Calculation and Implementation of the Gradient of the DSO Objective Function for the General Acoustic Model

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Abstract

We present in this paper the computation of the DSO objective function in the general acoustic case. In this model, the density and the velocity are functions of the space variables. We use a perturbationnal approach, justified by the separation of scales between the long and short wavelength components of the model. An extension of the adjoint state technique yields an accurate expression of the gradient of the DSO objective function. Then we use a finite difference approximation of the wave equation, and give in the discrete case the expression of the gradient. We show in that case how to apply the principle of images, so that the discrete operators involved are self-adjoint and give exact discrete integration by parts.
We define the reflectivities (cf [1]), as the relative perturbations, that is:
\[ r_\rho = \frac{\delta \rho}{\rho_0} \quad r_c = \frac{\delta c}{c_0} \]

Supposing that \( f, r_\rho \) and \( r_c \) do not have overlapping supports, we can write (3) as follows
\[
\left\{ \begin{aligned}
\frac{1}{\rho_0 c_0^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho_0} \nabla \delta u \right) &= (r_\rho + 2r_c) \nabla \left( \frac{1}{\rho_0} \nabla u_0 \right) - \nabla \left( \frac{r_\rho}{\rho_0} \nabla u_0 \right) \\
\delta u(x, z, 0) &= \frac{\partial \delta u}{\partial t}(x, z, 0) = 0 \\
\delta u(0, z, t) &= \delta u(X, z, t) = \delta u(x, 0, t) = \delta u(x, Z, t) = 0
\end{aligned} \right.
\]

We are now able to define the forward map \( F \) of our inverse problem. It maps the functions defining the medium, the density \( \rho \), the velocity \( c \), the reflectivity in density \( r_\rho \) and the reflectivity in velocity \( r_c \) to the seismogram produced in this medium at the array of receivers \( (x_r, z_r)_{r=1...R} \) by the source \( f(t) \) located in \( (x_s, z_s)_{s=1...S} \).
Therefore we define the forward map by:
\[
F(\rho, c, r_\rho, r_c)(t; x_s, z_s) = \sum_{s=1}^{S} \sum_{r=1}^{R} u(x_r, z_r, t; x_s, z_s)
\]
where \( u \) satisfies:
\[
\left\{ \begin{aligned}
\frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) &= f(x, z, t; x_s, z_s) \\
\frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u \right) &= (r_\rho + 2r_c) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{r_\rho}{\rho} \nabla u_0 \right) \\
u(x, z, 0) &= u_0(x, z, 0) = \frac{\partial u}{\partial t}(x, z, 0) = \frac{\partial u_0}{\partial t}(x, z, 0) = 0 \\
u(0, z, t) &= u_0(x, 0, t) = u_0(X, z, t) = u_0(x, Z, t) = 0 \\
u(0, z, t) &= u(x, 0, t) = u(X, z, t) = u(x, Z, t) = 0
\end{aligned} \right.
\]
Calculation of the Gradient for the DSO Objective Function

the derivative $D_{r_p}F$. This simply means that:

$$\begin{align*}
\nabla_{r_p} J_{DS} &= D_{r_p} F^*(F - F_{data}) - \sigma_p^2 \frac{\partial^2 r_p}{\partial x_s^2} + \lambda_p^2 W^T W r_p \\
\nabla_{r_c} J_{DS} &= D_{r_c} F^*(F - F_{data}) - \sigma_c^2 \frac{\partial^2 r_c}{\partial x_s^2} + \lambda_c^2 W^T W r_c
\end{align*}$$

(7)

When we use $W = \nabla_{x_s}$, we have to suppose that $r_p$ and $r_c$ belong to $H^2(\Omega)$. Setting the gradients to zero in (7) we get the following normal equations:

$$\begin{align*}
M_{r_p} F - \sigma_p^2 \frac{\partial^2 r_p}{\partial x_s^2} + \lambda_p^2 W^T W r_p &= M_{r_p} F_{data} \\
M_{r_c} F - \sigma_c^2 \frac{\partial^2 r_c}{\partial x_s^2} + \lambda_c^2 W^T W r_c &= M_{r_c} F_{data}
\end{align*}$$

