Several Procedures for Operator-Based
Averaging for Elliptic Equations

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Abstract: Numerical procedures are discussed for constructing averaged coefficients for elliptic differential operators. These procedures are intended for problems where the coefficients vary on a scale finer than can be resolved by a reasonable computational grid. Numerical methods for calculating locally averaged coefficients using mixed and Galerkin finite elements are presented. These methods involve solving local elliptic problems either to determine a pseudo-coefficient or as part of the overall solution procedure. The local problems are independent and can be solved in parallel. The procedures are formulated and numerical results demonstrating their performance are presented.

1. Introduction. Here we define several techniques for dealing with second order elliptic problems with rough coefficients. These procedures are designed to give reasonable solutions to problems using a mesh that is too coarse to resolve the detail produced by the rough coefficients. The "effective coefficients" used in approximating the solution are computed using local elliptic problems; local solutions to the differential equation are used to compute averaged coefficients for the coarse grid solution.

There are several schemes for producing discrete versions of an elliptic operator which involve forming averages of coefficients. Some of these are based on clever insight, some arise naturally in the formation of the scheme, and others are ad hoc.

For a one dimensional problem of the form

\[-(ay')' = f \text{ on } (0,1),\]

it is well known that difference operators using coefficients that are reciprocals of integral averages of \(\frac{1}{a}\) give outstanding results, even when \(a\) is very rough [3]. This "reciprocal averaging" gives exactly the right conductivities in this one-dimensional context. Because this averaging is so effective, it is tempting to use it in other contexts too, usually with less success. For Galerkin methods there is a natural type of averaging that comes directly from a weak form of the differential equation; when there are very large variations in the coefficients the discrete operators produced this way may not reflect the true nature of the equation unless the mesh is fine enough to resolve the roughness in the coefficients. Part of the motivation to examine different ways of producing discrete coefficients came from looking at the performance of multidimensional mixed methods which use a form of reciprocal averaging.

One way to view this work is that it is an attempt to produce an homogenized equation. Numerical homogenization should be useful in a wider range of situations than one would expect to be able to use standard homogenization. Instead of requiring highly

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regular periodic structures, these numerical schemes use local approximate solutions to extract the equivalent behavior, on the appropriate scale, of the rough coefficient operator.

There are two likely applications of the techniques given here. One is to give cheap but reasonable approximations of solutions to differential problems; these answers should have the right global behavior even though they miss some of the details of the solution. Another application is to give coarse grid operators that can be used in iterative methods for solving the algebraic problems associated with fine grid problems; this will not be studied in this paper. The methods described here involve large amounts of computation which can naturally be done in parallel, the solution of the local problems to define the coefficients for the homogenized operator.

**Differential Problem**

The methods will be applied to the following differential problem. Take $\Omega$ to be a bounded domain in $\mathbb{R}^2$ with polygonal boundary. Let $n_\Omega$ be the outward normal to $\partial \Omega$. Take $a$ to be a positive measurable function on $\Omega$, and suppose that $a$ is bounded above and below by positive constants. Take $p$ to be the solution of

$$
- \nabla \cdot (a \nabla p) = f, \quad (x, y) \in \Omega,
$$

$$
\quad u \cdot n_\Omega = g, \quad \text{on } \partial \Omega,
$$

where $u = -a \nabla p$. So that a solution to this problem exists, suppose that

$$
\int_{\partial \Omega} g d\omega = \int_\Omega f(x, y) dxdy,
$$

and so that $p$ is unique suppose that the solution has average value zero.

Let $\langle \cdot, \cdot \rangle_\Omega$ and $\langle \cdot, \cdot \rangle_{\partial \Omega}$ be the $L^2(\Omega)$ and $L^2(\partial \Omega)$ inner products, respectively. Usually we will shorten $\langle \cdot, \cdot \rangle_\Omega$ to $\langle \cdot, \cdot \rangle$ when the domain is clear from context. Take $H^1(\Omega)/\mathbb{R}$ to be the set of functions having average value zero such that the function and its gradient are square integrable. Then one weak form of the above problem is that $p \in H^1(\Omega)/\mathbb{R}$ satisfies

$$
\mathcal{A}(p, v) = \langle f, v \rangle - \langle g, v \rangle_{\partial \Omega} \quad \text{for } v \in H^1(\Omega)/\mathbb{R},
$$

where

$$
\mathcal{A}(\psi, \chi) = \int_\Omega a \nabla \psi \cdot \nabla \chi d\omega.
$$

**2. A mixed finite element-OBA approach.** In this section, we describe an OBA approach based on the mixed finite element method. We will restrict attention to the lowest-order Raviart-Thomas spaces on rectangular elements in $\mathbb{R}^2$ and to the case in which $\Omega$ is the unit square in $\mathbb{R}^2$.

