}) - f(S_{L}^{k,n})) \rangle , w \rangle 
\[ - \langle (a^{n+1} \xi_{\xi}^{n+1}) \rangle , w \rangle = \langle E^n , w \rangle. \]  

Let \( \kappa \) be defined as in (4.24), and let

\[ A_{j,n}^{k,n} = \begin{cases} 
(f(s_{j,L}^{k,n}) - f(S_{j,L}^{k,n}))/\xi_{j,L}^{k,n}, & \text{if } \xi_{j,L}^{k,n} \neq 0, \\
0, & \text{otherwise},
\end{cases} \]  

(4.144)
for \( j = 0, \ldots, J - 1 \).

Summing by parts in (4.143), using the facts

\[
\xi^{k,n}_{0,L} = 0, \quad k = 0, \ldots, K - 1, \quad (4.145)
\]

\[
\xi^0 = -\xi^1, \quad \xi^n = -\xi^n_{J-1}, \quad (4.146)
\]

and letting \( w = Z^n \in M^0_{-1}(\delta_z) \), we obtain

\[
\langle \xi^{n+1}_t, Z^n \rangle - \langle \sum_{k=0}^{K-1} \frac{\Delta t^{k,n}}{\Delta t^n} \kappa A^{k,n} \xi^{k,n}_L, Z^n \rangle
\]

\[
\quad - \langle \xi^{n+1}, (a^{n+1} Z^n)_z \rangle = \langle E^n, Z^n \rangle + B^n(Z^n),
\]

where

\[
B^n(Z^n) = B^n_D(Z^n) - \sum_{k=0}^{K-1} \frac{\Delta t^{k,n}}{\Delta t^n} \kappa A^{k,n} \xi^{k,n}_{J-1,L} (Z^n_j + Z^n_{J-1}),
\]

with \( B^n_D(Z^n) \) given by (4.27).

Now multiplying by \( \Delta t^n \) above and summing on \( n \), and summing by parts on \( n \) we obtain

\[
\langle \xi^N, Z^N \rangle = \sum_{n=0}^{N-1} \langle \xi^{n+1}_t, Z^{n+1}_t \rangle \Delta t^n
\]

\[
\quad + \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \langle \xi^{k,n}_L, \kappa A^{k,n} Z^n_x \rangle \Delta t^{k,n}_s
\]

\[
\quad + \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} B^n(Z^n) \Delta t^n
\]

\[
\quad + \langle \xi^0, Z^0 \rangle
\]

\[
\equiv I_1 + I_2 + I_3 + I_4 + I_5. \quad (4.148)
\]

We now choose \( Z^n \) exactly as it was chosen in Section 4.1; i.e., by (4.33)-(4.35). Thus \( I_1 = I_4 = 0 \), Lemma 4.1 holds, and

\[
||\xi^N||^2 = I_2 + I_3 + I_5. \quad (4.149)
\]
Estimate of $I_2$. Recall

$$I_2 = \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \langle \kappa \mathbf{A}^{k,n} \mathbf{\xi}^{k,n}_L, \mathbf{Z}^n_{x} \rangle \Delta t^k,n$$

$$\leq \max_{j,k,n} |A_j^{k,n}| \left( \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \| \mathbf{\xi}^{k,n}_L \|^2 \Delta t^k,n \right)^{\frac{1}{2}} \| \mathbf{Z}^n \|_{L^2(0,N-1;L^2)}$$

$$\leq \frac{CC_{2,3}}{a_s} \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \| \mathbf{\xi}^{k,n}_L \|^2 \Delta t^k,n + \frac{1}{8} \| \mathbf{\xi}^N \|^2$$

(4.150)

by Cauchy-Schwarz and Lemma 4.1. Here $C_{2,3}$ is a bound for $A_j^{k,n}$.

Now consider

$$\sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \| \mathbf{\xi}^{k,n}_L \|^2 \Delta t^k,n = \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \sum_{j=1}^{J-1} \| \mathbf{\xi}^{k,n}_L \|^2 \Delta x \Delta t^k,n.$$

Recalling (4.49), and assuming $f'$ is Lipschitz continuous with Lipschitz constant $L_f$, we have

$$|\mathbf{\xi}^{k,n}_L| \leq C \left( |\mathbf{\xi}^{k,n}_{j+1} - |\mathbf{\xi}^{k,n}_j| + |\mathbf{\xi}^{k,n}_{j-1}| \right)$$

$$+ C \Delta x \left( 1 - \omega^k,n \right) |\mathbf{\xi}^{k,n}_L|,$$

where $\omega^k,n$ satisfies

$$\omega^k,n = \omega^k,n \delta S^k,n.$$

Thus

$$\| \mathbf{\xi}^{k,n}_L \|^2 \leq C_L^2 (\| \mathbf{\xi}^{k,n}_L \|^2 + \Delta x^2),$$

(4.151)

where

$$C_L = C(L_f, C_{2,1})$$

(4.152)

with $C_{2,1}$ given by (4.51).

By integrating (2.1) over the region $B_j \times [t^k,n, t^{k+1,n}]$, we have

$$s^{k+1,n}_j = s^{k,n}_j - \Delta t^k,n \Delta x \left[ f(s^{k,n}_{j-1,L}) - f(s^{k,n}_j) \right] + \Delta t^k,n \Delta s^{k,n}_j,$$

(4.153)
where

\[ \tilde{E}_{j}^{k,n} = E_{T,j}^{k,n} + (E_{A,j}^{k,n})_z + \tilde{E}_{D,j}^{k,n}, \]  

(4.154)

with \( E_{A,j}^{k,n} \) given by (4.135),

\[ E_{T,j}^{k,n} = \frac{S_{j}^{k+1,n} - S_{j}^{k,n}}{\Delta t_{s}^{k,n}} - \frac{1}{\Delta t_{s}^{k,n}} \left[ \frac{1}{\Delta x} \int_{B_{j}} s^{k+1,n}(x)dx - \frac{1}{\Delta x} \int_{B_{j}} s^{k,n}(x)dx \right] , \]

and

\[ \tilde{E}_{D,j}^{k,n} = \frac{1}{\Delta t_{s}^{k,n}} \int_{t_{i,k,n}}^{t_{i+1,k,n}} \left( \left( a\frac{\partial g}{\partial z} \right)(x_{j+\frac{1}{2}}, \cdot) \right)_z dt. \]

Furthermore, recall from (2.52) that,

\[ S_{j}^{k+1,n} = S_{j}^{k,n} - \frac{\Delta t_{s}^{k,n}}{\Delta z} [f(S_{j,L}^{k,n}) - f(S_{j-1,L}^{k,n})]. \]  

(4.155)

Hence, subtracting (4.155) from (4.153) we obtain

\[ \xi_{j}^{k+1,n} = \xi_{j}^{k,n} - \frac{\Delta t_{s}^{k,n}}{\Delta z} \left[ A_{j}^{k,n} \xi_{j,L}^{k,n} - A_{j-1}^{k,n} \xi_{j-1,L}^{k,n} \right] + \Delta t_{s}^{k,n} \tilde{E}_{j}^{k,n}. \]  

(4.156)

By (4.156) and the fact that \( \frac{\Delta t_{s}^{k,n}}{\Delta z} A_{j}^{k,n} \leq C \), we have

\[ ||\xi^{k+1,n}||^2 \leq C ||\xi^{k,n}||^2 + C ||\xi_{L}^{k,n}||^2 + C(\Delta t_{s}^{k,n})^2 ||\tilde{E}^{k,n}||^2. \]

Thus, from (4.151) we obtain

\[ ||\xi^{k+1,n}||^2 \leq C(C_{2}^2) \left( ||\xi^{k,n}||^2 + \Delta z^2 \right) + C(\Delta t_{s}^{k,n})^2 ||\tilde{E}^{k,n}||^2 \]

\[ \leq C(k,C_{2}^2) \left( ||\xi^{n}||^2 + \Delta z^2 \right) \]

\[ + C(k) \Delta t_{s} \sum_{l=0}^{k} ||\tilde{E}_{l}^{k,n}||^2 \Delta t_{s}^{l,n}, \]  

(4.157)

where

\[ \Delta t_{s} = \max_{k,n} \Delta t_{s}^{k,n}. \]  

(4.158)
Hence, by (4.151) and (4.157) we have
\[
\sum_{n=0}^{N-1} \sum_{k=0}^{K-1} ||\xi_L^{k,n}||^2 \Delta t_s^{k,n} \leq C(K, C_L^2) \left( \sum_{n=0}^{N-1} ||\xi^n||^2 \Delta t^n + \Delta z^2 \right) \\
+ C(K) \Delta t_s \Delta t \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} ||\tilde{E}_s^{k,n}||^2 \Delta t_s^{k,n}.
\]
Under the smoothness assumptions of Theorem 4.4, we have by (4.154),
\[
\sum_{n=0}^{N-1} \sum_{k=0}^{K-1} ||\tilde{E}_s^{k,n}||^2 \Delta t_s^{k,n} \leq C \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \left[ ||E_T^{k,n}||^2 + ||(E_A^{k,n})_\varepsilon||^2 \right] \Delta t_s^{k,n} \\
+ C \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} ||\tilde{E}_B^{k,n}||^2 \Delta t_s^{k,n} \\
\leq C(CTE)^2.
\]
Thus,
\[
\sum_{n=0}^{N-1} \sum_{k=0}^{K-1} ||\xi_L^{k,n}||^2 \Delta t_s^{k,n} \\
\leq C(K, C_L^2, C_T^2) \left( \sum_{n=0}^{N-1} ||\xi^n||^2 \Delta t^n + \Delta z^2 + \Delta t_s^2 \right). \tag{4.159}
\]
Hence, by (4.150) and (4.159),
\[
I_2 \leq (C_2)^2 \left( \sum_{n=0}^{N-1} ||\xi^n||^2 \Delta t^n + \Delta z^2 + \Delta t_s^2 \right) + \frac{1}{8} ||\xi^N||^2, \tag{4.160}
\]
where
\[
C_2 = a_s^{-\frac{1}{2}} C(K, L_f', C_{2,1}, C_{2,3}, C_{TE}). \tag{4.161}
\]

Estimate of $I_3$. Recall from (4.148), (4.134),
\[
I_3 = \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n \\
\leq \sum_{n=0}^{N-1} \langle E_T^n + \sum_{k=0}^{K-1} \frac{\Delta t_s^{k,n}}{\Delta t^n} (E_A^{k,n})_\varepsilon + (E_B^n)_\varepsilon, Z^n \rangle \Delta t^n \\
\leq \sum_{n=0}^{N-1} \langle E_T^n, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \langle (E_A^{k,n})_\varepsilon, Z^n \rangle \Delta t_s^{k,n} \\
+ \sum_{n=0}^{N-1} \langle (E_B^n)_\varepsilon, Z^n \rangle \Delta t^n \\
\equiv I'_3 + I''_3 + I'''_3. \tag{4.162}
\]
The estimates for $I_3'$ and $I_3'''$ are precisely the same as they were in Section 4.1; i.e.

$$I_3' \leq C \|E_T\|_{L^2(L^2)}^2 + \frac{1}{24}\|\xi^N\|^2, \quad (4.163)$$

and

$$I_3''' \leq \frac{C}{a_*} \left[\|E_{DR}\|_{L^2(L^2)}^2 + \|E_{DL}\|_{L^2(L^2)}^2\right] + \frac{1}{24}\|\xi^N\|^2, \quad (4.164)$$

where $E_{DR}$ and $E_{DL}$ are given by (4.67).

Following essentially the same argument given in Section 4.1 for $I_3''$, we find

$$I_3'' \leq \frac{C}{a_*} \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \left[\|E'_{A,k,n}\|^2 + \|E''_{AR,k,n}\|^2 + \|E''_{AL,k,n}\|\right] \Delta t^{k,n}_s \quad (4.165)$$

where

$$E'_{A,j,k,n} = f(s^{k,n}_j) - f(s^{k+\frac{1}{2},n}_{j+\frac{1}{2}}), \quad (4.166)$$

$$E''_{AR,j,k,n} = f(s^{k+\frac{1}{2},n}_{j+\frac{1}{2}}) - \frac{1}{\Delta t^{k,n}_s} \int_{s^{k+1,n}_j}^{s^{k+1,n}_{j+1}} f(s(x_{j+\frac{1}{2}},t))dt, \quad (4.167)$$

and

$$E''_{AL,j,k,n} = E''_{AR,j-1,k,n}. \quad (4.168)$$

Assuming the regularity assumptions on $s$, $f$, $a$, $g_0$, $g_1$, and $s^0$ given in Theorem 4.4 hold and assuming $\Delta t^{k,n}_s = O(\Delta x)$, we have

$$\|E_T\|_{L^2(L^2)} \leq C_T \Delta x^2, \quad (4.169)$$

$$\left(\sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \|E'_{A,k,n}\|^2 \Delta t^{k,n}_s\right)^{\frac{1}{2}} \leq C'_A(\Delta x^2 + \Delta t_s), \quad (4.170)$$

$$\left(\sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \left[\|E''_{AL,k,n}\|^2 + \|E''_{AR,k,n}\|\right] \Delta t^{k,n}_s\right)^{\frac{1}{2}} \leq C'_A \Delta t_s^2, \quad (4.171)$$
and
\[
\|E_{DR}\|_{L^2(L^2)} + \|E_{DL}\|_{L^2(L^2)} \leq C_D(\Delta x^{3/2} + \Delta t).
\] (4.172)

Here $C_T$ is given by (A.10), $C'_A$ is given by (A.27),
\[
C''_A = C(C''_A, C''_A),
\] (4.173)
where $C''_A$ and $C''_A$ are given by (A.28) and (A.31), respectively, and
\[
C_D = C_{DL} + C_{DR},
\] (4.174)
where $C_{DL}$ and $C_{DR}$ are given by (A.37) and (A.43), respectively.

Thus, by (4.162)-(4.174),
\[
I_3 \leq (C_3)^2(\Delta x^3 + \Delta t^2) + \frac{1}{8}\|\xi^N\|^2,
\] (4.175)
where
\[
C_3 = a_2^{-\frac{1}{2}} C(C_T, C'_A, C''_A, C_D).
\] (4.176)

**Estimate of $I_5$.** The estimate of $I_5$ remains unchanged from Section 4.1. Thus
\[
I_5 \leq (C_5)^2 \Delta x^4 + \frac{1}{8}\|\xi^N\|^2,
\] (4.177)
where
\[
C_5 = C(\|s^0\|_{\mathcal{H}^2(I)}).
\] (4.178)

Combining (4.149), (4.160), (4.175), and (4.177), we obtain
\[
\|\xi^N\|^2 \leq (C_2)^2 \left( \sum_{n=0}^{N-1} \|\xi^n\|^2 \Delta t^n + \Delta x^2 + \Delta t^2 \right) + (C_3)^2(\Delta x^3 + \Delta t^2) + (C_5)^2 \Delta x^4.
\] (4.179)

By applying Gronwall's lemma to (4.179) we obtain the following theorem.
Theorem 4.8 Let $S$ be the GMM approximation to $s$ satisfying (2.1)-(2.3), (2.6) as given in Section 2.3. Assume the mesh is uniform, $\Delta t_s^{k,n}$ satisfies the hypothesis of Lemma 3.1, and $\Delta t_s^{k,n} = O(\Delta x)$. Let the regularity assumptions on $s$, $f$, $a$, $g_0$, $g_1$, and $s^0$ given in Theorem 4.4 hold, and assume $f' \geq 0$, $0 < a_* \leq a(x,t)$. Then

$$\|s - S\|_{L^\infty(L^2)} \leq C_\xi (\Delta x + \Delta t),$$

(4.180)

where

$$C_\xi = C(\sqrt{C_2}, C_3, C_8).$$

(4.181)

Remark 3.5. Remarks 3.1 and 3.2 of Section 4.1 apply also to this scheme. Thus, we can obtain estimates for the error similar to those given in Corollaries 4.2 and 4.3. Furthermore, all of these estimates hold if we make the modification $S_{0,L}^{k,n} = g_0^{k,n}$. Thus, a theorem analogous to Theorem 4.5 also holds in this case. Moreover, the results of this section can be extended to the case $a = a(x,t,s)$.

4.4 $L^\infty(L^1)$ estimate: no-flow at $x = 1$

In this section we derive error estimates in $L^\infty(L^1)$ for the GMM approximation to $s$ satisfying (2.1)-(2.3), (2.7). Thus, $S$ satisfies (2.22) with boundary data given in Section 2.2. We consider estimates for both definitions of $S_{\gamma-1,L}^n$, (2.44) and (2.45). Under the latter assumption, with a slight modification to the CFL time-step restriction, we obtain a $O(\Delta x + \Delta t)$ estimate. Under the former assumption, our time-step restriction is unchanged but we lose a half-power of $\Delta x$, obtaining a $O(\Delta x + \Delta t + \Delta t \Delta x^{-\frac{1}{2}})$ estimate in $L^1$. Maximum principles for this case can be derived by combining the proofs of Lemma 3.1 and Lemma 3.3.
For simplicity assume uniform mesh. As in Section 4.2, we must have

\[ C^* \geq \frac{\Delta t^n}{\Delta t^{n+1}} \geq C_*. \quad (4.182) \]

Thus we assume either \( S_{0,L}^n = g_0^{n+\frac{1}{2}} \) and \( \alpha_{l,1} = 0 \), or \( S_{0,L}^n = g_0^n \) and apply Lemma 3.1 or Lemma 3.2. Since up to now we have always made the former choice for \( S_{0,L}^n \), we will make the latter choice in this section.