(8)

Now we need to know the effect of the operator $D_{r_p} F^* = M_{r_p}$ and $D_{r_c} F^* = M_{r_c}$ on some seismogram $\varphi(x_r, z_r, t; x_s, z_s)$. Since $F$ is linear in $r_p$ and $r_c$, we have:

$$\begin{align*}
D_{r_p} F, \delta r_p &= F(p, c, \delta r_p, 0) \\
D_{r_c} F, \delta r_c &= F(p, c, 0, \delta r_c)
\end{align*}$$

therefore

$$\begin{align*}
(M_{r_p} \varphi, \delta r_p)_{L^2(\Omega)} &= \sum_{s=1}^{S} \sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), D_{r_p} F(p, c, r_p, r_c), \delta r_p)_{L^2(0; T)} \\
&= \sum_{s=1}^{S} \sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), F(p, c, r_p, 0), \delta r_p)_{L^2(0; T)}
\end{align*}$$
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \delta r_p \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{\delta r_p}{\rho} \nabla u_0 \right) \right) w_0(x, z, t; x_s, z_s) \, dx \, dz \, dt \\
= \int_{\Omega} \int_{0}^{T} \sum_{s=1}^{S} \left( w_0 \nabla \left( \frac{1}{\rho} \nabla u_0 \right) + \frac{1}{\rho} \nabla w_0 \nabla u_0 \right) dt \, \delta r_p \, dx \, dz \\
whence \frac{\partial}{\partial t} M_{r_p} \varphi = \sum_{s=1}^{S} \int_{0}^{T} w_0 \nabla \left( \frac{1}{\rho} \nabla u_0 \right) + \frac{1}{\rho} \nabla w_0 \nabla u_0 \, dt \\
in the same way we have:

\( (M_{r_p} \varphi, \delta r_c)_{L^2(\Omega)} = \sum_{s=1}^{S} \sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), D_{r_c} F(\rho, c, \rho, \varphi, r_c) \delta r_c)_{L^2(0; T)} \)

\( = \sum_{s=1}^{S} \sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), F(\rho, c, 0, r_c) \delta r_c)_{L^2(0; T)} \)

We have \( F(\rho, c, 0, r_c) = \delta u \) solution of:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) &= 2 \delta r_c \nabla \left( \frac{1}{\rho} \nabla u_0 \right) \\
\frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) &= f(x, z, t; x_s, z_s) \\
\delta u(x, z, 0) &= \frac{\partial \delta u}{\partial t}(x, z, 0) = 0 \\
\delta u(0, z, t) &= \delta u(X, z, t) = \delta u(x, 0, t) = \delta u(x, Z, t) = 0 \\
u_0(x, z, 0) &= \frac{\partial u_0}{\partial t}(x, z, 0) = 0 \\
u_0(0, z, t) &= u_0(X, z, t) = u_0(x, 0, t) = u_0(x, Z, t) = 0
\end{align*}
\]

then we have:

\( (M_{r_c} \varphi, \delta r_c)_{L^2(\Omega)} = \sum_{s=1}^{S} \int_{0}^{T} \sum_{r=1}^{R} \varphi(x_r, z_r, t; x_s, z_s) \delta u(x, z, t; x_s, z_s) \, dt \)
Calculation of the Gradient for the DSO Objective Function

\[
+ D_{\rho} J_{DS}(\rho, c, r_{\rho}(\rho, c), r_{c}(\rho, c)) (D_{\rho} r_{\rho}(\rho, c), \delta \rho + D_{\rho} r_{c}(\rho, c), \delta c) \\
+ D_{c} J_{DS}(\rho, c, r_{\rho}(\rho, c), r_{c}(\rho, c)) (D_{\rho} r_{\rho}(\rho, c), \delta \rho + D_{c} r_{c}(\rho, c), \delta c)
\]