In the discussion below, for a given spatial region $R$ and a given point $(x, y)$ on the boundary of $R$, let $n_R(=n_R(x, y))$ denote the outward normal vector to the boundary of $R$ at the point $(x, y)$. We will suppress the spatial dependence of $n_R$ in our notation.
Let $\delta_x$ and $\delta_y$ be partitions of $[0, 1]$:
\[
\delta_x : 0 = x_\frac{1}{2} < x_\frac{3}{2} < \cdots < x_{I+\frac{1}{2}} = 1,
\]
\[
\delta_y : 0 = y_\frac{1}{2} < y_\frac{3}{2} < \cdots < y_{J+\frac{1}{2}} = 1.
\]
Let $h^x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, $h^y_j = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$, and $h = \max_{i,j}(h^x_i, h^y_j)$. Let $x_i = (x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})/2$, with the analogous definition for $y_j$.

Let
\[
\mathcal{M}_{-1}^k(\delta_x) = \{g : g|_{B^x_i} \in P^k(B^x_i), \ i = 1, \ldots, I\},
\]
where $B^x_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and $P^k(B^x_i)$ is the set of all polynomials of degree less than or equal to $k$ defined on $B^x_i$. Let
\[
\mathcal{M}^k_0(\delta_x) = \mathcal{M}^k_{-1}(\delta_x) \cap \mathcal{C}^0(I).
\]

Then the lowest-order Raviart-Thomas spaces $V_h$ and $W_h$ are given by
\[
V_h = V^x_h \times V^y_h,
\]
with
\[
V^x_h = \mathcal{M}^0_0(\delta_x) \otimes \mathcal{M}^0_{-1}(\delta_y),
\]
and
\[
V^y_h = \mathcal{M}^0_{-1}(\delta_x) \otimes \mathcal{M}^0_0(\delta_y).
\]

Let $K_{i,j} = B^x_i \times B^y_j$.

The mixed finite element method applied to (1) and (2) involves determining $(P, U) \in W_h \times V_h$ which satisfy
\[
(a^{-1}U, v) = (P, \nabla \cdot v) = 0, \ v \in V^0_h,
\]
\[
(\nabla \cdot U, w) = (f, w), \ w \in W_h.
\]

Here $V^0_h$ are those functions in $V_h$ which satisfy $v \cdot n_\Omega = 0$. The boundary condition (2) is enforced strongly; i.e., we set
\[
U \cdot n_\Omega = g, \ \text{on} \ \partial \Omega.
\]

For simplicity we will assume that $g$ is such that this relation can hold exactly; this is certainly true in the important special case of $g = 0$.

Associated with the partitions $\delta_x$ and $\delta_y$ define coarser partitions $\delta^c_x$ and $\delta^c_y$ with $\delta^c_x \subset \delta_x$ and $\delta^c_y \subset \delta_y$. Let $K^c_{i,j}$ denote a rectangle in this coarser space, and let $V^c_h$ and $W^c_h$ denote the lowest-order Raviart-Thomas spaces. In the OBA approach outlined here, we will construct a "pseudo-coefficient" $A(x, y) = (A_{kl}(x, y))$ which is a constant, 2x2 matrix on each $K^c_{i,j}$. The construction of $A(x, y)$ is now described.

Consider an element $K^c_{i,j}$. For simplicity, denote this element by $K$. Let $K'$ denote a possibly larger rectangle containing $K$, obtained by padding the exterior of $K$ with a small number of cells. See Figure 1. Let $\Gamma_1$ and $\Gamma_2$ denote the left and right vertical
boundaries of $K$, respectively, and $\Gamma_3$ and $\Gamma_4$ the top and bottom horizontal boundaries, respectively. Similarly, let $\Gamma_1'$, $\Gamma_2'$, $\Gamma_3'$, and $\Gamma_4'$ denote the analogous boundaries of $K'$. Associated with each $K$, we solve two problems:

**Problem 1:**

\[
\nabla \cdot u \equiv -\nabla \cdot (a \nabla p) = 0, \quad \text{on } K',
\]

\[
(8)
\]

\[
u \cdot n_{K'} = 0, \quad \text{on } \Gamma_3' \cup \Gamma_4',
\]

\[
(9)
\]

\[
p = 0, \quad \text{on } \Gamma_1',
\]

\[
(10)
\]

\[
p = 1, \quad \text{on } \Gamma_2'.
\]

\[
(11)
\]

**Problem 2:**

\[
\nabla \cdot u \equiv -\nabla \cdot (a \nabla p) = 0, \quad \text{on } K',
\]

\[
(12)
\]

\[
u \cdot n_{K'} = 0, \quad \text{on } \Gamma_1' \cup \Gamma_2',
\]

\[
(13)
\]

\[
p = 0, \quad \text{on } \Gamma_3',
\]

\[
(14)
\]

\[
p = 1, \quad \text{on } \Gamma_4'.
\]

\[
(15)
\]

We solve Problems 1 and 2 using the lowest order mixed finite element method, which reduces to block-centered finite differences with the appropriate quadrature rules,
see [4]. The results of these solves are velocity-pressure pairs \((P_1, U_1)\) and \((P_2, U_2)\) in \(V_h \times W_h\) corresponding to Problems 1 and 2, respectively, from which we construct \(A(x, y)|_K\).

Suppose \((\bar{u}, \bar{p})\) satisfy
\[
\bar{u} = -A \nabla \bar{p},
\]
where \(\bar{u} = (\bar{u}^x, \bar{u}^y)\). Then, on \(K\),
\[
\begin{align*}
\bar{u}^x &= -A_{11} \bar{p}_x - A_{12} \bar{p}_y, \\
\bar{u}^y &= -A_{21} \bar{p}_x - A_{22} \bar{p}_y.
\end{align*}
\]
Integrating (17) and (18) over \(K\), we obtain
\[
\begin{align*}
\int_K \bar{u}^x \, dx \, dy &= -A_{11} \left[ \int_{\Gamma_2} \bar{p} \, dy - \int_{\Gamma_1} \bar{p} \, dy \right] - A_{12} \left[ \int_{\Gamma_3} \bar{p} \, dx - \int_{\Gamma_4} \bar{p} \, dx \right], \\
\int_K \bar{u}^y \, dx \, dy &= -A_{21} \left[ \int_{\Gamma_2} \bar{p} \, dy - \int_{\Gamma_1} \bar{p} \, dy \right] - A_{22} \left[ \int_{\Gamma_3} \bar{p} \, dx - \int_{\Gamma_4} \bar{p} \, dx \right],
\end{align*}
\]
The coefficients \(A_{kl}\) are determined by solving the 4x4 system:
\[
\begin{align*}
\int_K U^x_1 \, dx \, dy &= -A_{11} \left[ \int_{\Gamma_2} P^-_1 \, dy - \int_{\Gamma_1} P^+_1 \, dy \right] - A_{12} \left[ \int_{\Gamma_3} P^-_1 \, dx - \int_{\Gamma_4} P^+_1 \, dx \right], \\
\int_K U^y_1 \, dx \, dy &= -A_{21} \left[ \int_{\Gamma_2} P^-_1 \, dy - \int_{\Gamma_1} P^+_1 \, dy \right] - A_{22} \left[ \int_{\Gamma_3} P^-_1 \, dx - \int_{\Gamma_4} P^+_1 \, dx \right], \\
\int_K U^x_2 \, dx \, dy &= -A_{11} \left[ \int_{\Gamma_2} P^-_2 \, dy - \int_{\Gamma_1} P^+_2 \, dy \right] - A_{12} \left[ \int_{\Gamma_3} P^-_2 \, dx - \int_{\Gamma_4} P^+_2 \, dx \right], \\
\int_K U^y_2 \, dx \, dy &= -A_{21} \left[ \int_{\Gamma_2} P^-_2 \, dy - \int_{\Gamma_1} P^+_2 \, dy \right] - A_{22} \left[ \int_{\Gamma_3} P^-_2 \, dx - \int_{\Gamma_4} P^+_2 \, dx \right].
\end{align*}
\]
Here, for example, for \((\bar{x}, \bar{y}) \in \Gamma_2,\)
\[
\int_{\Gamma_2} P^-_1 \, dy = \lim_{\bar{x} \to \bar{x}^-} \int_{\Gamma_2} \lim_{\bar{y} \to \bar{y}} P_1(x, y) \, dy,
\]
and, for \((\bar{x}, \bar{y})\) on \(\Gamma_1,\)
\[
\int_{\Gamma_1} P^+_1 \, dy = \lim_{\bar{x} \to \bar{x}^+} \int_{\Gamma_1} \lim_{\bar{y} \to \bar{y}} P_1(x, y) \, dy.
\]