Integrating (2.1) again over the space-time domain \( B_j \times [t^n, t^{n+1}] \), we find that \( s \) satisfies, for all \( w \in \mathcal{M}_{-1}(\delta_x) \),

\[ (s_i^{n+1} + (f(s_i^n))_x - (s_i^{n+1} s_j^{n+1})_x, w) = (E^n, w), \quad (4.183) \]

where \( E^n_j \) is given by (4.3). Here for \( n = 0, \ldots, N^* \),

\[ s_j^n = s(x_j, t^n), \quad j = 1, \ldots, J - 1, \quad (4.184) \]

\[ s_0^n = 2g_0^n - s_1^n, \quad s_J^n = s_{J-1}^n, \quad (4.185) \]

\[ s_{0,L}^n = g_0^n, \quad (4.186) \]

\[ s_{j,L}^n = s_j^n + \frac{\Delta x}{2} (1 - h(s_j^n))\delta s_j^n, \quad j = 1, \ldots, J - 2, \quad (4.187) \]

\[ h(s) = \frac{\Delta t^n}{\Delta x} f'(s), \quad (4.188) \]

\[ \delta s_j^n = \frac{s_{j+1}^n - s_{j-1}^n}{2\Delta x}, \quad j = 2, \ldots, J - 2, \quad (4.189) \]

\[ \delta s_1^n = \frac{s_2^n + 3s_1^n - 4g_0^n}{3\Delta x}, \quad (4.190) \]

and

\[ \delta s_{J-1}^n = 0. \quad (4.191) \]

Moreover, if (2.44) holds, we define

\[ s_{j-1,L}^n = s_{j-1}^n, \quad (4.192) \]
while if (2.45) holds,

$$s_{j-1,L}^n = s_{j-1}^{n+1}. \quad (4.193)$$

The GMM approximation $S$ satisfies

$$\left( S_t^{n+1} + (f(S_L^n))_x - (a^{n+1}S_x^{n+1})_x, w \right) = 0, \quad w \in \mathcal{M}_{-1}(\delta_x). \quad (4.194)$$

Thus, defining $\xi$ and $\xi_L$ as before, subtracting (4.194) from (4.183) we obtain

$$\left( \xi_t^{n+1} + (A^n\xi_L^n)_x - (a^{n+1}\xi_x^{n+1})_x, w \right) = (E^n, w). \quad (4.195)$$

Here $A_j^n$ is again given by (4.25), and we note that

$$\xi_0^n = -\xi_1^n, \quad \xi_j^n = \xi_{j-1}^n, \quad \xi_{0,L}^n = 0, \quad (4.196) \quad (4.197)$$

and

$$\xi_{j-1,L}^n = \xi_{j-1}^n. \quad (4.198)$$

if (2.44) holds, and if (2.45) holds

$$\xi_{j-1,L}^n = \xi_{j-1}^{n+1}. \quad (4.199)$$

Summing by parts in (4.195), using (4.196)-(4.197) we find

$$\left( \xi_t^{n+1}, w \right) - \left( \xi_L^n, A^n w_x \right) - \left( \xi_x^{n+1} + (a^{n+1}w_x)_x, \left( E^n, w \right) + B^n(w), \right)$$

where

$$B^n(w) = a^{n+1}\xi_1^n \frac{w_1 + w_0}{\Delta x}$$

$$- \left( A_j^{n+1} \xi_{j-1,L}^n w_J + a^{n+1}_j \xi_{j-1}^{n+1} \frac{w_J - w_{j-1}}{\Delta x} \right). \quad (4.200)$$
Letting \( w = Z^n \in M_{-1}^{0}(\delta_x) \), where \( Z^n \) is chosen below, multiplying above by \( \Delta t^n \) and summing on \( n \) we obtain

\[
\langle \xi^N, Z^N \rangle = \sum_{n=0}^{N-1} \langle \xi^{n+1}, Z_t^{n+1} + (a^{n+1} Z_x^n)_{\xi} \rangle \Delta t^n
+ \sum_{n=0}^{N-1} \langle \xi^L_n, A^n Z_x^n \rangle \Delta t^n
+ \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} B^n(Z^n) \Delta t^n
+ \langle \xi^0, Z^0 \rangle.
\] (4.201)

Recalling that for \( j = 1, \ldots, J - 2 \),

\[
\xi_j^n = \xi_j^n + \frac{\Delta x}{2} (1 - h(s^n_j)) \delta s^n_j - \frac{\Delta x}{2} (1 - h(S^n_j)) \delta S^n_j
\equiv \xi_j^n + \Delta x (\gamma(s^n_j) - \gamma(S^n_j)),
\] (4.202)

and for \( j = J - 1 \), either (4.198) or (4.199) holds, we can rewrite (4.201) as

\[
\langle \xi^N, Z^N \rangle = \sum_{n=0}^{N-1} \langle \xi^{n+1}, Z_t^{n+1} + A^n Z_x^n + (a^{n+1} Z_x^n)_{\xi} \rangle \Delta t^n
+ \sum_{n=0}^{N-1} \langle \xi^n - \xi^{n+1} + \gamma(s^n) - \gamma(S^n), \kappa A^n Z_x^n \rangle \Delta t^n
+ \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} B^n(Z^n) \Delta t^n
+ \langle \xi^0, Z^0 \rangle.
\equiv I_1 + I_2 + I_3 + I_4 + I_5,
\] (4.203)

where \( \kappa_j = 1, j = 1, \ldots, J - 2 \), and \( \kappa_{J-1} = 1 \) if (4.198) holds, \( \kappa_{J-1} = 0 \) if (4.199) holds.

Choice of \( Z \). In this case, we choose \( Z \) to be an implicit upwinding-mixed method approximation to \( z \) satisfying

\[
\frac{\partial z}{\partial t} + p(x,t) \frac{\partial z}{\partial x} + \frac{\partial}{\partial x} \left( \tilde{a}(x,t) \frac{\partial z}{\partial x} \right) = 0, \quad (x,t) \in I \times [0,t^n),
\]

\[
z(x,t^N) = \text{sgn} \xi_j^N, \quad x \in B_j,
\]
\[ z(0, t) = 0, \]
\[ (p_z + a \frac{\partial z}{\partial x})(1, t) = 0, \quad 0 \leq t < t^N. \]

Here \( p \) and \( \tilde{a} \) are defined as before; i.e. \( p \) satisfies \( p(x_j + \frac{1}{2}, t^n) = A_j^n \), for \( j = 1, \ldots, J - 1 \), and \( \tilde{a}(x, t) = a(x, t + \Delta t^n), \quad t^n \leq t < t^{n+1} \).

Thus \( Z \) satisfies

\[ (Z_{i+1}^n + A^n Z^n_x + (a^{n+1} Z^n_x)_x, v) = 0, \quad v \in \mathcal{M}_0^0(\delta_x), \quad (4.204) \]
\[ Z_j^N = \text{sgn } \xi_j^N, \quad j = 1, \ldots, J - 1, \quad (4.205) \]
\[ Z_0^n = -Z_1^n, \quad (4.206) \]
\[ A_{j-1}^n Z_j^n + a^{n+1} \frac{Z_j^n - Z_{j-1}^n}{\Delta x} = 0, \quad n = 0, \ldots, N - 1. \quad (4.207) \]

We now demonstrate that \( Z \) is bounded. Let \( Y^n = Z^{N-n} \), \( \bar{A}^n = A^{N-n} \), \( \bar{a}^n = \bar{a}^{N-n} \), and \( \Delta t^n = \Delta t^{N-n} \). Then \( Y \) satisfies

\[ (Y_{i+1}^{n+1} - \bar{A}^{n+1} Y_x^{n+1} - (\bar{a}^{n+1} Y_x^{n+1})_x, v) = 0, \quad v \in \mathcal{M}_0^0(\delta_x), \quad (4.208) \]
\[ Y_j^0 = \text{sgn } \xi_j^N, \quad j = 1, \ldots, J - 1, \quad (4.209) \]
\[ Y_0^{n+1} = -Y_1^{n+1}, \quad (4.210) \]

and

\[ \bar{A}_{j-1}^{n+1} Y_j^{n+1} + \bar{a}_{j-\frac{1}{2}}^{n+1} \frac{Y_j^{n+1} - Y_{j-1}^{n+1}}{\Delta x} = 0, \quad n = 0, \ldots, N - 1. \quad (4.211) \]

Hence, for each \( j = 1, \ldots, J - 1 \),

\[ (1 + \beta_j + \sigma_j + \sigma_{j-1}) Y_{j+1}^{n+1} - (\beta_j + \sigma_j) Y_{j+1}^{n+1} - \sigma_{j-1} Y_{j-1}^{n+1} = Y_j^n, \quad (4.212) \]

where

\[ \beta_j = \bar{A}_j^{n+1} \frac{\Delta t^{n+1}}{\Delta x} \geq 0, \]
\[ \sigma_j = \frac{\bar{a}_{n+1}^{j} \Delta t^{n+1}}{\Delta x^j} > 0. \]

Assume \( |Y_j^{n+1}| = ||Y^{n+1}||_{L^\infty} \), for some \( 2 \leq j \leq J - 2 \), then it is easily seen from (4.212) that
\[ ||Y^{n+1}||_{L^\infty} \leq ||Y^n||_{L^\infty}. \] (4.213)

Next, assume \( |Y_1^{n+1}| = ||Y^{n+1}||_{L^\infty} \). By (4.212) and (4.210) we have
\[ (1 + \beta_1 + 2\sigma_0 + \sigma_1)Y_1^{n+1} - (\beta_1 + \sigma_1)Y_2^{n+1} = Y_1^n. \]

Thus
\[ |Y_1^{n+1}| \leq |Y_1^n| \leq ||Y^{n+1}||_{L^\infty}. \] (4.214)

Finally, assume \( |Y_{J-1}^{n+1}| = ||Y^{n+1}||_{L^\infty} \). By (4.212) and (4.211) we have
\[ (1 + \beta_{J-1} + \sigma_{J-1})Y_{J-1}^{n+1} - \sigma_{J-1}Y_{J-2}^{n+1} = Y_{J-1}^n. \]

Thus
\[ |Y_{J-1}^{n+1}| \leq |Y_{J-1}^n| \leq ||Y^n||_{L^\infty}. \] (4.215)

From (4.213)-(4.215) and the definition of \( Y^n \) we derive the following lemma.

**Lemma 4.5** Let \( Z \) satisfy (4.204)-(4.207) and assume \( 0 < a_* \leq a(x,t) \), then
\[ ||Z||_{L^\infty(0,N;L^\infty)} \leq 1. \] (4.216)

Furthermore
\[ ||Z_x||_{L^2(0,N-1;L^2)} + ||Z_\#||_{L^2(0,N-1;L^2)} \leq \frac{C(C_{2,3})}{a_*}, \] (4.217)

where \( C_{2,3} \) is a bound for \( ||A||_{L^\infty(L^\infty)} \).
Proof. The first result, (4.216), follows immediately from (4.213)-(4.215) by recalling
the definition of $Y$ and applying induction on $n$.

To prove (4.217), let $w = Y^{n+1}$ in (4.208), then summing by parts we have

$$
\frac{||Y^{n+1}||^2 - ||Y^n||^2}{2\Delta t^{n+1}} + \frac{a_*}{2} \left( ||Y_{x}^{n+1}||^2 + ||Y_{x}^{n+1}||^2 \right)
\leq \langle \tilde{A}_{n+1}Y_{x}^{n+1}, Y^{n+1} \rangle + \frac{1}{2}a_{j-\frac{1}{2}}^{n+1}(Y_{j+1}^{n+1} + Y_{j-1}^{n+1}) \frac{Y_{j+1}^{n+1} - Y_{j-1}^{n+1}}{\Delta x}
\leq \frac{CC_{2,3}}{a_*} + CC_{2,3} + \frac{a_*}{4} ||Y_{x}^{n+1}||^2.
$$

Here we have used (4.216) and (4.211) and the fact that

$$
|Y_{j+1}^{n+1}| = |Y_{j-1}^{n+1}| \frac{\tilde{A}_{j-\frac{1}{2}}^{n+1}}{\tilde{A}_{j-1}^{n+1}} \leq \frac{A_{j+1}^{n+1}}{\Delta x} \leq 1.
$$

Thus, multiplying above by $\Delta t^{n+1}$ and summing on $n$ we obtain (4.217).

Returning now to (4.203), we have by (4.204)-(4.207), $I_1 = 0$, and

$$
||\xi^N||_{L^1} = I_2 + I_3 + I_4 + I_5. \quad (4.218)
$$

Before estimating $I_2 - I_5$, we derive lemmas similar to Lemma 4.4.

**Lemma 4.6** Let the hypothesis of Lemma 4.4 hold, and assume $\xi^N_{j-1,L}$ is given by (4.198).

Then

$$
||\xi^N||_{L^2(L^2)} + ||\xi^N||_{L^2(L^2)} \leq \frac{C_{\xi^N}}{a_*}, \quad (4.219)
$$

and

$$
||\xi^{n+1}||_{L^2(L^2)} \leq \frac{C_{\xi^{n+1}}}{a_*}, \quad (4.220)
$$

where

$$
C_{\xi^N} = C(||s^0||_{L^\infty(I)}, ||g_0||_{L^\infty(0,T)}, ||g_1||_{L^\infty(0,T)}, C_{TE}, C_{2,3}), \quad (4.221)
$$
and

\[ C_{\xi} = C(C_{\xi}, a_{\ast}, \|\frac{\partial s}{\partial x}\|_{L^\infty(Q_T)}, \|s_0\|_{L^\infty(0,T)}), \|\frac{ds}{dx}\|_{L^\infty(T)}) \tag{4.222} \]

with \(C_{TE}\) given in Theorem 4.4.

**Proof.** We will derive (4.219). The proof of (4.220) is essentially the same as the proof of (4.103).

Let \(w = \xi^{n+1}\) in (4.195), then summing by parts and using (4.196)-(4.198) we have

\[
\frac{\|\xi^{n+1}\|^2 - \|\xi^n\|^2}{2\Delta t^n} + \frac{a_{\ast}}{2} \left( \|\xi^{n+1}_x\|^2 + \|\xi^{n+1}_x\|^2 \right)
\leq (A^n \xi^n, \xi^{n+1}) + (E^n, \xi^{n+1}) - A^n_{-1} \xi^{n+1}_x L \xi^{n+1}_x
\leq \frac{CC_{2.3}}{a_{\ast}} \|\xi^n_x\|^2 + \frac{a_{\ast}}{4} \|\xi^{n+1}_x\|^2 + \|E^n\|_{L^1} \|\xi^{n+1}\|_{L^\infty} + C_{2.3} \|\xi\|_{L^\infty(L^\infty)}
\leq \frac{CC_{2.3}}{a_{\ast}} + C_{\ast} + \|E^n\|^2 + \frac{a_{\ast}}{4} \|\xi^{n+1}_x\|^2,
\]

where \(C_{\ast}\) is a bound for \(\|\xi_L\|_{L^\infty(L^\infty)}\) and \(\|\xi\|_{L^\infty(L^\infty)}\) as given by (4.109). Using the regularity assumptions on \(s\) and the coefficients and boundary and initial data, and hiding the \(\frac{a_{\ast}}{4} \|\xi^{n+1}_x\|^2\) term on the left, multiplying by \(\Delta t^n\), summing on \(n\) and applying the quasi-uniformity assumption (4.182), we obtain (4.219).

**Lemma 4.7** Let the hypothesis of Lemma 4.6 hold with the modification that \(\xi^{n+1}_x\) satisfies (4.199). Furthermore, assume that for a chosen \(\epsilon\) satisfying \(0 < \epsilon < 1/3\), we have

\[
\sup f'(S) \leq \frac{\sqrt{1 - 3\epsilon}}{(1 + \epsilon)^{\frac{1}{2}}}, \tag{4.223}
\]

where \(\sup f'(S)\) is taken over all possible values of \(S\) as given by the maximum principles of Chapter 3. Then (4.219) holds, and (4.220) holds with the modification that \(C_{\xi}\) depends on \(1/\epsilon\).
Proof. For this case we derive (4.220). The proof of (4.219) is similar to the one given in Lemma 4.6. We will make repeated use of the following two inequalities, namely for \( b, c \in \mathbb{R}, \epsilon > 0, \)
\[
bc \leq \frac{\epsilon}{2}c^2 + \frac{1}{2\epsilon}b^2, \tag{4.224}
\]
and
\[
(c + b)^2 \leq (1 + \frac{1}{\epsilon})b^2 + (1 + \epsilon)c^2. \tag{4.225}
\]

Recall that by choosing \( w = \xi^{n+1}_t \) in (4.195), summing by parts and manipulating we obtain
\[
\left| \xi^{n+1}_t \right|^2 + \frac{1}{4} \left[ \sum_{j=1}^{J-1} \left( a_{n+1}^{j+\frac{1}{2}} \left| \xi^{n+1}_{j,x} \right|^2 \right)_t \Delta x + \sum_{j=1}^{J-1} \left( a_{n+1}^{j-\frac{1}{2}} \left| \xi^{n+1}_{j,x} \right|^2 \right)_t \Delta x \right]
\leq -\langle (f(s^*_L) - f(S^*_L))_x, \xi^{n+1}_t \rangle + \langle E^n, \xi^{n+1}_t \rangle
\]
\[+ \frac{1}{4} \left[ \sum_{j=1}^{J-1} \left( a_{n+1}^{j+\frac{1}{2}} \left| \xi^n_{j,x} \right|^2 + a_{n+1}^{j-\frac{1}{2}} \left| \xi^n_{j,x} \right|^2 \right) \Delta x \right]\]
\[\leq \frac{1}{2} \left| \langle (f(s^*_L) - f(S^*_L))_x \rangle \right|^2 + \frac{C}{\epsilon} ||E^n||^2 + \frac{(1 + \epsilon)}{2} ||\xi^{n+1}_t||^2
\]
\[+ C(L_t)(||\xi^n_x||^2 + ||\xi^n_t||^2) \]
\[\leq \frac{C}{\epsilon} \left| \langle (f(s^*_L))_x \rangle \right|^2 + \frac{(1 + \epsilon)}{2} \left| \langle (f(S^*_L))_x \rangle \right|^2 + \frac{C}{\epsilon} ||E^n||^2 + \frac{(1 + \epsilon)}{2} ||\xi^{n+1}_t||^2
\]
\[+ C(L_t)(||\xi^n_x||^2 + ||\xi^n_t||^2). \tag{4.226}
\]

Following the arguments given in the proof of Lemma 4.4, we have for \( j = 1, \ldots, J - 2, \)
\[
\left| \langle f(S^*_j, L) \rangle_x \right|^2 \leq C + C \left( ||\xi^n_{j,x}||^2 + ||\xi^n_{j,t}||^2 \right). \tag{4.227}
\]