But since we assumed that the normal equations have been solved exactly, cf (7) then:

\[
\begin{align*}
D_{\rho} J_{DS}(\rho, c, r_{\rho}(\rho, c), r_{c}(\rho, c)) &= 0 \\
D_{c} J_{DS}(\rho, c, r_{\rho}(\rho, c), r_{c}(\rho, c)) &= 0
\end{align*}
\]

and we get the following simpler expression for the derivative of \( \tilde{J} \):

\[
D \tilde{J}(\rho, c), (\delta \rho, \delta c) = D_{\rho} J_{DS}(\rho, c, r_{\rho}(\rho, c), r_{c}(\rho, c)) \delta \rho + D_{c} J_{DS}(\rho, c, r_{\rho}(\rho, c), r_{c}(\rho, c)) \delta c
\]

Since

\[
\tilde{J}(\rho, c) = \frac{1}{2} \|F(\rho, c, r_{\rho}, r_{c}) - F_{data}\|_{L_{2}(0; T)}^{2} + \xi(r_{\rho}, r_{c})
\]

where \( \xi \) does not depend explicitly on \( \rho \) and \( c \), we have:

\[
\begin{align*}
D_{\rho} J_{DS}(\rho, c, r_{\rho}, r_{c}) \delta \rho &= (D_{\rho} F(\rho, c, r_{\rho}, r_{c}) \delta \rho, F(\rho, c, r_{\rho}, r_{c}) - F_{data})_{L_{2}(0; T)} \\
D_{c} J_{DS}(\rho, c, r_{\rho}, r_{c}) \delta c &= (D_{c} F(\rho, c, r_{\rho}, r_{c}) \delta c, F(\rho, c, r_{\rho}, r_{c}) - F_{data})_{L_{2}(0; T)}
\end{align*}
\]

Now to compute the gradients of \( \tilde{J} \) with respect to \( \rho \) and \( c \), we must find two bilinear forms \( B_{\rho} \) and \( B_{c} \) such that

\[
(D_{\rho} F(\rho, c, r_{\rho}, r_{c}) \delta \rho, F(\rho, c, r_{\rho}, r_{c}) - F_{data})_{L_{2}(0; T)} = (\delta \rho, B_{\rho}(F(\rho, c, r_{\rho}, r_{c}) - F_{data}))_{L_{2}(\Omega)}
\]

\[
(D_{c} F(\rho, c, r_{\rho}, r_{c}) \delta c, F(\rho, c, r_{\rho}, r_{c}) - F_{data})_{L_{2}(0; T)} = (\delta c, B_{c}(F(\rho, c, r_{\rho}, r_{c}) - F_{data}))_{L_{2}(\Omega)}
\]

Remark

\( r_{\rho} \) and \( r_{c} \) being chosen as the solution of the normal equations, they are fixed. To enhance this fact, we use a different notation and from now on we will use \( q_{\rho} \) for \( r_{\rho} \) and \( q_{c} \) for \( r_{c} \).
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Given a seismogram \( \varphi \), we can now evaluate the following quantity:

\[
(D_p F(\rho, c, r_p, r_c), \delta \rho, \varphi)_{L^2(0,T)} = \sum_{s=1}^{S} \sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), \delta u(x_r, z_r; x_s, z_s))_{L^2(0,T)}
\]

\[
= \sum_{s=1}^{S} \int_0^T \int_{\Omega} \sum_{r=1}^{R} \varphi(x_r, z_r, t; x_s, z_s) \delta u(x, z, t; x_s, z_s) \delta(x - x_s, z - z_s) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_0^T \int_{\Omega} \left( \frac{1}{\rho c^2} \frac{\partial^2 w_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla w_0 \right) \right) \delta u(x, z, t; x_s, z_s) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_0^T \int_{\Omega} w_0(x, z, t; x_s, z_s) \left( \frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) \right) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_0^T \int_{\Omega} \frac{w_0}{\rho} \left( \nabla \left( \frac{1}{\rho} \nabla u \right) + (q_p + 2q_c) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{q_p}{\rho} \nabla u_0 \right) \right) \delta \rho \, dx \, dz \, dt
\]

\[
+ \int_0^T \int_{\Omega} \frac{1}{\rho^2} \left( \nabla w_0 \nabla u + \nabla((q_p + 2q_c)w_0) \nabla u_0 - q_p \nabla w_0 \nabla u_0 \delta \rho \right) \, dx \, dz \, dt
\]