Once the matrix \(A(x, y)\) is obtained on each coarse element \(K\), coarse grid velocities and pressure \((P_c, U_c) \in W^c_h \times V^c_h\) can be determined by solving
\[
\begin{align*}
(A^{-1} U_c, v) - (P_c, \nabla \cdot v) &= 0, \quad v \in V^c_h, \\
(\nabla \cdot U_c, w) &= (f, w), \quad w \in W^c_h.
\end{align*}
\]
3. Numerical results-mixed finite element case. In this section, we present numerical results for the OBA approach discussed above. We consider the problem

$$\nabla \cdot u = -\nabla \cdot (a \nabla p) = q,$$

on $\Omega = (0, 1) \times (0, 1)$, with $q = \delta(0, 0) - \delta(1, 1)$, where $\delta(x, y)$ is the Dirac delta function with its mass at $(x, y)$. We assume $u \cdot n_\Omega = 0$ on the boundary of $\Omega$; i.e., $g = 0$. We will consider different choices of the function $a(x, y)$.

First Test Problem. We first define $a(x, y)$ by

$$a(x, y) = \begin{cases} 1, & \frac{n}{10} < x < \frac{n+1}{10}, \quad n = 0, 2, 4, 6, 8, \\ 10^{-6}, & \frac{n}{10} < x < \frac{n+1}{10}, \quad n = 1, 3, 5, 7, 9. \end{cases}$$

In this case, the function $a$ is piecewise constant on a 10x10 uniform mesh. The velocity solution generated on this mesh using standard block-centered finite differences is given in Figure 2. We generated the effective coefficient $A$ on a 5 x 5 uniform grid. In this case, $A = \text{diag}(A_x, A_y)$ with $A_x << A_y$. A velocity field $U_c$ was generated on the coarse mesh by solving (27)-(28). These velocities are plotted in Figure 3. Note that these velocities capture the characteristics of the fine grid flow pattern.

Second Test Problem. Next, we define $a(x, y)$ as a piecewise constant on a 20 x 20 mesh with $a(x, y) = 1$ except for those blocks which intersect the lines

$$y = x + \alpha, \quad \alpha = -.9, -.7, -.5, \ldots, .5, .7, .9,$$
where $a = 10^{-6}$. The $20 \times 20$ finite difference solution for this case is given in Figure 4, which demonstrates that the flow is concentrated along the diagonal of $\Omega$, between the lines $y = x + .1$ and $y = x - .1$. The OBA solution with $A$ generated on a $5 \times 5$ mesh is given in Figure 5. Though the flow is smeared by the coarseness of the mesh, the essential features of the flow are still captured.

The effective coefficient $A$ in this case is

$$A = \begin{bmatrix} .272729 & .272275 \\ .272725 & .272729 \end{bmatrix}$$

This matrix has eigenvectors that point in the directions $(1, 1)$ and $(1, -1)$ and the associated eigenvalues are approximately $0.54$ and $27 \times 10^{-5}$, respectively. The solution of the differential problem associated with $A$ will therefore concentrate its flow strongly around the diagonal, more so than even the solution of the original problem. Call the flow field associated with this problem $W(x, y)$. The velocities shown in Figure 5, since they are computed from the mixed method, are the best fit in a weighted $L^2$ norm of $W$ in a certain space of functions. The best fit is taken in the space of divergence free velocities associated with the $5 \times 5$ Raviart-Thomas space.