For \( j = J - 1, \) we have
\[
\left| \langle f(S^*_j, L) \rangle_x \right|^2 = \left| \frac{f(S_{j-1}^n) - f(S_{j-2}^n)}{\Delta x} \right|^2
\]
\[
\begin{aligned}
&= \left| \frac{f(S_{j-1}^{n+1}) - f(S_{j-1}^n)}{\Delta x} + \frac{f(S_{j-1}^n) - f(S_{j-2}^n)}{\Delta x} \right|^2 \\
&\leq (1 + \epsilon)(h(\hat{S}))^2 \left| S_{j-1,1,t}^{n+1} \right|^2 \\
&\quad + \frac{C}{\epsilon} \left( \left| s_{j-1,1,t}^n \right|^2 + \left| \xi_{j-1,1,t}^n \right|^2 \right), \\
\end{aligned}
\] (4.228)

where \( h(s) = f'(s) \frac{\Delta t^n}{\Delta x} \) and \( \hat{S} \) is some point between \( S_{j-1}^{n+1} \) and \( S_{j-1}^n \). Thus, by (4.227)-(4.228) we have

\[
\| (f(S_{j}^{n})_{x}) \| \leq \frac{C}{\epsilon} (1 + \| \xi_{x}^{n} \|^{2} + \| \xi_{x}^{n} \|^{2}) \\
\quad + (1 + \epsilon)(h(\hat{S}))^2 \left| S_{j-1,1,t}^{n+1} + \xi_{j-1,1,t}^{n+1} \right|^2 \Delta x \\
\leq \frac{C}{\epsilon} (1 + \| \xi_{x}^{n} \|^{2} + \| \xi_{x}^{n} \|^{2}) \\
\quad + (1 + \epsilon)^2(h(\hat{S}))^2 \| \xi_{t}^{n+1} \|^{2} + \frac{C}{\epsilon} \| s_{t}^{n+1} \|^{2}. \\
\] (4.229)

Combining (4.226) and (4.229) we obtain

\[
\| \xi_{t}^{n+1} \|^{2} + \frac{1}{4} \sum_{j=1}^{J-1} \left( a_{j+1}^{n+1} \left| \xi_{j+1,1,t}^{n+1} \right|^{2} \right) \Delta x + \sum_{j=1}^{J-1} \left( a_{j+1}^{n+1} \left| \xi_{j,1,t}^{n+1} \right|^{2} \right) \Delta x \\
\leq \frac{C}{\epsilon} (1 + \| \xi_{x}^{n} \|^{2} + \| \xi_{x}^{n} \|^{2}) \\
\quad + \left( \frac{1 + \epsilon}{2}(h(\hat{S}))^2 + \frac{1 + \epsilon}{2} \right) \| \xi_{t}^{n+1} \|^{2}. \\
\] (4.230)

By our assumption on \( \Delta t^n \), (4.223), we have that

\[
\frac{(1 + \epsilon)^3}{2}(h(\hat{S}))^2 + \frac{1 + \epsilon}{2} \leq 1 - \epsilon.
\]

Thus, hiding the \((1 - \epsilon)\| \xi_{t}^{n+1} \|^{2}\) term on the left, multiplying by \( \Delta t^n \) and summing on \( n \), we obtain

\[
\epsilon \| \xi_{t}^{n+1} \|_{L^2(L^2)}^{2} \leq \frac{C}{\epsilon a_{t}^{2}},
\]

where we have used (4.219) and truncation error bounds. The lemma now follows.
Estimate of $I_2$. Recall from (4.203),

$$
I_2 = \sum_{n=0}^{N-1} \langle \xi^n - \xi^{n+1} + \Delta x(\gamma(s^n) - \bar{\gamma}(S^n)), \kappa A^n Z^n_x \rangle \Delta t^n
\leq C_{2,3} \Delta t \|\xi^{n+1}\|_{L^2(L^2)} \|Z_x\|_{L^2(L^2)}
+C_{2,3} \Delta x \|\gamma(s) - \bar{\gamma}(S)\|_{L^2(L^2)} \|Z_x\|_{L^2(L^2)}.
$$

(4.231)

Following the argument given by (4.121)-(4.122) with the modification that $\delta s^n_{j-1} = \delta\bar{S}^n_{j-1} = 0$, we obtain

$$
\|\gamma(s) - \bar{\gamma}(S)\|_{L^2(L^2)} \leq C \left( \|\xi_x\|_{L^2(L^2)} + \|\eta_x\|_{L^2(L^2)} \right) + C \|\delta s\|_{L^2(L^2)}
\leq a^{-1}_w C(C_{\xi}, C_{2,1}),
$$

where $C_{2,1}$ is a bound for $\|\delta s\|_{L^2(L^2)}$.

Thus by (4.231), Lemmas 4.6 and 4.5 we have

$$
I_2 \leq C_2 (\Delta t + \Delta x),
$$

(4.232)

where

$$
C_2 = a^{-2}_w C(C_{\xi}, C_{\xi}, C_{2,1}, C_{2,3}).
$$

(4.233)

Moreover, in the case (4.199) holds, $C_2$ also depends on $\frac{1}{\xi}$, where $\epsilon$ is chosen as described in the hypothesis of Lemma 4.7.

Estimate of $I_3$. Next, recall from (4.203)

$$
I_3 = \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n.
$$

Here we take essentially the same approach as we did in Sections 4.1-4.3 to obtain
\[
I_3 \leq \frac{C}{a_*} \left[ ||E_A'||_{L^2(L^2)} + ||E_{AR}'||_{L^2(L^2)} + ||E_{AL}'||_{L^2(L^2)} \right. \\
\left. + ||E_{DL}'||_{L^2(L^2)} + ||E_{DR}'||_{L^2(L^2)} \right] + ||E_T||_{L^2(L^2)} \\
+ \sum_{n=0}^{N-1} \left| Z_j^{n+1} \right| \left| f(s_j^{n+1/2}) - f(s_j^{n+1/2}) \right| \Delta t^n \\
+ \sum_{n=0}^{N-1} \left| Z_j^n \right| \left| f(s_j^{n+1/2}) - \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} f(s(x_j^{n+1/2}, t) dt \right| \Delta t^n, \\
+ \sum_{n=0}^{N-1} \left| Z_j^n \right| \left| f(g_0^n) - f(g_0^{n+1/2}) \right| \Delta t^n 
\] (4.234)

where \( E_A', E_{AR}' \), and \( E_{AL}' \), \( E_{DL} \), and \( E_{DR} \), and \( E_T \) are given by (4.62), (4.65), (4.67), and (4.4), respectively. The last three terms above are a result of summation by parts, see (4.64) and (4.65).

Bounds for the terms in (4.234) are given in Appendices A and B. We summarize these results here.

**Theorem 4.9** Let \( f, a, s^0 \), and \( g_0 \) satisfy the following:

(i) \( f \in C^2(\mathcal{R}) \),

(ii) \( a^{2s} \in C^1(Q_T), \ 0 < a_* \leq a(x, t) < a^* \),

(iii) \( s^0 \in C^2(I) \cap H^3(I) \),

(iv) \( g_0 \in C^2(0, T) \).

Let \( s \), the solution of (2.1)-(2.3), (2.7), have the following regularity.

(v) \( s \) and all first and second partials of \( s \) with respect to \( x \) and \( t \) exist everywhere in \( Q_T \) and are bounded,

(vi) \( \frac{\partial^2 s}{\partial x^2} \) and \( \frac{\partial^2 s}{\partial t^2} \) exist in \( Q_T \) and are in \( L^2(0, T; L^\infty(I)) \) and \( L^\infty(0, T; L^2(I)) \), respectively.
Assume $\Delta x$ is uniform, and $\Delta t^n = \mathcal{O}(\Delta x)$, then

\[
\| E'_A \|_{L^2(L^2)} \leq C'_A (\Delta x^2 + \Delta t), \quad (4.235)
\]
\[
\| E''_A \|_{L^2(L^2)} \leq C''_{AL} (\Delta t^2), \quad (4.236)
\]
\[
\| E''_{AR} \|_{L^2(L^2)} \leq C''_{AR} (\Delta t^2), \quad (4.237)
\]
\[
\| E_{DL} \|_{L^2(L^2)} \leq C_{DL} (\Delta x^{3/2} + \Delta t), \quad (4.238)
\]
\[
\| E_{DR} \|_{L^2(L^2)} \leq C_{DR} (\Delta x^2 + \Delta t), \quad (4.239)
\]

and

\[
\| E_T \|_{L^2(L^2)} \leq C_T \Delta x^2, \quad (4.240)
\]

where $C'_A$ is given by (A.27), $C''_{AL}$ is given by (A.31), $C''_{AR}$ is given by (B.1), $C_{DL}$ is given by (A.43), $C_{DR}$ is given by (A.37), and $C_T$ is given by (A.10).

**Proof.** See Appendix B.

The last three terms in (4.234) are either pieces in the estimates of $E'_A$ and $E''_{AR}$ or they are clearly $\mathcal{O}(\Delta t)$. Thus, combined these terms are $\mathcal{O}(\Delta x^2 + \Delta t)$. By the above theorem and (4.234) we have

\[
I_3 \leq C_{TE} (\Delta x^{3/2} + \Delta t), \quad (4.241)
\]

where

\[
C_{TE} = C(C'_A, C''_{AL}, C''_{AR}, C_{DL}, C_{DR}, C_T). \quad (4.242)
\]

**Estimate of $I_4$.** Next, we have

\[
I_4 = \sum_{n=0}^{N-1} B^n(Z^n) \Delta t^n
\]

\[
= - \sum_{n=0}^{N-1} \left[ A^{n}_{j-1} \xi^{n+1}_{j-1} \frac{Z^n_{j} - Z^n_{j-1}}{\Delta z} \right] \Delta t^n
\]
by (4.200) and (4.206). Clearly, if (4.199) holds, then by (4.207), \( I_4 = 0 \). Otherwise, if (4.198) holds, then by (4.207), and Lemmas 4.5 and 4.6,

\[
I_4 = - \sum_{n=0}^{N-1} A^n_{j-1} (\xi^n_{j-1} - \xi^{n+1}_{j-1}) Z^n_j \Delta t^n
\leq C_{2,3} \Delta t \sum_{n=0}^{N-1} \left| \frac{\xi^n_{j-1} - \xi^{n+1}_{j-1}}{\Delta t^n} \right| \Delta t^n
\leq C_{2,3} \Delta x^{-\frac{1}{2}} \Delta t \left( \sum_{n=0}^{N-1} \left| \frac{\xi^n_{j-1} - \xi^{n+1}_{j-1}}{\Delta t^n} \right|^2 \Delta x \Delta t^n \right)^{\frac{1}{2}}
\leq C_{2,3} \Delta x^{-\frac{1}{2}} \Delta t \| \xi^{n+1}_t \|_{L^2(L^2)}
\leq C_4 \Delta x^{-\frac{1}{2}} \Delta t,
\]

where

\[
C_4 = a_x^{-1} C(C_{2,3}, C_{\xi}).
\]

**Estimate of \( I_5 \).** Finally, as before, we have

\[
I_5 \leq C_5 \Delta x^2,
\]

where

\[
C_5 = C \left( \| \frac{d^2 s^0}{dx^2} \|_{L^1(T)} \right).
\]

Assuming (4.198) holds, then combining (4.203), (4.232), (4.241), (4.243), and (4.245), we obtain

\[
\| \xi^n \|_{L^1} \leq C_4 \Delta t \Delta x^{-\frac{1}{2}} + C_2 (\Delta x + \Delta t) + C_3 (\Delta x^\frac{3}{2} + \Delta t) + C_5 \Delta x^2.
\]

Otherwise, if (4.199) holds, then

\[
\| \xi^n \|_{L^1} \leq C_2 (\Delta x + \Delta t) + C_3 (\Delta x^\frac{3}{2} + \Delta t) + C_5 \Delta x^2.
\]

We now state our results.
Theorem 4.10 Let the hypothesis of Theorem 4.9 hold. Let $\Delta t^n$ be chosen in accordance with the hypothesis of Lemma 3.1. Moreover, assume $S^n_{j-1,L} = S^n_j - 1$, and $a$ is Lipschitz in $t$. Then

$$
\|s - S\|_{L^\infty(L^1)} \leq C_\xi (\Delta t \Delta x^{-\frac{1}{2}} + \Delta x + \Delta t),
$$

(4.249)

where

$$
C_\xi = C(C_2, C_3, C_4, C_5).
$$

(4.250)

Thus $\|s - S\|_{L^\infty(L^1)} = \mathcal{O}(a^{-2}(\Delta x + \Delta t + \Delta t \Delta x^{-\frac{1}{2}}))$.

Theorem 4.11 Let the hypothesis of Theorem 4.10 hold, with the exceptions that $S^n_{j-1,L} = S^n_{j-1} + 1$ and $\Delta t^n$ satisfies the additional $\epsilon$-constraint as described in Lemma 4.7. Then,

$$
\|s - S\|_{L^\infty(L^1)} \leq C'_\xi (\Delta x + \Delta t),
$$

(4.251)

where

$$
C'_\xi = C(C_2, C_3, C_5),
$$

(4.252)

with $C_2 = C(\epsilon^{-1} a^{-2})$.

### 4.5 $L^\infty(L^2)$ estimate-mixed boundary condition

We now derive a $\mathcal{O}(\Delta x + \Delta t)$ error estimate in $L^\infty(L^2)$ for the GMM applied to (2.1)-(2.2), (2.5) and (2.6). A discrete maximum principle for the GMM approximation with these boundary conditions is given in Section 3.3.

Again we assume the mesh is uniform. Assume $\Delta t^n$ satisfies the hypothesis of Lemma 3.4.
For this problem, \( s \) again satisfies an equation of the form (4.183), with truncation error \( E \) given by (4.3). Here, for \( n = 0, \ldots, N^* \), we again have

\[
s^n_j = s(x_j, t^n), \quad j = 1, \ldots, J - 1, \quad (4.253)
\]

and

\[
s^j_j = 2s^n_j - s^n_{j-1}. \quad (4.254)
\]

For ease of notation, we define

\[
f(s^{n}_{0,L}) \equiv \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} f(s(0,t))dt,
\]

and

\[
a^{n+1}_{\frac{1}{2}} \frac{s^{n+1}_{1} - s^{n+1}_{0}}{\Delta x} \equiv \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} \left( \frac{\partial s}{\partial x} \right) (0,t)dt. \quad (4.256)
\]

Thus, we have

\[
f(s^{n}_{0,L}) - a^{n+1}_{\frac{1}{2}} \frac{s^{n+1}_{1} - s^{n+1}_{0}}{\Delta x} = \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} f(g_0(t))dt. \quad (4.257)
\]

Moreover, we define

\[
s^n_{j,L} = s^n_j + \frac{\Delta x}{2} (1 - h(s^n_j)) \delta s^n_j, \quad j = 1, \ldots, J - 1, \quad (4.258)
\]

where

\[
h(s) = \frac{\Delta t^n}{\Delta x} f'(s), \quad (4.259)
\]

\[
\delta s^n_j = \frac{s^n_{j+1} - s^n_{j-1}}{2\Delta x}, \quad j = 2, \ldots, J - 2, \quad (4.260)
\]

\[
\delta s^n_{j-1} = \frac{4s^n_j - 3s^n_{j-1} - s^n_{j-2}}{3\Delta x}, \quad (4.261)
\]

and

\[
\delta s^n_1 = \frac{\partial s(x_1, t^n)}{\partial x}. \quad (4.262)
\]
Defining $\xi$ and $\xi_L$ as before, we again have

$$
(\xi^{n+1}_t + (A^n \xi^n_L)_x - (a^{n+1}_x \xi^{n+1}_x)_x, w) = (E^n, w).
$$

(4.263)

By (4.254), (4.257), (2.38), and (2.46),

$$
\xi^n_j = -\xi^n_{j-1},
$$

(4.264)

and

$$
A^n_0 \xi^n_0 - a^{n+1}_1 \frac{\xi^{n+1}_1 - \xi^n_0}{\Delta x} = 0.
$$

(4.265)

Here $A^n_j$ is again given by (4.25).

Summing by parts, we have

$$
\sum_{j=1}^{J-1} (A^n_j \xi^n_{j,L})_x w_j \Delta x = - \sum_{j=1}^{J-1} A^n_j \xi^n_{j,L} w_{j,x} \Delta x + A^n_{j-1} \xi^n_{j-1,L} w_j - A^n_0 \xi^n_0 w_1
$$

$$
= - \sum_{j=2}^{J-1} A^n_{j-1} \xi^n_{j-1,L} w_{j-1,x} \Delta x + A^n_{J-1} \xi^n_{J-1,L} w_{J-1} - A^n_0 \xi^n_0 w_1.
$$

Thus

$$
\langle (A^n \xi^n_L)_x, w \rangle = - (A^n \xi^n_L, \kappa w_x) + B^n_\lambda(w),
$$

(4.266)

where

$$
B^n_\lambda(w) = A^n_{J-1} \xi^n_{J-1,L} \frac{w_J + w_{J-1}}{2} - A^n_0 \xi^n_0 w_1
$$

(4.267)

and $\kappa$ is given by (4.24).

Moreover

$$
\langle (a^{n+1}_x \xi^{n+1}_x)_x, w \rangle = \langle \xi^{n+1}_x, (a^{n+1}_x w_x)_x \rangle + B^n_\delta(w),
$$

(4.268)
where, applying (4.264),

\[
B^n_D(w) = a_{j+\frac{1}{2}}^{n+1} \xi_{j-\frac{1}{2}}^{n+1} \frac{w_{j+1} + w_{j-1}}{\Delta x} - a_{j+\frac{1}{2}}^{n+1} \left( \xi_{j+\frac{1}{2}}^{n+1} - \xi_{j-\frac{1}{2}}^{n+1} \right) \frac{w_1 - w_0}{\Delta x}.
\]  

(4.269)

Substituting (4.266), (4.268) into (4.263) and letting \( w = Z^n \in M_{-1}^0(\delta x) \), we obtain

\[
\langle \xi_{j+1}^{n+1}, Z^n \rangle - \langle \xi_{j}^{n}, \kappa A^n Z_x^n \rangle - \langle \xi_{j+1}^{n+1}, (a_{j+1}^{n+1} Z_{x_j}^n) \rangle = \langle F^n, Z^n \rangle + B^n(Z^n),
\]  

(4.270)

where

\[
B^n(Z^n) = B^n_D(Z^n) - B_A^n(Z^n).
\]  

(4.271)

**Choice of \( Z^n \).** Let \( Z^n \) be the block-centered finite difference approximation to \( z \) satisfying

\[
\frac{\partial z}{\partial x} + \frac{\partial}{\partial x} \left( \tilde{a}(x,t) \frac{\partial z}{\partial x} \right) = 0, \quad (x,t) \in I \times [0,t^N),
\]

\[
z(x,t^N) = \xi_N^N, \quad x \in B_j,
\]

\[
\frac{\partial z(0,t)}{\partial x} = z(1,t) = 0, \quad 0 \leq t < t^N.
\]

Thus \( Z \) satisfies

\[
\langle Z_{j+1}^{n+1} + (a_{j+1}^{n+1} Z_x^N) \xi_j, v \rangle = 0, \quad v \in M_{-1}^0(\delta x),
\]  

(4.272)

\[
Z^N_j = \xi_j^N, \quad j = 1, \ldots, J - 1,
\]  

(4.273)

\[
Z_0^n = Z_1^n, \quad Z_j^N = -Z_{j-1}^N, \quad n = 0, \ldots, N - 1.
\]  

(4.274)

Multiplying (4.270) by \( \Delta t^n \), summing on \( n \), summing by parts on \( n \), applying (4.272)-(4.274) and (4.265), we have \( B^n(Z^n) = 0 \) for all \( n \), and
\[ \|\xi^N\|^2 = \sum_{n=0}^{N-1} \langle A^n \xi^0_L, \kappa Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n + \langle \xi^0, Z^0 \rangle \]
\[ \equiv I_2 + I_3 + I_5. \quad (4.275) \]

Following essentially the same argument that was used to prove Lemma 4.1 we find that

\[ \|Z\|_{L^\infty(0,N; L^2)} \leq \|\xi^N\|, \quad (4.276) \]

and

\[ \|Z_x\|_{L^2(0,N-1; L^2)} + \|Z_{xx}\|_{L^2(0,N-1; L^2)} \leq \frac{C}{a^2} \|\xi^N\|. \quad (4.277) \]

**Estimate of \( I_2 \).** Following the same argument used to derive (4.54), we obtain

\[ I_2 \leq (C_2)^2 \left( \|\xi\|^2_{L^2(0,N-1; L^2)} + \Delta x^2 \right) + \frac{1}{8} \|\xi^N\|^2, \quad (4.278) \]

where

\[ C_2 = a^{-\frac{1}{2}} C(C_{2,1}, C_{2,2}, C_{2,3}), \quad (4.279) \]

with

\[ \|\delta s\|_{L^4(L^2)} \leq C_{2,1}, \]
\[ \frac{\Delta t^n}{\Delta x} \|h(s) - h(S)\|_{L^\infty(L^\infty)} \leq C_{2,2}, \]

and

\[ \|A\|_{L^\infty(L^\infty)} \leq C_{2,3}. \]

**Estimate of \( I_3 \).** Next, consider

\[ I_3 = \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n \]
\[ I_3' = \sum_{n=0}^{N-1} \langle E_T^n, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} \langle (E^*_A)_n, Z^n \rangle \Delta t^n \\
+ \sum_{n=0}^{N-1} \langle (E^*_D)_n, Z^n \rangle \Delta t^n \]
\[ = I_3' + I_3'' + I_3''', \quad (4.280) \]

where \( E_T, E_A, \) and \( E_D \) are given by (4.4)-(4.6), respectively.