\[
+ \int_0^T \int_{\Omega} \left( \nabla \left( \frac{1}{\rho} \nabla (q_p + 2q_c)w_0 \right) - \nabla \left( \frac{q_p}{\rho} \nabla w_0 \right) \right) \delta u_0 \, dx \, dz \, dt
\]

where \( w_0 \) is the solution of (10). To find the expression of the gradient of \( \tilde{J} \) with respect to \( \rho \), we need to work on the last integral. We introduce another adjoint state \( w \) defined by:

\[
\begin{cases}
\frac{1}{\rho c^2} \frac{\partial^2 w}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla w \right) = \nabla \left( \frac{1}{\rho} \nabla (q_p + 2q_c)w_0 \right) - \nabla \left( \frac{q_p}{\rho} \nabla w_0 \right) \\
w(x, z, T) = \frac{\partial w}{\partial t}(x, z, T) = 0 \\
w(0, z, t) = w(X, z, t) = w(x, 0, t) = w(x, Z, t) = 0
\end{cases}
\]

(15)
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From (13) we can derive the equation verified by \( \delta u = D_c F(\rho, c, q_\rho, q_c) \cdot \delta c \):

\[
\frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) = 2 \frac{\delta c}{c} \nabla \left( \frac{1}{\rho} \nabla u \right) + 2 \frac{\delta c}{c} \left\{ (q_\rho + 2q_c) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{q_\rho}{\rho} \nabla u_0 \right) \right\} + (q_\rho + 2q_c) \nabla \left( \frac{1}{\rho} \nabla \delta u_0 \right) - \nabla \left( \frac{q_\rho}{\rho} \nabla \delta u_0 \right)
\]

(17)

\[
\frac{1}{\rho c^2} \frac{\partial^2 \delta u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u_0 \right) = 2 \frac{\delta c}{c} \nabla \left( \frac{1}{\rho} \nabla u_0 \right)
\]

\[
\delta u(x, z, 0) = \delta u_0(x, z, 0) = \frac{\partial \delta u}{\partial t} (x, z, 0) = \frac{\partial \delta u_0}{\partial t} (x, z, 0) = 0
\]

\[
\delta u_0(0, z, t) = \delta u_0(x, 0, t) = \delta u_0(X, z, t) = \delta u_0(x, Z, t) = 0
\]

\[
\delta u(0, z, t) = \delta u(x, 0, t) = \delta u(X, z, t) = \delta u(x, Z, t) = 0
\]

Given a seismogram \( \varphi \), we can now evaluate the following quantity:

\[
(D_c F(\rho, c, r_\rho, r_c) \cdot \delta c, \varphi)_{L^2(0,T)} = \sum_{s=1}^{S} \sum_{r=1}^{R} (\varphi(x, z; x_r, z_s), \delta u(x, z; x_r, z_s))_{L^2(0,T)}
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \sum_{r=1}^{R} \varphi(x, z; x_r, t; x_s, z_s) \delta u(x, z; t; x_s, z_s) \delta(x - x_s, z - z_s) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \frac{1}{\rho c^2} \frac{\partial^2 w_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla w_0 \right) \right) \delta u(x, z; t; x_s, z_s) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} w_0(x, z; t; x_s, z_s) \left( \frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) \right) \, dx \, dz \, dt
\]
4 Implementation of the Gradient

We now turn to the implementation aspects of the computation of the two gradients obtained above. We are going to use a finite difference method to compute the different wave fields we need. We summarize below the expressions of the two gradients and the equations we need to discretize to compute them.

\[
\nabla_p \bar{J} = \sum_{s=1}^{S} \int_0^T \frac{w_0}{\rho} \left( \nabla \left( \frac{1}{\rho} \nabla u \right) + (q_\rho + 2q_c) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{q_c}{\rho} \nabla u_0 \right) \right) \, dt \\
+ \sum_{s=1}^{S} \int_0^T \frac{1}{\rho^2} \left( \nabla w_0 \nabla u + \nabla ((q_\rho + 2q_c)w_0) \nabla u_0 - q_\rho \nabla w_0 \nabla u_0 \right) \, dt \\
+ \sum_{s=1}^{S} \int_0^T \frac{w}{\rho} \nabla \left( \frac{1}{\rho} \nabla u_0 \right) + \frac{1}{\rho^2} \nabla w \nabla u_0 \right) \, dt \\