*Third Test Problem.* In the description of operator based averaging above, we allowed for embedding each element $K$ in a larger element $K'$. We now demonstrate with an example that this padding procedure can produce better results. As in the second test problem, we generate a coefficient $a$ which is piecewise constant on a $20 \times 20$ mesh, with $a = 1$ except in certain cells. In particular, consider a $5 \times 5$ coarse mesh, with
Fig. 4. "True" velocities for second test problem

Fig. 5. OBA-mixed method velocities for second test problem
each coarse cell containing a $4 \times 4$ submesh. We will construct the same permeability pattern within each coarse grid block. Thus, let $a_{k,l}$, $k = 1, \ldots, 4$, $l = 1, \ldots, 4$ denote the values of $a$ on block $k$ in the x-direction and $l$ in the y-direction on each coarse cell. We set $a_{k,l} = 1$ except in blocks $(1, 3)$, $(1, 4)$, $(2, 4)$, $(2, 1)$, $(4, 1)$, $(4, 2)$, and $(4, 3)$, where $a_{k,l} = 10^{-6}$. In constructing the effective coefficient $A$, we use a $4 \times 4$ grid on each coarse grid block, and pad each block by 2 on all sides. Thus $K'$ is discretized using an $8 \times 8$ grid. With padding, we obtain an effective coefficient which allows for almost no flow in the $x$-direction; without padding, there is only slightly less flow in the $x$-direction than in the $y$-direction, which is clearly non-physical based on our choice of $a$. The result without padding can be seen in Figure 6, and the result with padding can be seen in Figure 7. The $20 \times 20$ finite difference solution is given in Figure 8. The figures demonstrate that by padding each coarse element we obtain velocities which are much closer to the "correct" flow pattern.

**Post-processing of velocities**

In many applications, the flow equation (29) is coupled with an equation describing the transport of some quantity. For example, in contaminant transport in groundwater, $u$ describes the advection of some contaminant through an aquifer. Ignoring diffusive effects, the transport process can be described by the linear advection equation

$$c_t + \nabla \cdot (uc) = q\tilde{c},$$

where $c$ represents the concentration of some contaminant, and $\tilde{c}$ is a specified concentration at sources, $\tilde{c} = c$ at sinks. In numerical simulation, the velocities generated from
Fig. 7. OBA-mixed method velocities with padding

Fig. 8. "True" velocities for third test problem
the flow code must be coupled with a numerical procedure for solving (32). On the same computational grid, solving (32) is generally less expensive than solving (29), thus one can often afford finer mesh for the solution of (32). We now describe two methods for post-processing the coarse mesh velocities generated by OBA to obtain velocities on a refined mesh.

Post-Processing Method 1. As in Section 2, let \( U_c = (U_c^x, U_c^y) \) denote the velocities obtained using OBA on a coarse mesh, and let \( K \) denote a coarse mesh element. A simple method for obtaining velocities on a refined mesh is to resolve the original equation

\[
\nabla \cdot u = -\nabla \cdot (a \nabla p) = f,
\]

on \( K \), with \( u \cdot n_K = U_c \cdot n_K \) on the boundary of \( K \). We applied this procedure to the second test problem above to obtain velocities on a 20 \( \times \) 20 mesh from a 5 \( \times \) 5 coarse mesh. These velocities were then substituted into a transport simulator (described in [1]). In Figures 9-12 concentration profiles after 10, 20, 30, and 40 time steps are given. The "true" solution on a 20 \( \times \) 20 mesh (using velocities generated on the same mesh), after 40 time steps, is given in Figure 13. While some features of the flow are captured, the results are less than satisfactory.

Post-Processing Method 2. A more complicated but decidedly better method for obtaining fine grid velocities involves a two-step process. First, we pad each coarse grid element \( K \) to form an enlarged element \( K' \). We determine normal velocities on
Fig. 10. Concentration contours after 20 time steps-postprocessing method 1

Fig. 11. Concentration contours after 30 time steps-postprocessing method 1
Fig. 12. Concentration contours after 40 time steps-postprocessing method 1

Fig. 13. Concentration contours after 40 time steps using "true" velocities
the boundary of $K'$ from $U_c$, recalling $U_c(x, y)$ is a piecewise polynomial function. We then solve (33) on $K'$, with the boundary condition $u \cdot n_{K'} = U_c \cdot n_{K'}$. Assuming the overlapping is not greater than the width of one grid block, we now have two sets of fine-grid velocities on $K$, obtained from two different local solves. In order to determine one fine grid velocity which satisfies (3) on each fine grid cell, we proceed to the next step.

On the boundary of $K$, we arithmetically average the normal components of the two fine-grid velocities obtained from the two local solves; denote this average by $U_{f,a}$. Let $\alpha_K$ denote the integral average of $U_{f,a} \cdot n_K$ over the boundary of $K$. We now resolve (33) over the element $K$, with $u \cdot n_K = U_c \cdot n_K + U_{f,a} \cdot n_K - \alpha_K$ on the boundary of $K$.