The estimate for \( I_3' \) is precisely the same as in (4.61), thus

\[ I_3' \leq C \| E_T \|_{L^2(L^2)}^2 + \frac{1}{24} \| \xi^N \|^2. \quad (4.281) \]

Summing by parts, applying (4.274), (4.277), and (4.255), we have

\[ I_3'' \leq \frac{C}{a_*} \left[ \| E_A' \|_{L^2(L^2)}^2 + \| E_A'' \|_{L^2(L^2)}^2 \right] + \frac{1}{24} \| \xi^N \|^2 \quad (4.282) \]

where \( E_A'' \) is given by (4.62) and \( E_A'' \) by (4.65).

Similarly, summing by parts and applying (4.256) we have

\[ I_3''' \leq \frac{C}{a_*} \| E_{DR} \|_{L^2(L^2)}^2 + \frac{1}{24} \| \xi^N \|^2, \quad (4.283) \]

where \( E_{DR} \) is given by (4.67).

Combining (4.280)-(4.283), we find that

\[ I_3 \leq \frac{C}{a_*} \left[ \| E_A' \|_{L^2(L^2)}^2 + \| E_A'' \|_{L^2(L^2)}^2 + \| E_{DR} \|_{L^2(L^2)}^2 \right] \\
+ \| E_T \|_{L^2(L^2)}^2 + \frac{1}{8} \| \xi^N \|^2. \quad (4.284) \]

The truncation error analysis for this problem is essentially given in Appendix A. The only difference is in the estimate of \( f(s^n_{1,\xi}) - f(s^{n+\frac{1}{2}}_{\frac{1}{2}}) \) with the term involving \( \delta s^n_{1,\xi} - \frac{\delta s_{1,\xi}(t^n)}{\delta x} \).

Here, this term is zero by (4.262). Thus we have the following theorem.

**Theorem 4.12** Let \( f, a, s^0, g_0, \) and \( g_1 \) satisfy the hypothesis of Theorem 4.4. Let \( s \) have the regularity given in Theorem 4.4. Assume the mesh is uniform and \( \Delta t^n = \mathcal{O}(\Delta x) \), then

\[ I_3 \leq (C_3)^2 (\Delta x^3 + \Delta t^2) + \frac{1}{8} \| \xi^N \|^2, \quad (4.285) \]
where

\[ C_3 = a_\frac{1}{2}C(C_T, C_{AR}', C_A, C_{DR}) \] (4.286)

with \( C_T, C_{AR}', C_A', \) and \( C_{DR} \) given by (A.10), (A.28), (A.27), and (A.37), respectively.

**Estimate of \( I_5 \).** As before,

\[
I_5 \leq C \Delta z^4 \|s\|^2_{(II)^2} + \frac{1}{8}\|\xi^N\|^2.
\]

\[ \equiv (C_5)^2 \Delta z^4 + \frac{1}{8}\|\xi^N\|^2. \] (4.287)

By (4.275), (4.278), (4.285), and (4.287) we have

\[
\|\xi^N\|^2 \leq (C_2)^2 \left( \sum_{n=0}^{N-1} \|\xi^n\|^2 \Delta t^n + \Delta z^2 \right) + (C_3)^2(\Delta z^3 + \Delta t^2) + (C_5)^2 \Delta z^4.
\] (4.288)

Thus by Gronwall's lemma applied to (4.288) we have the following theorem.

**Theorem 4.13** Let the hypothesis of Theorem 4.12 hold. Let \( S \) be the GMM approximation to \( s \) satisfying (2.1)-(2.2), (2.5), and (2.6). Then

\[
\|s - S\|_{L^\infty(L^2)} \leq C_\xi(\Delta x + \Delta t),
\] (4.289)

where

\[
C_\xi = C(\sqrt{e(C_2)^2}, C_3, C_5).
\] (4.290)

### 4.6 Nonuniform spatial mesh

We claim the the estimates given in Sections 4.1-4.5 hold when \( \Delta x_j \) is nonuniform. The only major change in the estimates is in the truncation error term involving \( E_D \), see (4.6).
In this term we lose a power of $\Delta x$ at interior points due to the nonuniformity of the grid. However, the global error remains unchanged.

As a means of demonstration, we will derive an $L^\infty(L^2)$ estimate for the problem given in Section 4.1 assuming $\Delta x_j$ is nonuniform.

Thus, let $\Delta t^n$ satisfy the hypothesis of Lemma 3.1. We must assume the grid is quasi-uniform; i.e., there exist positive constants $c_*, c^*$ independent of the mesh such that

$$c_* \leq \frac{\Delta x_j}{\Delta x_i} \leq c^*, \quad j, i = 1, \ldots, J - 1. \tag{4.291}$$

As in Section 4.1, the true solution $s$ to (2.1)-(2.3), (2.6) satisfies

$$(s_t^{n+1} + (f(s_L^n))_{xx} - (a^{n+1} s_x^{n+1})_{xx}, w) = (E^n, w), \tag{4.292}$$

for all $w \in M_{-1}^0(\delta_x)$. Here again

$$E^n_j = E^n_{T,j} + \left( E^n_{A,j} \right)_x + \left( E^n_{D,j} \right)_x, \tag{4.293}$$

where $E_T$, $E_A$, and $E_D$ are given by (4.4)-(4.6), with the modifications that in (4.4), $\Delta x = \Delta x_j$, and in (4.6), $\Delta x = \Delta x_{j+\frac{1}{2}}$.

In (4.292), we have, for $n = 0, \ldots, N^*$,

$$s^n_j = s(x_j, t^n), \quad j = 1, \ldots, J - 1, \tag{4.294}$$

$$s^0_n = 2s^0_n - s^n_1, \quad s^n_j = 2s^n_j - s^n_{j-1}, \tag{4.295}$$

$$s^n_{0,L} = g_0^{n+\frac{1}{2}}, \tag{4.296}$$

and

$$s^n_{j,L} = s^n_j + \frac{\Delta x}{2} (1 - h_j(s^n_j)) \delta s^n_j, \quad j = 1, \ldots, J - 1. \tag{4.297}$$
Here

\[ h_j(s) = f'(s) \frac{\Delta t^n}{\Delta x_j'}, \quad (4.298) \]

\[ \delta s^n_j = \frac{s^n_{j+1} - (1 - \mu_j^2)s^n_j - \mu_j^2 s^n_{j-1}}{\mu_j(\Delta x_{j-\frac{1}{2}} + \Delta x_{j+\frac{1}{2}})} , \quad j = 2, \ldots, J - 2, \quad (4.299) \]

\[ \delta s^n_1 = \frac{s^n_2 - (1 - 4\mu_1^2)s^n_1 - 4\mu_1^2 g^n_0}{\mu_1(\Delta x_1 + 2\Delta x_{\frac{1}{2}})} , \quad (4.300) \]

and

\[ \delta s^n_{j-1} = \frac{4\mu_{j-1}^2 g^n_0 - (4\mu_{j-1}^2 - 1)s^n_{j-1} - s^n_j}{\mu_{j-1}(2\Delta x_{j-\frac{1}{2}} + \Delta x_{j-1})} , \quad (4.301) \]

with

\[ \mu_j = \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}} , \quad j = 2, \ldots, J - 2, \quad (4.302) \]

\[ \mu_1 = \frac{\Delta x_{\frac{1}{2}}}{\Delta x_1} , \quad (4.303) \]

and

\[ \mu_{j-1} = \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_{j-1}} . \quad (4.304) \]

The GMM approximation \( S \) to \( s \) satisfies

\[ \langle S^{n+1}_t + (f(S^n_L))_t - (a^{n+1} S^{n+1}_x)_t, w \rangle = 0, \quad w \in M^0_{-1}(\delta_x) . \quad (4.305) \]

Defining \( \xi \) and \( \xi_L \) as before and subtracting (4.305) from (4.292) we obtain

\[ \langle \xi^{n+1}_t + (f(s^n_L) - (S^n_L))_t - (a^{n+1} \xi^{n+1}_x)_t, w \rangle = \langle E^n, w \rangle , \quad (4.306) \]

with

\[ \xi^n_{0,L} = 0 , \quad (4.307) \]

\[ \xi^n_0 = -\xi^n , \quad (4.308) \]
and

\[ \xi^j - \xi^j_{j-1}. \quad (4.309) \]

Summing by parts, recalling \( s^n_{0,L} = S^n_{0,L} = g_0^{n+\frac{1}{2}} \), we have

\[
\langle(f(s^n_L) - f(S^n_L))\xi, w \rangle = - \sum_{j=1}^{J-1} (f(s^n_j,L) - f(S^n_j,L))w_{j,\xi} \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} \Delta x_j \\
+ (f(s^n_{J-1,L} - f(S^n_{J-1,L}))w_J \\
= - \sum_{j=1}^{J-1} (f(s^n_{j-1,L}) - f(S^n_{j-1,L}))w_{j,\xi} \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_{j-1}} \Delta x_{j-1} \\
+ (f(s^n_{J-1,L} - f(S^n_{J-1,L}))w_{J-1}.
\]

Thus,

\[
\langle(f(s^n_L) - f(S^n_L))\xi, w \rangle = \langle A^n \xi^n_L, \kappa w \rangle + B^n_\lambda(w), \quad (4.310)
\]

where \( A^n_j \) is given by (4.25),

\[
B^n_\lambda(w) = (f(s^n_{J-1,L}) - f(S^n_{J-1,L})) \frac{w_J + w_{J-1}}{2}, \quad (4.311)
\]

and

\[
\kappa_j = \begin{cases} \\
\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j}, & j = 1, \ldots, J - 2, \\
\frac{\Delta x_{J-\frac{1}{2}}}{2\Delta x_{J-1}}, & j = J - 1.
\end{cases} \quad (4.312)
\]

Summing by parts twice we have

\[
\langle(a^{n+1}\xi^{n+1})\xi, w \rangle = \langle \xi^{n+1}, (a^{n+1}w)\xi \rangle - B^n_D(w), \quad (4.313)
\]

where, by applying (4.308) and (4.309),

\[
B^n_D(w) = \frac{1}{\Delta x_{J-\frac{1}{2}}} a^{n+1}_J \xi^{n+1}_J(w_J + w_{J-1}) \\
- \frac{1}{\Delta x_{\frac{1}{2}}} a^{n+1}_1 \xi^{n+1}_1(w_1 + w_0). \quad (4.314)
\]
Let \( w = Z^n \in \mathcal{M}_{-1}^0(\delta_x) \), then by (4.305), (4.310), and (4.313)

\[
\langle \xi_t^{n+1}, Z^n \rangle - \langle \xi_t^n, \kappa A^n Z^n \rangle - \langle \xi_t^{n+1}, (a^{n+1} Z^n) \rangle = \langle E^n, Z^n \rangle + B^n(Z^n),
\]

where

\[
B^n(Z^n) = B^n_D(Z^n) - B^n_A(Z^n).
\]

Multiplying (4.315) by \( \Delta t^n \), summing on \( n \), and summing by parts on \( n \) we obtain

\[
\langle \xi^N, Z^N \rangle = \sum_{n=0}^{N-1} \langle \xi_t^{n+1}, Z_t^{n+1} + (a^{n+1} Z^n) \rangle \Delta t^n + \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} B^n(Z^n) \Delta t^n + \langle \xi^0, Z^0 \rangle
\]

\[
\equiv I_1 + I_2 + I_3 + I_4 + I_5.
\]

**Choice of \( Z^n \).** We choose \( Z^n \) precisely as we did in Section 4.1. Thus \( Z \) satisfies

\[
\langle Z_t^{n+1} + (a^{n+1} Z^n), v \rangle = 0, \quad v \in \mathcal{M}_{-1}^0(\delta_x),
\]

\[
Z_j^N = \xi_j^N, \quad j = 1, \ldots, J - 1,
\]

\[
Z_0^n = -Z_1^n, \quad Z_j^n = -Z_{j-1}^n, \quad n = 0, \ldots, N.
\]

Moreover, assuming \( 0 < a_* \leq a(x, t) \), Lemma 4.1 holds, hence

\[
\|Z\|_{L^\infty(0,N-1; L^2)} \leq \|\xi^N\|,
\]

and

\[
\|Z_t\|_{L^2(0,N-1; L^2)} + \|Z\|_{L^2(0,N-1; L^2)} \leq \frac{C}{a_*^2} \|\xi^N\|.
\]
Therefore, by (4.318)-(4.320), \( I_1 = I_4 = 0 \) in (4.317), and
\[
\|\xi^N\|^2 = I_2 + I_3 + I_5. \tag{4.323}
\]

**Estimate of \( I_2 \).** Following the argument given in (4.43)-(4.51) and using the quasi-uniformity assumption (4.291), we have
\[
I_2 \leq (C_2)^2 \left( \sum_{n=0}^{N-1} \|\xi^n\|^2 \Delta t^n + \Delta x^2 \right) + \frac{1}{8} \|\xi^N\|^2. \tag{4.324}
\]
Here
\[
C_2 = a_*^{-\frac{1}{2}} C(c_*, c^n, C_{2,1}, C_{2,2}, C_{2,3}), \tag{4.325}
\]
where as before, \( C_{2,1}, C_{2,2} \), and \( C_{2,3} \) are positive constants satisfying
\[
\|\delta s\|_{L^2(L^2)} \leq C_{2,1},
\]
\[
\frac{\Delta t^n}{\Delta x_j} \|h(s) - h(S)\|_{L^\infty(L^\infty)} \leq C_{2,2},
\]
and
\[
\|A\|_{L^\infty(L^\infty)} \leq C_{2,3}.
\]

**Estimate of \( I_3 \).** Recall from (4.317), (4.293),
\[
I_3 = \sum_{n=0}^{N-1} \langle E^n, Z^n \rangle \Delta t^n
\]
\[
= \sum_{n=0}^{N-1} \langle E^n_T, Z^n \rangle \Delta t^n + \sum_{n=0}^{N-1} \langle (E^n_A), Z^n \rangle \Delta t^n
\]
\[
+ \sum_{n=0}^{N-1} \langle (E^n_D), Z^n \rangle \Delta t^n
\]
\[
\equiv I_3' + I_3' + I_3'''. \tag{4.326}
\]
The terms \( I_3', I_3'', \) and \( I_3''' \) are handled precisely as they were in Section 4.1; i.e., we sum by parts. Thus following the arguments given in Section 4.1 and applying (4.291) we obtain
\[
I_3' \leq C\|E_T\|_{L^2(L^2)}^2 + \frac{1}{24} \|\xi^N\|^2, \tag{4.327}
\]
\[ I_3'' \leq \frac{C(c, c^*)}{a_*} \left[ ||E_A'||^2_{L^2(L^2)} + ||E_{AR}'||^2_{L^2(L^2)} + ||E_{AL}'||^2_{L^2(L^2)} \right] + \frac{1}{24} ||\xi^N||^2, \quad (4.328) \]

where \( E_A' \) is given by (4.62), \( E_{AR}' \), and \( E_{AL}' \) are given by (4.65), and

\[ I_3''' \leq \frac{C(c, c^*)}{a_*} \left[ ||E_{DR}'||^2_{L^2(L^2)} + ||E_{DL}'||^2_{L^2(L^2)} \right] + \frac{1}{24} ||\xi^N||^2, \quad (4.329) \]

where \( E_{DR} \) and \( E_{DL} \) are given by (4.67).

Assuming the hypothesis of Theorem 4.4 holds, we obtain from Appendix A and (4.326)-(4.329),

\[ I_3 \leq (C_3)^2 (\Delta x^2 + \Delta t^2) + \frac{1}{8} ||\xi^N||^2, \quad (4.330) \]

where

\[ C_3 = a_*^{-\frac{3}{2}} C(T, C_A', C_{AR}', C_{AL}', C_{DR}, C_{DL}, c, c^*), \quad (4.331) \]

with \( C_T \) given by (A.10), \( C_A' \) by (A.27), \( C_{AR}' \) by (A.28), \( C_{AL}' \) by (A.31), \( C_{DR} \) by (A.39), and \( C_{DL} \) by (A.41).

**Estimate of \( I_5 \).** Finally,

\[ I_5 \leq (C_5)^2 \Delta x^4 + \frac{1}{8} ||\xi^N||^2, \quad (4.332) \]

where

\[ C_5 = C(||s^0||_{H^2(I)}). \quad (4.333) \]

Combining (4.323), (4.324), (4.330), (4.332) and using the discrete Gronwall Lemma we obtain the following.