\nabla_c \bar{J} = \sum_{s=1}^{S} \int_0^T \frac{w_0}{c} \left( \nabla \left( \frac{1}{\rho} \nabla u \right) + (q_\rho + 2q_c) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \frac{w_0}{c} \nabla \left( \frac{q_c}{\rho} \nabla u_0 \right) \right) \, dt \\
+ \sum_{s=1}^{S} \int_0^T \frac{w}{c} \nabla \left( \frac{1}{\rho} \nabla u_0 \right) \, dt
\]

where the two direct states \( u_0, u \) are solutions of:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) &= f(x, z, t; x_s, z_s) \\

u_0(x, z, 0) &= \frac{\partial u_0}{\partial t} (x, z, 0) = 0 \\
u_0(0, z, t) &= u_0(X, z, t) = u_0(x, 0, t) = u_0(x, Z, t) = 0 \\
\frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u \right) &= (q_\rho + 2q_c) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{q_c}{\rho} \nabla u_0 \right) \\
u(x, z, 0) &= \frac{\partial u}{\partial t} (x, z, 0) = 0 \\
u(0, z, t) &= u(X, z, t) = u(x, 0, t) = u(x, Z, t) = 0
\end{align*}
\]
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\[ L_{o,o}^2 = \left\{ \varphi \in L^2(\Omega) \mid \varphi = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \varphi_{i,j+\frac{1}{2}} \mathbf{1}_{[(i-\frac{1}{2})\Delta x, (i+\frac{1}{2})\Delta x) \times [(j-\frac{1}{2})\Delta z, (j+\frac{1}{2})\Delta z]}(x, z) \right\} \]

\[ L_{*,o}^2 = \left\{ \varphi \in L^2(\Omega) \mid \varphi = \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \varphi_{i,j+\frac{1}{2}} \mathbf{1}_{[(i+1)\Delta x, (i+1+\frac{1}{2})\Delta x) \times [(j-\frac{1}{2})\Delta z, (j+\frac{1}{2})\Delta z]}(x, z) \right\} \]

where

\[ \mathbf{1}_{[a,b] \times [c,d]}(x, z) = \begin{cases} 1 & (x, z) \in [a, b] \times [c, d] \\ 0 & (x, z) \notin [a, b] \times [c, d] \end{cases} \]

We approximate the first derivative by the following operator:

\[ A_x^o : L_{o,o}^2 \longrightarrow L_{*,o}^2 \]

\[ u \longmapsto A_x^o u(i + \frac{1}{2}, j) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta x} [u(i + l, j) - u(i - l + 1, j)] \]

\[ A_x^o \] is a finite difference approximation of order 2L in \((i + \frac{1}{2})\Delta x, j \Delta z\) of the quantity \( \frac{\partial u}{\partial x} \), with the coefficients \((\beta_l)_{l=1,L}\) defined in appendix 1. The exponent refers to the departure set \( L_{o,o}^2 \); the subscript to the direction of differentiation. Similarly we define:

\[ A_z^o : L_{o,o}^2 \longrightarrow L_{*,o}^2 \]

\[ u \longmapsto A_z^o u(i, j + \frac{1}{2}) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta z} [u(i, j + l) - u(i, j - l + 1)] \]

\[ A_z^o \]

\[ v \longmapsto A_x^o v(i, j + \frac{1}{2}) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta x} [v(i + l + \frac{1}{2}, j + \frac{1}{2}) - v(i - l + \frac{1}{2}, j + \frac{1}{2})] \]

\[ A_z^o : L_{*,o}^2 \longrightarrow L_{*,o}^2 \]

\[ v \longmapsto A_z^o v(i, j + \frac{1}{2}) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta z} [u(i + \frac{1}{2}, j + l + \frac{1}{2}) - u(i + \frac{1}{2}, j - l + \frac{1}{2})] \]