Numerical results for this approach for the second test problem are given in Figures 14-17, for 10, 20, 30, and 40 time steps, respectively. Though the solution is somewhat smeared, the gross features of the flow are much closer to the "true" solution given in Figure 13. In this experiment, we padded each coarse grid element by one coarse grid element on each side and diagonally.

4. A Galerkin OBA Method. Here we describe a method which uses local solutions of elliptic equations to form a Galerkin scheme. In this section $\Omega$ is allowed to be a domain in \( \mathbb{R}^2 \) with polygonal boundary, because that level of generality does not complicate the description significantly.

Take $T$ to be a collection of closed polygonal regions $T_\alpha$ such that (1) the interior of each $T_\alpha$ is contained in $\Omega$, (2) the union of the $T_\alpha$'s is the closure of $\Omega$, and (3) the
Fig. 15. Concentration contours after 20 time steps-postprocessing method 2

Fig. 16. Concentration contours after 30 time steps-postprocessing method 2
intersection of two $T_\alpha$'s is void, a vertex of each, or an entire edge of each. For example $T$ could be a triangulation of $\Omega$. The primary reason for wanting more generality in $T$ than just a triangulation is that it may be convenient to use a mixture of triangles and quadrilaterals.

Let $\mathcal{H}$ be the subspace of $H^1(\Omega)$ consisting of functions $\varphi$ where the trace of $\varphi$ on each edge of a region in $T$ is a polynomial of degree at most one. (The generalization to higher degrees is straightforward.) It follows that the restriction of such a function $\varphi$ to the boundary of $T_\alpha$ is continuous.

The Galerkin OBA approximation of $p$ is a function $P$ in $\mathcal{H}/\mathbb{R}$ such that

$$\mathcal{A}(P, v) = (f, v), \quad v \in \mathcal{H}/\mathbb{R}. \quad (34)$$

On the interior of each $T_\alpha$, $P$ satisfies the same differential equation as $p$. The function $P$ can be decomposed as

$$P = W + V,$$

where $W$ satisfies the differential equation in the interior of each $T_\alpha$ and vanishes on each edge of the $T_\alpha$'s. The function $V$ satisfies the differential equation with $f = 0$ in the interior of each $T_\alpha$, and $V$ is completely characterized by its values at the vertices of elements of $T$. In some cases $V$ is just a piecewise polynomial of degree at most one in each variable separately. In general $V$ will be more complicated because of variations in $\alpha$. 

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Since $W$ is completely locally determined on each $T_\alpha$, it can be approximated using calculations that are entirely local to the cell. Thus the approximation of $W$ is a problem that could be very effectively attacked in parallel. The set of all functions which are linear on the edges of $T$ and satisfy the homogeneous differential equation inside each subregion is a finite dimensional space $\mathcal{M}$. The function $V$ is determined by a finite dimensional linear system that is like the set of linear equations that define a usual Galerkin approximation over $T$, provided the function space for the Galerkin method is parametrized by the values of the functions at vertices.

In practical use the function $P$ will be approximated by using a fine grid on each $T_\alpha$. A basis for $\mathcal{M}$ can be computed locally and the equations that define $V$ are built up from small sets of coefficients that are determined on each $T_\alpha$

Because the OBA Galerkin method is just a usual Galerkin method with a special choice of the approximating space, standard results on finite element methods imply that $P$ is close to $p$ if anything in the approximating space is close to $p$.

**Special Case**

Suppose that we use a tensor product mesh like the one used in Section 2. Consider the case in which $a$ is constant on each subregion $T_\alpha$. In those subregions in which $f = 0$ the elements of $\mathcal{H}$ are just tensor polynomials of degree at most one in each variable. This special case is one that has been widely studied and can be used to provide insights into the behavior of this method.

**Extension**

Since the process discussed in Section 2 for the mixed method produced numerically an homogenized coefficient matrix $A$ for the differential operator, we can use those coefficient matrix in a Galerkin method (or even a finite difference method). One simply uses the given mixed method technique to compute the effective diffusivity matrix on each subdivision; this diffusivity is then used to generate a discrete Galerkin method using standard piecewise polynomial subspaces.

**Local Reconstruction**

The OBA Galerkin method provides a function which has the possibility of having rich local structure because of the flexibility of the function space inside the subregions $T_\alpha$. After solving the global system that defines $V$ one can combine $V$ with the locally defined function $W$. It is possible that $W$ will be quite small and it may be appropriate to neglect it in regions where $f$ is moderate. (If $f$ is zero, $W$ will be too. If $f$ is bounded, $W$ will be of the size of the square of the diameter of the subregion $T_\alpha$.)