**Theorem 4.14** Let the hypothesis of Theorem 4.4 hold. Let \( \Delta t^n \) satisfy the hypothesis of Lemma 3.1 and assume \( \Delta x_j \) satisfies (4.291). Then the error between the GMM approximation \( S \) and \( s \) satisfies

\[ ||s - S||_{L^\infty(L^2)} \leq C(\Delta x + \Delta t), \quad (4.334) \]
where

\[ C = C(\sqrt{e(C_2)^2}, C_3, C_5). \]  

(4.335)

4.7 \( L^\infty(L^1) \) estimate for first-order Godunov

We conclude this chapter by deriving a \( O(\Delta x + \Delta t) \) estimate in \( L^\infty(L^1) \) for the first-order Godunov-mixed method applied to (2.56)-(2.58), (2.61). The particular form of the scheme we are going to analyze is given by (2.81)-(2.84). Here we take a different approach than in previous sections. This approach, suggested to us by B. Engquist, can also be used to derive error estimates for the first-order Godunov scheme for \( a \equiv 0 \), assuming smooth flow.

With only slight modification to the proof of Lemma 3.2, one can show that the GMM approximation \( S \) satisfies in this case

\[
\begin{align*}
\max_j S_j^{n+1} &\leq \max_j S_j^n + \Delta t \sup_I r(x, t^{n+1}, S_j^{n+1}(x)), \\
\min_j S_j^{n+1} &\geq \min_j S_j^n + \Delta t \inf_I r(x, t^{n+1}, S_j^{n+1}(x)).
\end{align*}
\]

Thus for any \( n \geq 1 \) and \( 1 \leq j \leq J - 1 \),

\[
T \inf_{Q_T} r(x, t, S) + \min_j S_j^n \leq S_j^n \leq \max_j S_j^n + T \sup_{Q_T} r(x, t, S).
\]

Assuming \( r \) is bounded, we have boundedness of \( S \).

Assume uniform mesh and assume \( \Delta t^n \equiv \Delta t \), with \( \Delta t \) satisfying

\[
\sup_{S} f'(s) \frac{\Delta t}{\Delta x} \leq 1,
\]

(4.336)

where \( S \subset \mathbb{R} \) is defined by

\[
S = [\min_{Q_T}(\inf S, \min_j S_j^n), \max_{Q_T}(\sup S, \max_j S_j^n)].
\]
The true solution $s$ to (2.56)-(2.58), (2.61) satisfies for $j = 1, \ldots, J-1$ and $n = 0, \ldots, N^*$,

$$s^{n+1}_{j,t} + (f(s^n_{j,L}))_x - \left( a^{n+1}_{j+\frac{1}{2}}(s^{n+1}_{j+\frac{1}{2}}) s^{n+1}_{j,x} \right)_x = r^{n+1}_j + E^n_j. \tag{4.337}$$

Here $r^{n+1}_j \equiv r(x_j, t^{n+1}, s^{n+1}_j)$, and

$$E^n_j = E^n_{T,j} + (E^n_{A,j})_x + (E^n_{D,j})_x + E^n_{R,j}, \tag{4.338}$$

where $E^n_{T,j}$ is given by (4.4), $E^n_{A,j}$ is given by (4.5),

$$E^n_{D,j} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} a(x_{j+\frac{1}{2}}, t, s(x_{j+\frac{1}{2}}, t)) \frac{\partial s(x_{j+\frac{1}{2}}, t)}{\partial x} dt - a^{n+1}_{j+\frac{1}{2}}(s^{n+1}_{j+\frac{1}{2}}) s^{n+1}_{j,x}, \tag{4.339}$$

and

$$E^n_{R,j} = \frac{1}{\Delta x \Delta t} \int_{t^n}^{t^{n+1}} \int_{B_j} r(x, t, s) dx dt - r^{n+1}_j. \tag{4.340}$$

In the above equations, for $n = 0, \ldots, N^*$,

$$s^n_j = s(x_j, t^n), \quad j = 1, \ldots, J-1, \tag{4.341}$$

$$s^n_0 = 2g^n_0 - s^n_1, \quad s^n_j = 2g^n_1 - s^n_{j-1}, \tag{4.342}$$

$$s^n_{0,L} = g^n_0, \tag{4.343}$$

$$s^n_{j,L} = s^n_j, \quad j = 1, \ldots, J-1, \tag{4.344}$$

and

$$s^{n+1}_{j+\frac{1}{2}} = \frac{s^n_j + s^{n+1}_{j+1}}{2}, \quad j = 0, \ldots, J-1. \tag{4.345}$$

Define

$$\bar{s}^{n+1}_j = s^n_j - \Delta t(f(s^n_{j,L}))_x. \tag{4.346}$$
and let
\[ \eta_j^n = S_j^n - S_j^p, \]  
\[ \bar{\eta}_j^n = \bar{S}_j^n - \bar{S}_j^p. \]  

Then, by (4.346), (4.343)-(4.344), and (2.81)-(2.82),
\[ \bar{\eta}_j^{n+1} = \eta_j^n - \Delta t (A_j^n \eta_j^n)_{x}, \]  
where
\[ A_j^n = \begin{cases} 
(f(s_j^n) - f(S_j^n))/\eta_j^n, & j = 1, \ldots, J - 1, \eta_j^n \neq 0, \\
0, & j = 0, \text{ or } \eta_j^n = 0. \end{cases} \]  

Let
\[ \beta_j^n = \frac{\Delta t}{\Delta x} A_j^n. \]  

Then \( \beta_0^n = 0, \beta_j^n \geq 0, \) and by applying the Mean Value Theorem to (4.350) and using (4.336), we have
\[ \beta_j^n = \frac{\Delta t}{\Delta x} f'(\bar{\xi}_j^n) \leq 1, \]  
where \( \bar{\xi}_j^n \) is some point between \( s_j^n \) and \( S_j^n. \)

By (4.349)-(4.352) then
\[ |\bar{\eta}_j^{n+1}| \leq (1 - \beta_j^n) |\eta_j^n| + \beta_{j-1}^n |\eta_{j-1}^n|. \]  

Multiplying above by \( \Delta x \) and summing on \( j, j = 1, \ldots, J - 1, \) we have
\[ ||\bar{\eta}^{n+1}||_{L^1} \leq \sum_{j=1}^{J-1} (1 - \beta_j^n) |\eta_j^n| \Delta x + \sum_{j=1}^{J-1} \beta_{j-1}^n |\eta_{j-1}^n| \Delta x \]
\[ = \sum_{j=1}^{J-1} (1 - \beta_j^n) |\eta_j^n| \Delta x + \sum_{j=1}^{J-2} \beta_j^n |\eta_j^n| \Delta x \]
\[ = \sum_{j=1}^{J-1} \kappa_j^n |\eta_j^n| \Delta x, \]  
(4.353)
since $\beta_0^n = 0$, where

$$\kappa_j^n = \begin{cases} 
1, & j = 1, \ldots, J - 2, \\
1 - \beta_{j-1}^n, & j = J - 1.
\end{cases}$$

Since $0 \leq \kappa_j^n \leq 1$, we obtain from (4.353),

$$||\bar{\eta}^{n+1}||_{L^1} \leq ||\eta^n||_{L^1}. \quad (4.354)$$

Now, subtracting (2.84) from (4.337) we obtain

$$\eta_j^{n+1} = \bar{\eta}_j^{n+1} + \Delta t \left( a_{j+\frac{1}{2}}^{n+1} \frac{(S_{j+\frac{1}{2}}^{n+1}) \eta_j^{n+1}}{\nu_j^{n+1}} \right)_x$$
$$+ \Delta t \left( a_{j+\frac{1}{2}}^{n+1} \frac{(\bar{s}_{j+\frac{1}{2}}^{n+1}) - a_{j+\frac{1}{2}}^{n+1} (S_{j+\frac{1}{2}}^{n+1}) \eta_j^{n+1}}{\nu_j^{n+1}} \right)_x$$
$$+ \Delta t R_j^{n+1} \eta_j^{n+1} + \Delta t E_j^n, \quad (4.355)$$

where

$$R_j^{n+1} = \begin{cases} 
(r(x_j, t^{n+1}, s_j^{n+1}) - r(x_j, t^{n+1}, S_j^{n+1}))/\eta_j^{n+1}, & \text{if } \eta_j^{n+1} \neq 0, \\
0, & \text{otherwise}.
\end{cases} \quad (4.356)$$

Let $\eta_j^{n+1} = s_j^{n+1} - S_j^{n+1}$ and let

$$\sigma_{j+\frac{1}{2}}^{n+1} = \begin{cases} 
(a_{j+\frac{1}{2}}^{n+1} \frac{(s_{j+\frac{1}{2}}^{n+1}) - a_{j+\frac{1}{2}}^{n+1} (S_{j+\frac{1}{2}}^{n+1})}{\eta_j^{n+1}})/\eta_j^{n+1}, & \text{if } \eta_j^{n+1} \neq s_j^{n+1}, \\
0, & \text{otherwise}.
\end{cases} \quad (4.357)$$

Then

$$\sigma_{\frac{1}{2}}^{n+1} = \sigma_{-\frac{1}{2}}^{n+1} = 0, \quad (4.358)$$

by (4.342) and (2.37)-(2.38). Moreover,

$$\eta_j^{n+1} = \bar{\eta}_j^{n+1} + \Delta t \left( a_{j+\frac{1}{2}}^{n+1} \frac{(S_{j+\frac{1}{2}}^{n+1}) \eta_j^{n+1}}{\nu_j^{n+1}} \right)_x$$
$$+ \frac{\Delta t}{2} \left( a_{j+\frac{1}{2}}^{n+1} (\eta_j^{n+1} + \eta_j^{n+1}) s_j^{n+1} \right)_x$$
$$+ \Delta t R_j^{n+1} \eta_j^{n+1} + \Delta t E_j^n.$$
Finally, defining

\[ \mu^{n+1}_{j+\frac{1}{2}} = \frac{\Delta t}{\Delta x^2} a^{n+1}_{j+\frac{1}{2}} (s^{n+1})_{j+\frac{1}{2}}, \quad (4.359) \]
\[ \gamma^{n+1}_{j+\frac{1}{2}} = \frac{\Delta t}{2\Delta x} \sigma^{n+1}_{j+\frac{1}{2}} s^{n+1}_{j+\frac{1}{2}}, \quad (4.360) \]

we have

\[
(1 + \mu^{n+1}_{j-\frac{1}{2}} + \mu^{n+1}_{j+\frac{1}{2}} - \gamma^{n+1}_{j+\frac{1}{2}} + \gamma^{n+1}_{j-\frac{1}{2}}) \eta^{n+1}_j \\
= \eta^{n+1}_j + (\mu^{n+1}_{j+\frac{1}{2}} + \gamma^{n+1}_{j+\frac{1}{2}}) \eta^{n+1}_{j+1} + (\mu^{n+1}_{j-\frac{1}{2}} - \gamma^{n+1}_{j-\frac{1}{2}}) \eta^{n+1}_{j-1} \\
+ R^{n+1}_j \eta^{n+1}_j \Delta t + E^{n+1}_j \Delta t. \quad (4.361)
\]

Assume that, either \( a \equiv 0 \), or

1. positive constants \( a_* \) and \( a^* \) exist such that \( 0 < a_* \leq a(x, t, s) \leq a^* \),

2. \( a \) is Lipschitz continuous in \( s \) with Lipschitz constant \( L_a \),

3. \( s \) is Lipschitz continuous in \( x \) with Lipschitz constant \( L_s \),

4. \( \Delta x \) is sufficiently small so that

\[ \Delta x \leq \frac{2a_*}{L_a L_s}. \]

Then, by (4.360), (4.357), (4.359), and Assumptions (1)-(4),

\[ \left| \gamma^{n+1}_{j+\frac{1}{2}} \right| \leq \frac{\Delta t}{2\Delta x} L_a L_s \leq \frac{\Delta t}{\Delta x^2} a_* \leq \mu^{n+1}_{j+\frac{1}{2}}. \quad (4.362) \]

Thus, the coefficients of \( \eta^{n+1}_j, \eta^{n+1}_{j+1}, \eta^{n+1}_{j-1} \) in (4.361) are all positive.

Taking the absolute value of both sides in (4.361), multiplying by \( \Delta x \) and summing on \( j \) we obtain...
\[
\sum_{j=1}^{J-1} \left( 1 + \mu_{j+\frac{1}{2}}^{n+1} + \mu_{j+\frac{1}{2}}^{n+1} - \gamma_{j+\frac{1}{2}}^{n+1} + \gamma_{j-\frac{1}{2}}^{n+1} \right) \left| \eta_{j}^{n+1} \right| \Delta x \\
\leq ||\eta^{n+1}||_{L^1} + \sum_{j=1}^{J-1} (\mu_{j+\frac{1}{2}}^{n+1} + \gamma_{j+\frac{1}{2}}^{n+1}) \left| \eta_{j}^{n+1} \right| \Delta x + \sum_{j=1}^{J-1} (\mu_{j-\frac{1}{2}}^{n+1} - \gamma_{j-\frac{1}{2}}^{n+1}) \left| \eta_{j}^{n+1} \right| \Delta x \\
+ \Delta t \left( ||R^{n+1} \eta^{n+1}||_{L^1} + ||E^n||_{L^1} \right) \\
= ||\eta^{n+1}||_{L^1} + \Delta t \left( ||R^{n+1} \eta^{n+1}||_{L^1} + ||E^n||_{L^1} \right) \\
+ \sum_{j=1}^{J-1} (\mu_{j+\frac{1}{2}}^{n+1} + \mu_{j-\frac{1}{2}}^{n+1} - \gamma_{j+\frac{1}{2}}^{n+1} + \gamma_{j-\frac{1}{2}}^{n+1}) \left| \eta_{j}^{n+1} \right| \Delta x + B^{n+1} \Delta x, \tag{4.363}
\]

by summation by parts. Here

\[
B^{n+1} = \mu_{j+\frac{1}{2}}^{n+1} \left| \eta_{j+\frac{1}{2}}^{n+1} \right| - \mu_{j+\frac{1}{2}}^{n+1} \left| \eta_{j+\frac{1}{2}}^{n+1} \right| + \gamma_{j+\frac{1}{2}}^{n+1} \left| \eta_{j+\frac{1}{2}}^{n+1} \right| - \gamma_{j+\frac{1}{2}}^{n+1} \left| \eta_{j+\frac{1}{2}}^{n+1} \right| \\
- \mu_{j-\frac{1}{2}}^{n+1} \left| \eta_{j-\frac{1}{2}}^{n+1} \right| + \mu_{j-\frac{1}{2}}^{n+1} \left| \eta_{j-\frac{1}{2}}^{n+1} \right| + \gamma_{j-\frac{1}{2}}^{n+1} \left| \eta_{j-\frac{1}{2}}^{n+1} \right| - \gamma_{j-\frac{1}{2}}^{n+1} \left| \eta_{j-\frac{1}{2}}^{n+1} \right| \\
= 0, \tag{4.364}
\]

since by (4.342), (2.37)-(2.38), and (4.358), (4.360), we have \( \left| \eta_{0}^{n+1} \right| = \left| \eta_{1}^{n+1} \right|, \left| \eta_{J}^{n+1} \right| = \left| \eta_{J-1}^{n+1} \right|, \) and \( \gamma_{n+1}^{n+1} = \gamma_{n+1}^{n+1} = 0. \)

Thus, by (4.363), (4.364), and (4.354),

\[
||\eta^{n+1}||_{L^1} \leq ||\eta^n||_{L^1} + \Delta t ||R^{n+1}||_{L^\infty} ||\eta^{n+1}||_{L^1} + \Delta t ||E^n||_{L^1}. 
\]

Summing on \( n \) above we obtain

\[
||\eta^{N}||_{L^1} \leq ||R||_{L^\infty(L^\infty)} \sum_{n=0}^{N} ||\eta^n||_{L^1} \Delta t + ||E||_{L^1(L^1)} + ||\eta^0||_{L^1}, \tag{4.365}
\]

where \( N \leq N^* \) is arbitrary.

As we have seen previously, we have

\[
||\eta^0||_{L^1} \leq C_0 \Delta x^2, \tag{4.366}
\]

where

\[
C_0 = C \left( ||d^2s^0/dx^2||_{L^1(I)} \right). \tag{4.367}
\]
Moreover, we have the following theorem, which is derived in Appendix C.

**Theorem 4.15** Let \( f, a, s^0, g_0, g_1, \) and \( r \) satisfy the following regularity assumptions:

(i) \( f \in C^2(\mathbb{R}), f' \geq 0, \)

(ii) \( a \in C^2(I \times [0,T] \times \mathbb{R}), \)

(iii) \( r \in C^1(I \times [0,T] \times \mathbb{R}), \)

(iv) \( s^0 \in C^1(I), \frac{\partial s^0}{\partial x} \in L^1(I), \) and \( \frac{\partial^2 s^0}{\partial x^2} \in L^1(I), \)

(v) \( g_0, g_1 \in C^1(0,T), g_0', g_1' \in L^1(0,T). \)

Let \( s \) have the following regularity:

(i) \( s \) and all partial derivatives of \( s \) of order \( \leq 2 \) exist and are bounded in \( Q_T, \)

(ii) \( \frac{\partial^2 s}{\partial x^2} \) and \( \frac{\partial^2 s}{\partial x^2} \) exist in \( Q_T \) and are in \( L^\infty(0,T;L^1(I)) \), and \( \frac{\partial^2 s}{\partial x^2} \) exists in \( Q_T \) and is in \( L^1(0,T;L^\infty(I)) \).

Then \( E \) given by (4.338) satisfies

\[
\|E\|_{L^1(I)} \leq C_E(\Delta x + \Delta t), \tag{4.368}
\]

where

\[
C_E = C(C_T, C'_A, C''_A, C_D, C''_D, C''', C_R) \tag{4.369}
\]

with \( C_T \) given by (C.10), \( C'_A \) by (C.18), \( C''_A \) by (C.22), \( C'_D \) by (C.26), \( C''_D \) by (C.28), \( C'''_D \) by (C.30), and \( C_R \) by (C.34).

Thus, by (4.365)-(4.369), (4.356), and assuming \( r \) is Lipschitz continuous in \( s \), we have

\[
\|\eta^N\|_{L^1} \leq L_T \sum_{n=0}^{N} \|\eta^n\|_{L^1} \Delta t + C_E(\Delta x + \Delta t) + C_0 \Delta x^2, \tag{4.370}
\]
where $L_r$ is a Lipschitz constant for $r$. Thus, by Gronwall’s Lemma applied to (4.370) we obtain the following theorem.

**Theorem 4.16** Let the hypothesis of Theorem 4.15 hold. Let Assumptions (1)-(4) hold. Let $\Delta t$ satisfy (4.336), and let $L_r$ be a Lipschitz constant for $r$ with respect to $s$. Then

$$||s - S||_{L^\infty(L)} \leq C_\eta(\Delta x + \Delta t),$$

where

$$C_\eta = C(e^{L_r}, C_E, C_0).$$

Finally, we note that the estimate above is indirectly dependent on the lower bound $a_\ast$ of $a(x, t, s)$ through Assumption (4). However, if $a = a(x, t)$; i.e., $a$ does not depend on $s$, Assumptions (2)-(4) are unnecessary and our estimate thus holds for any sufficiently smooth, nonnegative $a(x, t)$.

We also want to remark that, while the estimates for the first-order Godunov procedure are of the same order as those obtained for the higher-order MUSCL scheme, the MUSCL scheme works much better in practice. The superior performance of the higher-order scheme can be traced to our heuristic arguments given in Remarks 3.1 and 3.2. In particular, the first-order scheme is truly locally and globally first-order, while the MUSCL scheme is only first-order in areas where the solution is particularly sharp and slope-limiting is in effect. Obtaining more precise estimates which reflect this behavior is something we hope to do in the near future.
Appendix A

Truncation Error—Dirichlet

In this section we analyze the truncation error terms in (4.69) for $s$ satisfying (2.1)-(2.3), (2.6). Bounds for these terms in $L^2(L^2)$ are derived.

For simplicity, we denote $\frac{\partial s}{\partial x}$ by $\partial_x s$, $\frac{\partial s}{\partial t}$ by $\partial_t s$, $\frac{\partial^2 s}{\partial x^2}$ by $\partial_{xx}^2 s$, etc. We will assume throughout that $s$ has as many derivatives as we need, but we make an effort to keep the dependence on the smoothness of $s$ as minimal as possible.

Recall from (4.69),

\[(TE)^2 = ||E_A'||^2_{L^2(L^2)} + ||E_{AR}'||^2_{L^2(L^2)} + ||E_{AL}'||^2_{L^2(L^2)} + ||E_{DL}'||^2_{L^2(L^2)} + ||E_T||^2_{L^2(L^2)},\]

where

\[E_{A,j}^n = f(s_{j,L}^n) - f(s_{j+\frac{1}{2}}^{n+\frac{1}{2}}),\]

\[E_{AR,j}^n = f(s_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} f(s(x_{j+\frac{1}{2}}, t))dt,\]

\[E_{AL,j}^n = E_{AR,j-1}^n,\]
\[ E_{DR,j}^n = \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} \left( a \frac{\partial s}{\partial x} \right) (x_{j+\frac{1}{2}}, t) dt - a_{j+\frac{1}{2}}^{n+1} s_{j+\frac{1}{2}}^{n+1}, \] (A.5)

and

\[ E_{DL,j}^n = E_{DR,j-1}^n, \] (A.6)

\[ E_{T,j}^n = \frac{1}{\Delta t^n} \left( s_j^{n+1} - \frac{1}{\Delta x_j} \int_{B_j} s^{n+1} dx - \left( s_j^n - \frac{1}{\Delta x_j} \int_{B_j} s^n dx \right) \right). \] (A.7)

Let

\[ \Delta x = \max_j \Delta x_j \]

and

\[ \Delta t = \max_n \Delta t^n, \]

and assume \( \Delta x_j \) is at least quasi-uniform; i.e., there exist positive constants \( c_*, c^* \) which are independent of the mesh such that

\[ c_* \leq \frac{\Delta x_j}{\Delta x_i} \leq c^*, \quad i, j = 1, \ldots, J - 1. \] (A.8)

**Bound for \( E_T \).** We begin by bounding \( ||E_T||_{L^2(L^1)} \). By Taylor's formula with integral remainder

\[ s_j^{n+1} - \frac{1}{\Delta x_j} \int_{B_j} s^{n+1} dx = \frac{1}{\Delta x_j} \partial_{xx}s^{n+1}(x_j) \int_{B_j} \frac{(x - x_j)^2}{2} dx \]

\[ + \frac{1}{2\Delta x_j} \int_{B_j} \int_{x_j}^x (y - x_j)^2 \partial_{xxx}s^{n+1}(y) dy \]

\[ = \frac{\Delta x_j^2}{24} \partial_{xx}s^{n+1}(x_j) + W_{1,j}^{n+1}, \]

where

\[ |W_{1,j}^{n+1}| \leq C \Delta x_j^\frac{1}{2} ||\partial_{xxx}s^{n+1}||_{L^2(B_j)}. \]
Thus
\[
|E_{T,j}^n| \leq C \Delta x^2 \left| \int_{t^n}^{t^{n+1}} \partial_{xx}^3 s(x_j, t) dt \right| \\
+ \frac{1}{\Delta t^n} \left( |W_{1,j}^{n+1}| + |W_{1,j}^n| \right) \\
\leq \frac{C \Delta x^2}{(\Delta t^n)^2} \|\partial_{xx}^3 s(x_j)\|_{L^2(t^n, t^{n+1})} \\
+ \frac{C \Delta x^{3/2}}{\Delta t^n} \left( \|\partial_{xxx}^3 s^{n+1}\|_{L^2(B_j)} + \|\partial_{xxx}^3 s^n\|_{L^2(B_j)} \right).
\]

Hence, since $\Delta t^n = O(\Delta x)$,
\[
\|E_T\|^2_{L^2(L^2)} \leq C \Delta x^4 \sum_{j=1}^{J-1} \left( \|\partial_{xx}^3 s(x_j)\|^2_{L^2(0,T)} \Delta x_j \\
+ C \Delta x^4 \sum_{n=0}^{N+1} \left( \|\partial_{xxx}^3 s^{n+1}\|^2_{L^2(I)} + \|\partial_{xxx}^3 s^n\|^2_{L^2(I)} \right) \Delta t^n \right) \\
\leq (C_T)^2 \Delta x^4, \tag{A.9}
\]

where
\[
C_T = C(\|\partial_{xx}^3 s\|_{L^2(0,T;L^\infty(I))}, \|s^0\|_{H^2(I)}, \|\partial_{xxx}^3 s\|_{L^\infty(0,T;L^2(I))}). \tag{A.10}
\]

Estimate of $E_{A'}^n$. Next consider
\[
E_{A',j}^n = f(s_{j,L}^n) - f(s_{j+1,\frac{1}{2}}^{n+\frac{1}{2}}) \\
= f'(\bar{s})(s_{j,L}^n - s_{j+1,\frac{1}{2}}^{n+\frac{1}{2}}), \tag{A.11}
\]

where $\bar{s}$ is a point between $s_{j,L}^n$ and $s_{j+1,\frac{1}{2}}^{n+\frac{1}{2}}$.

First, let $n > 0$, then by Taylor expansion and the differential equation (2.1), we have

for $n > 0$,
\[
s_{j+1,\frac{1}{2}}^{n+\frac{1}{2}} = s_{j,L}^n + \frac{\Delta x_j}{2} \partial_x s^n(x_j) + \frac{\Delta t^n}{2} \partial_t s^n(x_j) \\
+ \left( \frac{\Delta x_j}{2} \partial_x + \frac{\Delta t^n}{2} \partial_t \right)^2 s(\bar{z}, \bar{t})
\]
\[
\begin{align*}
&= s^n_j + \frac{\Delta x_j}{2} (1 - h_j(s^n_j)) \partial_x s^n_j - \frac{\Delta t^n}{2} \partial_x (a^n_j \partial_x s^n_j) \\
&\quad + \left( \frac{\Delta x_j}{2} \partial_x + \frac{\Delta t^n}{2} \partial_t \right)^2 s(\bar{x}, \bar{t}),
\end{align*}
\]
(A.12)

where
\[
h_j(s^n_j) = f'(s^n_j) \frac{\Delta t^n}{\Delta x_j},
\]
(A.13)

and \((\bar{x}, \bar{t})\) is a point on the line between \((x_j, t^n)\) and \((x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}})\).

Thus, recalling
\[
s^n_{j,L} = s^n_j + \frac{\Delta x_j}{2} (1 - h_j(s^n_j)) \delta s^n_j, \quad j = 1, \ldots, J - 1,
\]
we have
\[
\begin{align*}
s^n_{j+\frac{1}{2}} - s^n_{j,L} &= \frac{\Delta x_j}{2} (1 - h_j(s^n_j)) (\partial_x s^n_j - \delta s^n_j) - \frac{\Delta t^n}{2} \partial_x (a^n_j \partial_x s^n_j) \\
&\quad + O(\Delta x^2 + \Delta t^2).
\end{align*}
\]
(A.14)

Recall from Sections 2.1 and 2.2, that for \(j = 2, \ldots, J - 2\),
\[
\delta s^n_j = \frac{s^n_{j+1} - (1 - \mu^2_j) s^n_j - \mu^2_j s^n_{j-1}}{\mu_j (\Delta x_{j-\frac{1}{2}} + \Delta x_{j+\frac{1}{2}})},
\]
where
\[
\mu_j = \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}}.
\]

By Taylor expansion,
\[
s^n_{j+1} - s^n_j = \Delta x_{j+\frac{1}{2}} \partial_x s^n_j + \frac{\Delta x_{j+\frac{1}{2}}}{2} \partial^2_{xx} s(\bar{x}_{j+1}),
\]
(A.15)

and
\[
s^n_{j-1} - s^n_j = -\Delta x_{j-\frac{1}{2}} \partial_x s^n_j + \frac{\Delta x_{j-\frac{1}{2}}}{2} \partial^2_{xx} s(\bar{x}_j),
\]
(A.16)
where \( \bar{x}_{j+1} \) is a point between \( x_j \) and \( x_{j+1} \), similarly for \( \bar{x}_j \). Thus

\[
|\delta s^n_j - \partial_x s^n_j| \leq \frac{C(\Delta x_{j+\frac{1}{2}}^2 + \Delta x_{j-\frac{1}{2}}^2)}{\mu_j(\Delta x_{j+\frac{1}{2}} + \Delta x_{j-\frac{1}{2}})} \|\partial_{xx} s^n\|_{L^\infty(\Omega_T)} \leq C \Delta x \|\partial_{xx} s^n\|_{L^\infty(\Omega_T)} \tag{A.17}
\]

by the quasi-uniformity assumption (A.8).

For \( j = 1 \), recall

\[
\delta s^n_1 = \frac{s^n_2 - (1 - 4\mu^2) s^n_1 - 4\mu^2 g_0^n}{\mu_1(\Delta x_1 + 2\Delta x_{1/2})},
\]

where

\[
\mu_1 = \frac{\Delta x_{1/2}}{\Delta x_1}.
\]

Thus by Taylor expansion

\[
|\delta s^n_1 - \partial_x s^n(x_1)| \leq C \Delta x \|\partial_{xx} s^n\|_{L^\infty(\Omega_T)}. \tag{A.18}
\]

Similarly

\[
|\delta s^n_{j-1} - \partial_x s^n(x_{j-1})| \leq C \Delta x \|\partial_{xx} s^n\|_{L^\infty(\Omega_T)}. \tag{A.19}
\]

Note above that we could derive a \( O(\Delta x^2) \) estimate for \( |\delta s^n_j - \partial_x s^n(x_j)| \) by a third-order Taylor expansion. However, this isn't necessary to obtain a \( O(\Delta x^2 + \Delta t) \) estimate for \( E_A \).

In fact, as we can see above we can obtain such an estimate for \( E_A \) if we only approximate \( \partial_x s^n(x_j) \) by a one-sided difference. Thus, the observation in Remark 3.2 is valid.

For \( j = 0 \), we have either

\[
s^n_{\frac{1}{2}} - s^n_{0,L} = g^n_{0,L} - g^n_{0} = 0, \tag{A.20}
\]

or

\[
s^n_{\frac{1}{2}} - s^n_{0,L} = g^n_{0,L} - g^n_{0} \leq C \Delta t \|g_0^n\|_{L^\infty(0,T)}, \tag{A.21}
\]
depending on whether \( S_{0,L}^n = g_0^{n+\frac{1}{2}} \), or \( S_{0,L}^n = g_0^n \).

Thus, by (A.14)-(A.21), for \( n > 0 \), and \( j = 0, \ldots, J - 1 \),

\[
|s_j^{n+\frac{1}{2}} - s_j^n| \leq C(\Delta x^2 + \Delta t), \tag{A.22}
\]

where

\[
C = C(\|s\|_{\mathcal{W}_x^2(Q_T)}, \|g_0\|_{\mathcal{W}_x^1(0,T)}, \|\partial_x(a \partial_x s)\|_{L^\infty(Q_T)}). \tag{A.23}
\]

For \( n = 0 \), \( s^0 \) doesn’t necessarily satisfy (2.1), thus for \( j > 0 \),

\[
s_j^{\frac{3}{2}} - s_j^0, \quad \Delta x_j \frac{\Delta t_j}{2} (1 - h_j(s_j^0))(\partial_x s_j^0 - \delta s_j^0) + \Delta x_j h_j(s_j^0) \partial_x s_j^0 + \frac{\Delta t_j}{2} \partial_t s_j^0 + O(\Delta x^2 + \Delta t^2)
\]

\[
\leq C(\Delta x^2 + \Delta t^0), \tag{A.24}
\]

since \( h_j(s_j^0) \Delta x_j = f'(s_j^0) \Delta t^0 \), where

\[
C = C(\sup_{Q_T} f'(s), \|s^0\|_{\mathcal{W}_x^1(L)}, \|s\|_{\mathcal{W}_x^2(Q_T)}). \tag{A.25}
\]

Thus, by (A.11), (A.22)-(A.25),

\[
\|E'_A\|_{L^2(L^2)} \leq C'_A(\Delta x^2 + \Delta t), \tag{A.26}
\]

where

\[
C'_A = C(\sup_{s} f'(s), \|s^0\|_{\mathcal{W}_x^1(L)}, \|s\|_{\mathcal{W}_x^2(Q_T)},
\|\partial_x(a \partial_x s)\|_{L^\infty(Q_T)}, \|g_0\|_{\mathcal{W}_x^1(0,T)}), \tag{A.27}
\]

and where \( S \subset \mathbb{R} \) is such that

\[
S \supset [\min_{j,n}(\min_{j,n} s_{j,L}^n, \min_{j,n} s_{j+\frac{1}{2},L}^n), \max_{j,n}(\max_{j,n} s_{j,L}^n, \max_{j,n} s_{j+\frac{1}{2},L}^n)].
\]
A crude bound for $s_{j,L}^n$ gives

$$|s_{j,L}^n| \leq \frac{2(1 + c^*)}{1 + c_*} ||s||_{L^\infty(Q_T)} = s^*.$$ 

Thus, we can set $\mathcal{S} = [-s^*, s^*]$.

Estimates for $E_{AR}''$ and $E_{AL}''$. Now, consider

$$E_{AR,j}'' = f(s_{j+\frac{1}{2}}^{n+1}) - \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} f(s(x_{j+\frac{1}{2}}, t)) dt, \quad j = 1, \ldots, J - 1.$$ 

By Taylor expansion,

$$|E_{AR,j}''|^n \leq C \Delta t^2 ||\partial_t^2 f(s(x_{j+\frac{1}{2}}, \cdot))||_{L^\infty(0,T)}$$

$$\leq \Delta t^2 C \left( \sup_{Q_T} |f'(s)|, \sup_{Q_T} |f''(s)|, ||s||_{W^2_2(Q_T)}, ||s||_{W^2_2(0,T)} \right)$$

$$\equiv T^{-1} C_{AR}'\Delta t^2.$$  \hspace{1cm} (A.28)

Thus

$$||E_{AR}''||_{L^2(L^2)} \leq C_{AR}'\Delta t^2.$$ \hspace{1cm} (A.29)

Similarly,

$$||E_{AL}''||_{L^2(L^2)} \leq C_{AL}'\Delta t^2.$$ \hspace{1cm} (A.30)

where

$$C_{AL}' = C \left( \sup_{Q_T} |f'(s)|, \sup_{Q_T} |f''(s)|, ||s||_{W^2_2(Q_T)}, ||s||_{W^2_2(0,T)} \right).$$ \hspace{1cm} (A.31)

Estimates for $E_{DR}$ and $E_{DL}$. Finally, consider

$$E_{DR,j}'' = a_{j+\frac{1}{2}}^{n+1} s_{j+\frac{1}{2}}^{n+1} - s_{j+\frac{1}{2}}^{n+1} - \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} \left( \frac{\partial s}{\partial x} (x_{j+\frac{1}{2}}, t) \right) dt$$

$$= a_{j+\frac{1}{2}}^{n+1} \left( s_{j+\frac{1}{2}}^{n+1} - s_{j+\frac{1}{2}}^{n+1} - \partial_x s_{j+\frac{1}{2}}^{n+1} \right) + a_{j+\frac{1}{2}}^{n+1} \partial_x s_{j+\frac{1}{2}}^{n+1} - \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} \left( \frac{\partial s}{\partial x} (x_{j+\frac{1}{2}}, t) \right) dt$$

$$\equiv V_{1,j}'' + V_{2,j}''.$$ \hspace{1cm} (A.32)
For \( j = 1, \ldots, J - 2 \), we have

\[
\left| V^n_{1,j} \right| \leq \begin{cases} 
C \Delta x^{\frac{3}{2}} \| \partial^2_{x\overline{x}x}s^{n+1} \|_{L^2(x_j, x_{j+1})}, & \text{if } \Delta x \text{ is uniform,} \\
C \Delta x \| \partial^2_{x\overline{x}s} \|_{L^\infty(Q_T)}, & \text{if } \Delta x \text{ is nonuniform.}
\end{cases}
\]  

(A.33)

For \( j = J - 1 \), recall

\[
\frac{s_j^{n+1} - s_j^{n+1}}{\Delta x_{J-\frac{1}{2}}} = 2 \frac{(g_1^{n+1} - s_j^{n+1})}{\Delta x_{J-1}}.
\]

Thus

\[
\left| V^n_{1,J-1} \right| \leq C \Delta x \| \partial^2_{x\overline{x}s} \|_{L^\infty((x_{J-1,1}) \times (0,T))}.
\]  

(A.34)

Next, we have

\[
\left| V^n_{2,j} \right| \leq C \Delta t \| \partial_t (a \partial_x s) \|_{L^\infty(Q_T)}
\]  

(A.35)

by Taylor expansion. Hence by (A.32)-(A.35), we have for uniform \( \Delta x \)

\[
\| E_{\text{DR}} \|^2_{L^2(L^2)} \leq C \sum_{n=0}^{N^*-1} \left| V^n_{1,J-1} \right|^2 \Delta x \Delta t^n + C \sum_{n=0}^{N^*-1} \sum_{j=1}^{J-2} \left| V^n_{1,j} \right|^2 \Delta x \Delta t^n + C \| V^n_2 \|^2_{L^2(L^2)}
\]

\[
\leq C (\Delta x^3 + \Delta x^4 + \Delta t^2).
\]

Thus

\[
\| E_{\text{DR}} \|_{L^2(L^2)} \leq C_{\text{DR}} (\Delta x^{\frac{3}{2}} + \Delta t),
\]  

(A.36)

where

\[
C_{\text{DR}} = C (\| \partial^2_{x\overline{x}x}s \|_{L^\infty(0,T; L^2(I))}, \| s \|_{W^2_\infty((1-\epsilon,1) \times (0,T))},
\]

\[
\| \partial_t (a \partial_x s) \|_{L^\infty(Q_T))},
\]  

(A.37)
for \( \epsilon > 0 \) and \( \Delta x \) sufficiently small so that \( \Delta x < \epsilon \).