**Reconstruction of Flux**

Suppose that the flux $u = -a \nabla p$ is really the quantity of interest in solving (1). The flux defined by the Galerkin method is discontinuous at the boundaries of the $T_\alpha$ subregions. In the special case mentioned above we know that this flux is not very accurate except at the centers of the grid blocks. In the case of Galerkin methods built on piecewise polynomials we would like to use a flux reconstruction technique based on the differential equation like the one discussed in [5] and [2].

**5. Numerical results-Galerkin case.** To test the OBA Galerkin method we looked at a variant of the second test problem of section 3. The coefficient $a$ is defined
by the following formula for all integer values of $k$:

$$a(x, y) = \begin{cases} 
10^{-6}, & 5k + 2 < 25(y - x) < 5k + 3, \\
1, & \text{otherwise};
\end{cases}$$

see Figure 18. This problem was solved using a $5 \times 5$ mesh. The solution on the 25 subregions was actually approximated using a piecewise linear over triangles Galerkin method on a finer mesh; with compatible meshes this is really just a 5-point difference scheme on the fine mesh.

It is interesting to observe that if we use the standard Galerkin method based on tensor product piecewise linear functions on this coarse mesh, the fluxes are just those associated with the constant conductivity problem. In fact the matrix for the problem with the tight streaks is just a constant $k$ times the matrix with $a \equiv 1$. The constant $k$ is just the average of $a$ on any horizontal or vertical line, $k = 0.6 + 0.4 \times 10^{-6}$.

The flux was reconstructed as discussed above. This gives an $x$-flux and a $y$-flux, each of which is a tensor product piecewise linear function over the $5 \times 5$ mesh. The flux is shown in Figure 19. The flow is more strongly focused in the center region than it is with a uniform conductivity, but still quite far from the true solution in which essentially all the flow would be in the center strip from $(0, 0)$ to $(1, 1)$.

To give a different view of the ability of these schemes to mimic the behavior of the true solution even on this very coarse mesh we show in Figure 20 the flux across a line $L$ that is the diagonal on which $y - x = 1$. The flux across this line has integral one
Fig. 19. OBA-Galerkin fluxes for test problem

and in the differential problem is almost entirely concentrated in the center “tube” that connects the source and sink. One can clearly see that the OBA Galerkin scheme has more flow down the center than the usual Galerkin method, which is the same as the uniform conductivity case shown. Even though the second test problem for the mixed method is slightly different from this test problem it has the same character. The flux across $L$ is also shown for the mixed OBA method; it certainly is more effective at putting the flow in the center than the Galerkin method is, for the same mesh. (It is the case that the mixed method solution is defined by 85 unknowns and the Galerkin solutions are defined by 36.)

The integral under the curves on Figure 20 should be $1/\sqrt{2}$, since they are plotted as functions of $x$ instead of arc length along $L$. For the mixed method this is exactly true, since the flow field is divergence free in the interior. For the Galerkin schemes a similar identity holds, but only for lines parallel to the axes. For the cases shown, the flux for the Galerkin methods is slightly lower than the correct value.

6. Conclusions. Several numerical methods for incorporating “averaged” fine scale information about rough coefficients in elliptic operators have been presented. The basic idea is to solve local elliptic problems to determine the effective behavior of the coefficient over a subregion. These local problems are independent and can be solved in parallel. The pseudo-coefficients can then be used to calculate coarse-grid solutions which capture the essential features of the flow. The averaging procedures are based on Galerkin and mixed finite element discretizations. For the mixed finite element case, we
have restricted our attention to the lowest-order Raviart-Thomas spaces on rectangles. The effectiveness of these procedures on several test problems has been presented. For the cases considered here, the OBA procedure based on the mixed finite element method was found to be somewhat superior to the Galerkin variant; however the Galerkin method has more flexibility in approximating spaces and is defined for general triangulations/rectangulations. For the mixed method, it was also observed that padding the subregions was often helpful in calculating a more effective pseudo-coefficient.

The coarse-grid velocities calculated using the OBA pseudo-coefficient can be post-processed to obtain velocities on a finer grid. Two post-processing method were discussed, with the more complicated method giving better results. These post-processing methods also involve solving local problems in parallel.

REFERENCES