For nonuniform \( \Delta x \), we have

\[
\|E_{DR}\|_{L^2(L^2)} \leq C_{DR}(\Delta x + \Delta t),
\]
(A.38)

where

\[
C_{DR} = C(\|\partial^2_{xx}s\|_{L^\infty(Q_T)}, \|\partial_t(a\partial_x s)\|_{L^\infty(Q_T)}).
\]
(A.39)

For nonuniform \( \Delta x \), we obtain by a similar argument

\[
\|E_{DL}\|_{L^2(L^2)} \leq C_{DL}(\Delta x + \Delta t),
\]
(A.40)

where

\[
C_{DL} = C_{DR}.
\]
(A.41)

For uniform \( \Delta x \), we obtain

\[
\|E_{DL}\|_{L^2(L^2)} \leq C_{DL}(\Delta x^{\frac{1}{2}} + \Delta t),
\]
(A.42)

where

\[
C_{DL} = C(\|\partial^3_{xxx}s\|_{L^\infty(0,T; L^2(I))}, \|s\|_{W^3_\beta((0,\epsilon) \times (0,T))}, \|\partial_t(a\partial_x s)\|_{L^\infty(Q_T)}),
\]
(A.43)

for \( \epsilon > 0 \) and \( \Delta x \) sufficiently small so that \( \Delta x < \epsilon \).
Appendix B

**TE for no-flow at** \(x = 1\).

The truncation error for the problem with a Dirichlet boundary condition at \(x = 0\) and a Neumann boundary condition at \(x = 1\) varies only slightly from that given in Appendix A.

The estimate for \(E_T\) remains the same, i.e. (A.9) and (A.10) hold.

The estimate for \(E_A'\) varies only in the term

\[
f(s_{-1,L}^n) - f(s_{-1/2}^{n+1}).
\]

Assume \(S_{-1,L}^n = S_{-1}^n\), then \(s_{-1,L}^n = s_{-1}^n\). Recalling \(\delta s_{-1}^n = 0\), we have, following (A.12) and (A.14),

\[
s_{-1/2}^{n+1} - s_{-1,L}^n = \frac{\Delta x_{-1}}{2}(1 - h_{-1}(s_{-1}^n)) \frac{\Delta t^n}{2} \partial_x s_{-1}^n - \frac{\Delta t^n}{2} \partial_x (a_{-1} s_{-1}^n s_{-1}^n) + O(\Delta x^2 + \Delta t^2).
\]

Since \(\partial_x s(1, t^n) = 0\), we have

\[
\left| s_{-1/2}^{n+1} - s_{-1,L}^n \right| \leq C \Delta x^2 \| \partial_x^2 s \|_{L^\infty(Q_T)} + C \Delta t \| \partial_x (a \partial_x s) \|_{L^\infty(Q_T)} + C(\|s\|_{W^2(Q_T)})(\Delta x^2 + \Delta t^2).
\]

Thus (A.22)-(A.25) hold so that the bound for \(\|E_A'\|_{L^2(L^2)}\) given by (A.26)-(A.27) is valid.
When \( S_{j-1,L}^n = S_{j-1}^{n+1} \) and \( s_{j-1,L}^n = s_{j-1}^{n+1} \), we have

\[
\frac{s_{j-\frac{1}{2}}^{n+1} - s_{j-1,L}^n}{2} = \frac{\Delta x_{j-1}}{2} (1 + h_{j-1}(s_{j-1}^{n+1})) \partial_x s_{j-1}^{n+1} + \frac{\Delta t^n}{2} \partial_x (a_{j-1}^{n+1} \partial_x s_{j-1}^{n+1}) + O(\Delta x^2 + \Delta t^2)
\]

\[
\leq C\Delta x^2 \| \partial_x^2 s \| \mathcal{L}^\infty(Q_T) + C\Delta t \| \partial_x (a \partial_x s) \| \mathcal{L}^\infty(Q_T)
\]

\[
+ C(\| s \| \mathcal{W}^2_\infty(Q_T))(\Delta x^2 + \Delta t^2).
\]

Thus (A.26)-(A.27) also hold in this case.

The estimate for \( E_{AR}'' \) differs only in the constant. Hence (A.29) holds with

\[
C_{AR}'' = C(\| s \| \mathcal{W}^2_\infty(Q_T), \sup_{Q_T} f'(s), \sup_{Q_T} |f''(s)|).
\]  

(B.1)

The bound for \( ||E_{AR}''||_{L^2(L^2)} \) is unchanged, thus (A.30) and (A.31) hold.

For this case, since \( E_{BRJ-1}'' = 0 \) (recall we set \( s_{j} = s_{j-1}^{n+1} \)), the estimate of \( ||E_{DR}||_{L^2(L^2)} \) is improved in the uniform grid case and unchanged in the nonuniform grid case. For uniform grid, we obtain

\[
||E_{DR}||_{L^2(L^2)} \leq C_{DR}(\Delta x^2 + \Delta t),
\]

where \( C_{DR} \) is given by (A.37). The estimate of \( ||E_{DL}||_{L^2(L^2)} \) is again given by (A.40)-(A.43).
Appendix C

First-order GMM

We now derive bounds for the $L^1(L^1)$ norm of the truncation error for the scheme analyzed in Section 4.7. What follows constitutes the proof of Theorem 4.15.

Recall the truncation error $E$ is given by

$$E_j^n = E^n_{T,j} + (E^n_{A,j})_x + (E^n_{D,j})_x + E^n_{R,j}.$$  \hfill (C.1)

Here

$$E^n_{T,j} = \frac{1}{\Delta t} \left( s_j^{n+1} - \frac{1}{\Delta x_j} \int_{B_j} s^{n+1} dx - \left( s_j^n - \frac{1}{\Delta x_j} \int_{B_j} s^n dx \right) \right),$$  \hfill (C.2)

$$E^n_{A,j} = E'_{A,j}^n + E''_{A,j}^n,$$  \hfill (C.3)

where

$$E'_{A,j}^n = f(s_{j+\frac{1}{2}}^n) - f(s_{j+\frac{1}{2}}^{n+\frac{1}{2}}),$$  \hfill (C.4)

and

$$E''_{A,j}^n = f(s_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(s(x_{j+\frac{1}{2}}, t)) dt.$$  \hfill (C.5)
Moreover,

\[ E_{D,j}^n = \frac{1}{\Delta t} \int_0^{t_{n+1}} \left( \frac{\partial s}{\partial x} \right) (x_{j+\frac{1}{2}}, t) dt - a_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left( s_{j,x}^{n+\frac{1}{2}} \right) s_j^{n+1}, \]  

(C.6)

and

\[ E_{R,j}^n = \frac{1}{\Delta x \Delta t} \int_{B_j} \int_0^{t_{n+1}} r(x, t, s(x, t)) dt dx - r(x_j, t^{n+1}, s_j^{n+1}). \]  

(C.7)

Thus, by (C.1) we have

\[ ||E||_{L^1(L^1)} \leq ||E_T||_{L^1(L^1)} + ||(E_{D,j}^n)||_{L^1(L^1)} + ||(E_{D,j})^n||_{L^1(L^1)} + ||E_{R,j}^n||_{L^1(L^1)} . \]  

(C.8)

We now estimate the terms on the right-hand side of (C.8).

**Estimate of \( ||E_T||_{L^1(L^1)} \).** Consider \( E_{T,j}^n \). By Taylor expansion we have

\[ \frac{1}{\Delta x} \int_{B_j} s^n dx - s_j^n = \frac{1}{\Delta x} \int_{B_j} \int_{x_j}^x (y - x_j) \partial_{xx}^2 s^n(y) dy \]

\[ \leq C \Delta x ||\partial_{xx}^2 s^n||_{L^1(B_j)}. \]

Thus by (C.2) and the above, we have for \( n = 0, 1, \ldots, N^* - 1 \),

\[ |E_{T,j}^n| \leq C \frac{\Delta x}{\Delta t} \left[ ||\partial_{xx}^2 s^{n+1}||_{L^1(B_j)} + ||\partial_{xx}^2 s^n||_{L^1(B_j)} \right]. \]

Hence,

\[ ||E_T||_{L^1(L^1)} \leq C \frac{\Delta x^2}{\Delta t} = C \Delta x, \]  

(C.9)

since \( \Delta t = \mathcal{O}(\Delta x) \), where

\[ C_T = C( ||\partial_{xx}^2 s||_{L^\infty(0,T, L^1(I))}, ||\partial_{xx}^2 s^0||_{L^1(I)} ). \]  

(C.10)

**Estimate of \( ||(E_A)_z||_{L^1(L^1)} \).** Next, consider \((E_{A,j})_z\). By Taylor expansion in (C.4), we have
\[(E'_{A,j}^{n})_x = f'(s^n_{j+\frac{1}{2}})(s^n_{j,L} - s^n_{j+\frac{1}{2}}) + \frac{f'(s^n_{j+\frac{1}{2}}) - f'(s^n_{j-\frac{1}{2}})}{\Delta x}(s^n_{j-1,L} - s^n_{j-\frac{1}{2}})
+ \frac{f''(\bar{s}^n_j)}{2\Delta x}(s^n_{j,L} - s^n_{j+\frac{1}{2}})^2 - \frac{f''(\bar{s}^n_{j-1})}{2\Delta x}(s^n_{j-1,L} - s^n_{j-\frac{1}{2}})^2,\]  

(C.11)  

where \(\bar{s}^n_j\) is some point between \(s^n_{j,L}\) and \(s^n_{j+\frac{1}{2}}\), similarly for \(\bar{s}^n_{j-1}\).

First, consider \(W^n_j = s^n_{j,L} - s^n_{j+\frac{1}{2}}\). Recall that

\[s^n_{0,L} = s^n_0,\]
\[s^n_{j,L} = s^n_j, \quad j = 1, \ldots, J - 1.\]

Thus

\[W^n_0 = 0,\]  

(C.12)

and

\[W^n_j = -\frac{\Delta x}{2} \partial_x s^n_j + O(\Delta x \|\partial^2_{xx} s^n\|_{L^1(x_j, x_{j+\frac{1}{2}})}).\]  

(C.13)

Hence, for \(j = 2, \ldots, J - 1\),

\[\frac{|W_j - W_{j-1}|}{\Delta x} \leq C \int_{x_{j-1}}^{x_j} \left| \partial^2_{xx} s^n(x) \right| ds
+ C \|\partial^2_{xx} s^n\|_{L^1(x_{j-1}, x_{j-\frac{1}{2}})} + C \|\partial^2_{xx} s^n\|_{L^1(x_j, x_{j+\frac{1}{2}})} \leq C \|\partial^2_{xx} s^n\|_{L^1(x_{j-1}, x_{j+\frac{1}{2}})}.\]  

(C.14)  

For \(j = 1\), we have \(|W^n_1| \leq C \Delta x \|\partial_x s\|_{L^\infty(Q_T)}\), thus

\[\frac{|W_1 - W_0|}{\Delta x} \leq C \|\partial_x s\|_{L^\infty(Q_T)}.\]  

(C.15)

By (C.14) and (C.15) we have

\[\|W^n_{j}\|_{L^1} = \sum_{j=2}^{J-1} |W^n_{j,x}| \Delta x + |W^n_{1,x}| \Delta x \leq C(\|\partial^2_{xx} s^n\|_{L^1(1)}, \|\partial_x s\|_{L^\infty(Q_T)}) \Delta x.\]  

(C.16)
Furthermore
\[
\left| \frac{f'(s_{j+\frac{1}{2}}) - f'(s_{j-\frac{1}{2}})}{\Delta x} \right| \leq \sup_{Q_T} |f''(s)| \left\| \partial_x s \right\|_{L^\infty(Q_T)}. \tag{C.17}
\]

Hence, by (C.11), (C.16), and (C.17),
\[
\left\| (E_A')_2 \right\|_{L^1(L^1)} \leq \sup_{Q_T} f'(s) \left\| W_2 \right\|_{L^1(L^1)} + \sup_{Q_T} |f''(s)| \left( \left\| W \right\|_{L^1(L^1)} \left\| \partial_x s \right\|_{L^\infty(Q_T)} + \Delta x^{-1} \left\| (W)^2 \right\|_{L^1(L^1)} \right)
\leq \Delta x C \left( \left\| \partial_x^2 s \right\|_{L^\infty(0,T; L^1(I))}, \left\| \partial_x s \right\|_{L^\infty(Q_T)}, \left\| \partial_x^2 s \right\|_{L^1(I)}, \right.
\left. \left\| \partial_x s \right\|_{L^\infty(Q_T)}, \sup_{Q_T} f'(s), \sup_{Q_T} |f''(s)| \right)
\equiv C_A' \Delta x. \tag{C.18}
\]

Now consider \((E_{A,j}''^n)_2\). By Taylor expansion in (C.5) we have
\[
E_{A,j}''^n = -\partial_t f(s_{j+\frac{1}{2}}) \left( \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} (t - t^n) dt \right) - \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_{t^n}^t (\tau - t^n) \partial^2_{tt} f(s(x_{j+\frac{1}{2}}, \tau)) d\tau
dt
\equiv \frac{\Delta t}{2} \partial_t f(s_{j+\frac{1}{2}}) + O(\Delta t \left\| \partial^2_{tt} f(s_{j+\frac{1}{2}}) \right\|_{L^1(t^n, t^{n+1})}). \tag{C.19}
\]

Thus
\[
\left| (E_{A,j}''^n)_2 \right| \leq C \frac{\Delta t}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} |\partial^2_{tt} f(s^n(x))| dx
\leq C \frac{\Delta t}{\Delta x} \left\| \partial^2_{tt} f(s_{j+\frac{1}{2}}) \right\|_{L^1(t^n, t^{n+1})}. \tag{C.20}
\]

Hence,
\[
\left\| (E_{A,j}''^n)_2 \right\|_{L^1(L^1)} \leq \Delta t C \left( \left\| \partial^2_{tt} f \right\|_{L^\infty(0,T; L^1(I))}, \right.
\left. \left\| \partial^2_{tt} f \right\|_{L^1(0,T; L^\infty(I))} \right)
\equiv C_A'' \Delta t, \tag{C.21}
\]
where

\[ C''_A = C\left( \sup_{Q_T} f'(s), \sup_{Q_T} |f''(s)|, \|s\|_{W^1_{\infty}(Q_T)} \right), \]

\[ \|\partial_{tt}^2 s\|_{L^\infty(0,T; L^1(I))}, \|\partial_{tt}^2 s\|_{L^1(0,T; L^\infty(I))}, \]

\[ \|g_0\|_{L^\infty(0,T)}, \|g_1\|_{L^\infty(0,T)}, \|g_0''\|_{L^1(0,T)}, \|g_1''\|_{L^1(0,T)}. \]

Combining (C.3), (C.18), and (C.21) we obtain

\[ \|\langle E_A \rangle_2\|_{L^1(L^1)} \leq C(C', C''_A)(\Delta x + \Delta t) \]

\[ \equiv C_A(\Delta x + \Delta t). \]  

(C.22)

Estimate of \( \|\langle E_D \rangle_2\|_{L^1(L^1)} \). Next, consider

\[ E_{D,j}^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \left( a \frac{\partial s}{\partial x} \right) (x_{j+\frac{1}{2}}, t) dt - a^n_{j+\frac{1}{2}} \sigma_{n+1}^{j+\frac{1}{2}} \]

\[ + a^n_{j+\frac{1}{2}} \left( \frac{\partial s}{\partial x} (x_{j+\frac{1}{2}}, t) dt - a^n_{j+\frac{1}{2}} \right) \sigma_{n}^{j+\frac{1}{2}} \]

\[ + (a^n_{j+\frac{1}{2}} \sigma_{n}^{j+\frac{1}{2}} - a^n_{j+\frac{1}{2}} \sigma_{n+1}^{j+\frac{1}{2}}) s_{n+1}^{j+\frac{1}{2}} \]

\[ \equiv E'_{D,j}^n + E''_{D,j}^n + E''_{D,j}^n. \]  

(C.24)

First of all, let \( \alpha(x, t) = a(x, t, s(x, t)) \partial_x s(x, t) \), then by Taylor expansion

\[ E'_{D,j}^n = \partial_t \alpha_{n+\frac{1}{2}} \left( \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} (t - t^n) dt \right) \]

\[ + \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_{t^n}^t (\tau - t^n) \partial_{tt}^2 \alpha_{j+\frac{1}{2}}(\tau) d\tau. \]

Thus, as in the estimate of \( E'_A \); i.e., (C.20)-(C.22), we have

\[ \|E'_D\|_{L^1(L^1)} \leq \Delta t C(\|\partial_{tt}^2 \alpha\|_{L^\infty(0,T; L^1(I))}, \|\partial_{tt}^2 \alpha\|_{L^1(0,T; L^\infty(I))}) \]

\[ \equiv C_D' \Delta t, \]  

(C.25)
where

\[ C_D' = C(\|a\|_{W_2^1(Q_T)}, \|s\|_{W_2^1(Q_T)}), \|\partial_{xx}^2 s\|_{L^\infty(0,T;L^1(\Omega))}, \|\partial_{xx}^2 s\|_{L^1(0,T;L^\infty(\Omega))}). \]  \hspace{1cm} (C.26)

Now consider \( E_{D,j}'' \). By Taylor expansion, for \( j = 1, \ldots, J - 1 \),

\[ \left| \partial_x s_{j+\frac{1}{2}}^n - s_{j,x}^n \right| \leq C \Delta x (\|\partial_{xx}^2 s\|_{L^\infty(\Omega)}). \]

Thus, by (C.24) and the above we have

\[ \|(E''_{D,j})_n\|_{L^1(\Omega)} \leq C_D'' \Delta x, \] \hspace{1cm} (C.27)

where

\[ C_D'' = C(\|a\|_{W_2^1(Q_T)}), \|s\|_{W_2^1(Q_T)}). \] \hspace{1cm} (C.28)

For the estimate of \( E_{D,j}''' \) we note

\[ s_{j+\frac{1}{2}}^n - s_{j+\frac{1}{2}}^n = \begin{cases} (s_j^n + s_{j+1}^n)/2 - s_{j+\frac{1}{2}}^{n+1}, & j = 1, \ldots, J - 2, \\ 0, & j = 0, J - 1, \end{cases} \]

by (4.345) and (4.342). Thus

\[ \left| s_{j+\frac{1}{2}}^n - s_{j+\frac{1}{2}}^n \right| \leq C \Delta x^2 (\|\partial_{xx}^2 s\|_{L^\infty(\Omega)}). \]

Hence

\[ \left\| (E_{D,j}'''^n) \right\|_{2} \leq \frac{1}{\Delta x} \sup_{Q_T} |\partial_x a| \times \left[ \left| s_{j+\frac{1}{2}}^{n+1} - s_{j+\frac{1}{2}}^{n+1} \right| \left| s_{j+\frac{1}{2}}^{n+1} - s_{j+\frac{1}{2}}^n \right| + \left| s_{j-\frac{1}{2}}^{n+1} - s_{j-\frac{1}{2}}^n \right| \left| s_{j-\frac{1}{2}}^{n+1} - s_{j-\frac{1}{2}}^n \right| \right] \]

\[ \leq C(\sup_{Q_T} |\partial_x a|, \|s\|_{W_2^1(Q_T)}) \Delta x. \]
Thus

\[
\| (E_{B,D}^{n})_{e} \|_{L^{1}(L^{1})} \leq C_{B}^{n} \Delta x,
\]

where

\[
C_{B}^{n} = C(\sup_{Q_{T}} |\partial_{x} a|, \|s\|_{W_{a}^{2}(Q_{T})}).
\]

Combining (C.24), (C.25)-(C.26), (C.27)-(C.28), and (C.29)-(C.30) we obtain

\[
\| (E_{D})_{e} \|_{L^{1}(L^{1})} \leq C_{D}(\Delta x + \Delta t),
\]

where

\[
C_{D} = C(C_{B}, C_{B}^{n}, C_{B}^{n}).
\]

Estimate of \( \|E_{R}\|_{L^{1}(L^{1})} \). Finally, by Taylor expansion in (C.7), we find that

\[
|E_{R,j}^{n}| \leq C \left( \| \frac{dr}{dx} \|_{C^{\infty}(Q_{T})}, \| \frac{dr}{dt} \|_{C^{\infty}(Q_{T})} \right) (\Delta x + \Delta t).
\]

Thus

\[
\|E_{R}\|_{L^{1}(L^{1})} \leq C_{R}(\Delta x + \Delta t),
\]

where

\[
C_{R} = C(\|r\|_{W_{a}^{1}(Q_{T})}, \|s\|_{W_{a}^{1}(Q_{T})}).
\]
Appendix D

Regularity of solutions

In this thesis we have restricted our attention to one-dimensional scalar equations, however, the fluid-flow problems we are interested in solving fall under the broader category of nonlinear (quasi-linear) parabolic equations (or systems of equations) of the form

$$s_t - \nabla \cdot a(x, t, s)\nabla s + b(x, t, s, \nabla s) = 0,$$

(D.1)

where $a = (a_{ij})$, $x = (x_1, x_2, \ldots, x_n)$, $(x, t) \in D \times (0, T] \equiv Q_T$. Here

$$\nabla = \left(\frac{d}{dx_1}, \ldots, \frac{d}{dx_n}\right),$$

where $d/dx_j$ means total derivative with respect to $x_j$, and $D$ is a bounded subset of $\mathbb{R}^n$.

We assume $s$ satisfies certain initial and boundary data; i.e., either

$$s|_{\Lambda_T} = \Psi|_{\Lambda_T},$$

(D.2)

where $\Lambda_T = S_T \cup S$ is the boundary of $Q_T$, with $S$ the boundary of $D$ and $S_T = S \times (0, T]$, or

$$s|_{t=0} = \Psi_0,$$

(D.3)
and

\[ [\sigma a(x, t, s) \nabla s \cdot \eta + \Psi(x, t, s)]|_{S_T} = 0. \quad (D.4) \]

Here \( \eta = \eta(x) \) is the outward normal to \( S_T \) and is assumed to be sufficiently smooth; \( \sigma \) is a nonnegative scalar. Note that (D.2) incorporates both initial and boundary data.

In our analysis we have assumed a certain amount of smoothness on the data and the solution of the problem. We now state theorems, given in Ladyzhenskaja, et. al. [26], which give us sufficient conditions for (D.1) to have a unique, smooth solution in a certain class of Hölder continuous functions.

Before stating these theorems we recall the definition of the Hölder spaces \( H^l(\tilde{D}) \) and \( H^{1,1/2}(\tilde{Q}_T) \), where \( l > 0 \) is nonintegral.

Denote by \( H^l(\tilde{D}) \) the Banach space whose elements are continuous functions \( s(x) \) in \( \tilde{D} \) having in \( \tilde{D} \) continuous derivatives up to order \([l]\) (largest integer) inclusively and a finite value for the quantity

\[ |s|^{(l)}_{\tilde{D}} = |s|^{(l)}_{\tilde{D}} + \sum_{j=0}^{[l]} |s|^{(j)}_{\tilde{D}}, \]

where

\[ |s|^{(0)}_{\tilde{D}} = |s|^{(0)}_{\tilde{D}} = \|s\|_{L^\infty(D)}, \]

\[ |s|^{(j)}_{\tilde{D}} = \sum_{(j)} |D^j x s|^{(0)}_{\tilde{D}}, \]

\[ |s|^{(l)}_{\tilde{D}} = \sum_{(l)} |D^l x s|^{(l-\alpha)}_{\tilde{D}}. \]

Here, for \( \alpha < 1 \),

\[ \langle s \rangle^D = \sup_{x, x' \in \tilde{D}, |x - x'| \leq \rho_0} \frac{|s(x) - s(x')|}{|x - x'|^{\alpha}}. \]

When the last quantity above is bounded for some \( \rho_0 > 0 \), we say that \( s \) satisfies a Hölder condition in \( x \) with exponent \( \alpha \).
The symbol $D^j_x$ denotes any derivative with respect to $x$ of order $j$, while $\sum_{(i)}$ denotes summation over all possible derivatives of order $j$. Also $|\cdot|$ denotes the standard $l^2$ norm; i.e.,

$$|x|^2 = \sum_{i=1}^{n} |x_i|^2.$$

Also, denote by $H^{l, l/2}(\mathring{Q}_T)$ the Banach space of functions $s(x, t)$ that are continuous in $\mathring{Q}_T$, together with all derivatives of the form $D_p^q D_x^s$ for $2p + q < l$, $p, q$ integers, and have a finite norm

$$|s|_{\mathring{Q}_T}^{(l)} = (s)_{\mathring{Q}_T}^{(l)} + \sum_{j=0}^{[l]} (s)_{\mathring{Q}_T}^{(j)},$$

where

$$(s)_{\mathring{Q}_T}^{(0)} = |s|_{\mathring{Q}_T}^{(0)} = ||s||_{L^\infty(\mathring{Q}_T)},$$

$$(s)_{\mathring{Q}_T}^{(j)} = \sum_{(2p+q=j)} |D_p^q D_x^s s|_{\mathring{Q}_T}^{(0)},$$

$$(s)_{\mathring{Q}_T}^{(j)} = (s)_{x, \mathring{Q}_T}^{(j)} + (s)_{t, \mathring{Q}_T}^{(j)},$$

$$(s)_{x, \mathring{Q}_T}^{(l)} = \sum_{(2p+q=[l])} (D_p^q D_x^s x)_{\mathring{Q}_T}^{(l-[l])},$$

$$(s)_{t, \mathring{Q}_T}^{(l)} = \sum_{0 < \alpha < 2} (D_p^q D_x^s t)_{\mathring{Q}_T}^{(l-\alpha)},$$

$$(s)_{x, \mathring{Q}_T}^{(\alpha)} = \sup_{(x, t), (x', t') \in \mathring{Q}_T, \|x-x'|^\alpha \leq \rho_0} \frac{|s(x, t) - s(x', t)|}{|x-x'|^\alpha}, \quad 0 < \alpha < 1,$$

$$(s)_{t, \mathring{Q}_T}^{(\alpha)} = \sup_{(x, t), (x', t') \in \mathring{Q}_T, \|t-t'|^\alpha \leq \rho_0} \frac{|s(x, t) - s(x, t')|}{|t-t'|^\alpha}, \quad 0 < \alpha < 1.$$

Moreover, we say that a surface $S$ belongs to class $H^l$ for $l > 1$ if there exists a number $\rho > 0$ such that the intersection of $S$ with a ball $K_\rho$ of radius $\rho$ with center at an arbitrary point $x^0 \in S$ is a connected surface $B$, the equation of which in the local coordinate system $(y_1, \ldots, y_n)$ with origin at $x^0$ has the form $y_n = \omega(y_1, \ldots, y_{n-1})$, where $\omega(y, \ldots, y_{n-1})$ is a function of class $H^l$ in the domain $\bar{B}$ that is the projection of $B$ on the surface $y_n = 0$. 
Below we state a general theorem which relates to the solvability of problem (D.1)-(D.2); we let
\[ a_i(x, t, s, \nabla s) = \sum_{j=1}^{n} a_{ij}(x, t, s) \frac{\partial s}{\partial x_j}. \]
Then (D.1) can be written as
\[ s_t + \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x, t, s, \nabla s) + b(x, t, s, \nabla s) = 0, \quad (D.5) \]
where \( d/dx_i \) means the total derivative with respect to \( x_i \).

Let \( M \) and \( M_1 \) be a priori bounds for \( \max_{Q_T} |s| \) and \( \max_{Q_T} |\nabla s| \), respectively. Then, the following conditions must hold. Assume that for \( (x, t) \in \bar{Q}_T \) and arbitrary \( s \) we have
\[ \sum_{i,j=1}^{n} \frac{\partial a_i(x, t, s, p)}{\partial p_j} \xi_i \xi_j |p=0 | \geq 0, \quad sA(x, t, s, 0) \geq -b_1 s^2 - b_2, \quad (D.6) \]
for \( \xi = (\xi_1, \ldots, \xi_n) \) an arbitrary vector in \( \mathbb{R}^n \), and \( b_1 \) and \( b_2 \) nonnegative constants, where
\[ A = b - \frac{\partial a_i}{\partial s} \frac{\partial s}{\partial x_i} - \frac{\partial a_i}{\partial x_i}. \]
Furthermore, for \( (x, t) \in \bar{Q}_T, |s| \leq M, \) and arbitrary \( p, \)
\[ \nu |\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial p_j} \xi_i \xi_j \leq \mu |\xi|^2, \quad \nu, \mu > 0, \quad (D.7) \]
\[ \sum_{i=1}^{n} \left( |a_i| + \left| \frac{\partial a_i}{\partial s} \right| (1 + |p|) + \sum_{i,j=1}^{n} \left| \frac{\partial a_i}{\partial x_j} \right| + |b| \right) \leq \mu (1 + |p|)^2, \quad (D.8) \]
and for \( (x, t) \in \bar{Q}_T, |s| \leq M, |p| \leq M_1, \) and \( h \to 0, \)
\[ \left| \frac{\partial a_i}{\partial s} \frac{a_i(x, t + h, s, p) - a_i(x, t, s, p)}{h} \right| \leq \mu, \quad (D.9) \]

We now state a theorem about existence and uniqueness of a classical solution to (D.5), (D.2).
Theorem D.1 Let $0 < \beta < 1$, and suppose that the following conditions hold.

(i) For $(x, t) \in \bar{Q}_T$ and arbitrary $s$, (D.6) holds.

(ii) For $(x, t) \in \bar{Q}_T$, $|s| \leq M$, and arbitrary $p$ the functions $a_i(x, t, s)$ and $b(x, t, s, p)$ are continuous, the $a_i$ are differentiable with respect to $x$, $s$, and $p$ and satisfy (D.7)-(D.8).

(iii) For $(x, t) \in \bar{Q}_T$, $|s| \leq M$, and $|p| \leq M_1$, the functions $a_i$, $b$, $\partial a_i / \partial p_j$, $\partial a_i / \partial s$, and $\partial a_i / \partial x_j$ are continuous functions satisfying a Hölder condition in $x$, $t$, $s$, and $p$ with exponents $\beta$, $\beta/2$, $\beta$, and $\beta$, respectively.

(iv) For $(x, t) \in \bar{Q}_T$, $|s| \leq M$, and $|p| \leq M_1$ the function $b(x, t, s, p)$ has the partial derivatives $\partial b / \partial p_i$ and $\partial b / \partial s$ and $a_i$ and $b$ satisfy (D.9).

(v) The boundary condition (D.2) is given by a function $\Psi(x, t) \in H^{2+\beta, 1+\beta/2}(\bar{Q}_T)$ and satisfying on $S_0 = \{(x, t) : x \in S, t = 0\}$

$$
\Psi_t - \sum_{i=1}^{n} \frac{d}{dz_i} a_i(x, 0, \Psi(x, 0), \nabla \Psi(x, 0))
+ b(x, 0, \Psi(x, 0), \nabla \Psi(x, 0))|_{x \in S} = 0,
$$

in other words, the compatibility conditions of zero and first order are assumed to be fulfilled.

(v) $S \in H^{2+\beta}$.

Then there exists a unique solution of problem (D.1)-(D.2) in $H^{2+\beta, 1+\beta/2}$. Moreover, the derivatives $\frac{\partial^2 S}{\partial t \partial z_j}$ are square integrable on $Q_T$.

We now concern ourselves with the second boundary value problem; i.e. (D.1), and (D.3)-(D.4). In this case we modify the definition of $b$ above and rewrite (D.1) as

$$
s_t - \sum_{i,j=1}^{n} a_{ij}(x, t, s) \frac{\partial^2 s}{\partial x_i \partial x_j} + b(x, t, s, \nabla s) = 0.
$$
Suppose the functions $a_{ij}$, $b$ and $\Psi$ are subject for arbitrary $s$ to the conditions

\begin{align}
0 \leq & \sum_{i,j} a_{ij}(x, t, s)\xi_i \xi_j \leq \mu_1 |\xi|^2, \quad \text{for } (x, t) \in \mathcal{D} \cap (0, T], \quad (D.10) \\
-sb(x, t, s, p) \leq & c_0 |p|^2 + c_1 s^2 + c_2, \quad \text{for } (x, t) \in \mathcal{Q}_T - \Lambda_T, \quad (D.11) \\
\nu_1 |\xi|^2 \leq & \sum_{i,j=1}^n a_{ij}(x, t, s)\xi_i \xi_j, \quad -s\Psi(x, t, s) \leq c_3 s^2 + c_4, \quad for (x, t) \in S_T, \quad (D.12)
\end{align}

where $\mu_1, \nu_1$ are positive constants, and $c_i = \text{const.}$, $i = 0, \ldots, 4$. Furthermore, for $(x, t) \in \mathcal{Q}_T, |s| \leq M$, and arbitrary $p$,

\begin{align}
\nu |\xi|^2 \leq & \sum_{i,j=1}^n a_{ij}(x, t, s)\xi_i \xi_j \leq \mu |\xi|^2, \quad \nu, \mu > 0, \quad (D.13) \\
\left| \frac{\partial a_{ij}}{\partial s} , \frac{\partial a_{ij}}{\partial x_j}, \frac{\partial \Psi}{\partial s}, \frac{\partial \Psi}{\partial x_j} \right| \leq & \mu, \quad (D.14) \\
|b(x, t, s, p)| \leq & \mu(1 + |p|^2), \quad (D.15) \\
\left| \frac{\partial^2 \Psi}{\partial s^2}, \frac{\partial^2 \Psi}{\partial s \partial x_j}, \frac{\partial^2 \Psi}{\partial t \partial x_j}, \frac{\partial^2 \Psi}{\partial t^2} \right| \leq & \mu, \quad (D.16) \\
|\nabla_p b|(1 + |p|) + |\frac{\partial b}{\partial s}| + |\frac{\partial b}{\partial t}| \leq & \mu(1 + |p|^2), \quad (D.17)
\end{align}

and

\begin{align}
\left| \frac{\partial^2 a_{ij}}{\partial s^2}, \frac{\partial^2 a_{ij}}{\partial s \partial t}, \frac{\partial^2 a_{ij}}{\partial s \partial x_j}, \frac{\partial^2 a_{ij}}{\partial t \partial x_j} \right| \leq & \mu. \quad (D.18)
\end{align}

Here

$$\nabla_p = \left( \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n} \right).$$

We now formulate a theorem.

**Theorem D.2** Suppose the following conditions are fulfilled.

(i) The functions $a_{ij}, b,$ and $\Psi$ satisfy $(D.10)$-$(D.12)$ for arbitrary $s$. 

(ii) For \((x, t) \in \tilde{Q}_T, |s| \leq M, \) and arbitrary \(p\) the functions \(a_{ij}, b, \) and \(\Psi\) are continuous in their arguments, and possess the derivatives entering into conditions \((D.18)-(D.18)\) and satisfy these conditions.

(iii) For \((x, t) \in \tilde{Q}_T, |s| \leq M \) and \(|p| \leq M_1, \) the functions \(D_x^1 a_{ij}\) are Hölder continuous in the variables \(x\) with exponent \(\beta, \) \(D_x^1 \Psi\) is Hölder continuous in \(x\) and \(t\) with exponents \(\beta\) and \(\beta/2\) respectively, and \(b\) is Hölder continuous in \(x\) with exponent \(\beta.\)

(iv) \(S \in H^{2+\beta}; \) \(\Psi(x, 0, 0)|_{x \in S} = \Psi_0(x).\)

Then \((D.1), (D.3)-(D.4)\) has a unique solution \(s(x, t)\) in the class \(H^{2+\beta, 1+\beta/2}.)\)

In our estimates we have assumed more smoothness on the solution than is given here. However, as remarked earlier, for the one-dimensional problems we consider one can obtain a \(O(\Delta x + \Delta t)\) estimate for the truncation error when \(s\) is a classical solution, with slightly stronger assumptions on the derivatives \(\frac{\partial s}{\partial x}\) and \(\frac{\partial^2 s}{\partial t^2}.\) In particular, we need these derivatives to exist pointwise and be bounded.
Bibliography