We approximate the quantity \( \nabla (\frac{1}{\rho} \nabla u) \) by \( \nabla_h (\frac{1}{\rho} \nabla_h u) = -t A_x^o (\frac{1}{\rho} A_x^o u) - t A_z^o (\frac{1}{\rho} A_z^o u) \)
with \( a_i = \beta_i / \Delta x \) and:

\[
L^2(\Omega_o) = \left\{ \varphi \in L^2(\Omega) \ / \ \varphi = \sum_{i=1}^{N_x} \varphi_i 1_{(i-\frac{1}{2})\Delta x, (i+\frac{1}{2})\Delta x}(x) \right\}
\]

\[
L^2(\Omega_\ast) = \left\{ \varphi \in L^2(\Omega) \ / \ \varphi = \sum_{i=1}^{N_x-1} \varphi_{i+\frac{1}{2}} 1_{i\Delta x, (i+1)\Delta x} \right\}
\]

Those spaces are provided with the usual scalar products defined by:

\[
(f, g)_{L^2(\Omega_o)} = (f, g)_o = \sum_{i=1}^{N} f_i g_i \Delta x \Delta z
\]

\[
(f, g)_{L^2(\Omega_\ast)} = (f, g)_\ast = \sum_{i=1}^{N-1} f_{i+\frac{1}{2}} g_{i+\frac{1}{2}} \Delta x \Delta z
\]

We suppose that the boundaries of the domain are located in \( i = 1 \) and \( i = N \), so that \( u_1 = u_N = 0 \). We extend \( u \in \Omega_o \), outside \( \Omega_o \) by the following procedure:

\[
u_{1-k} = -u_{1+k} \quad k = 0..L - 1
\]

\[
u_{N+k} = -u_{N-k} \quad k = 0..L - 1
\]

That is we skew-symetrize \( u \), at the boundary. Now we want to find under what conditions we have

\[(Au, v)_\ast = (v, \, ^tAu)_o\]
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We have

\[
B_1 = \sum_{l=1}^{L} a_l \left( \sum_{j=1}^{i} u_j \ v_{j-l+\frac{1}{2}} + \sum_{j=2-l}^{0} u_j \ v_{j+l-\frac{1}{2}} \right) \\
= \sum_{l=1}^{L} a_l \left( \sum_{j=2}^{i} u_j \ v_{j-l+\frac{1}{2}} + \sum_{k=1}^{i-1} u_{1-k} \ v_{k+l+\frac{1}{2}} \right) \\
= \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l-1} u_{1+k} \ v_{j-l+\frac{1}{2}} + u_{1-k} \ v_{k+l+\frac{1}{2}} \right) \\
= \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l-1} u_{1+k} \left( v_{j-l+\frac{1}{2}} - v_{k+l+\frac{1}{2}} \right) \right)
\]

and

\[
B_2 = \sum_{l=1}^{L} a_l \left( \sum_{j=N+1}^{N+l-1} u_j \ v_{j-l+\frac{1}{2}} + \sum_{j=N-l+1}^{N} u_j \ v_{j+l-\frac{1}{2}} \right) \\
= \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l-1} u_{N+k} \ v_{N+k-l+\frac{1}{2}} + \sum_{k=0}^{l-1} u_{N-k} \ v_{N-k+l-\frac{1}{2}} \right) \\
= \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l-1} u_{N+k} \ v_{N+k-l+\frac{1}{2}} + \sum_{k=1}^{l-1} u_{N-k} \ v_{N-k+l-\frac{1}{2}} \right) \\
= \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l-1} u_{N-k} \left( v_{N-k+l-\frac{1}{2}} - v_{N+k-l+\frac{1}{2}} \right) \right)
\]

We see that if we extend \( v \) outside \( \Omega_+ \) by symmetry, that is:

\[
v_{-k+\frac{1}{2}} = v_{k+\frac{1}{2}} \quad k = 0..L-1 \\
v_{N+k+\frac{1}{2}} = v_{N+k-\frac{1}{2}} \quad k = 0..L-1
\]

we annihilate the boundary terms \( B_1 \) and \( B_2 \).
Calculation of the Gradient for the DSO Objective Function

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 w_{0,h}}{\partial t^2} - \nabla_h\left(\frac{1}{\rho} \nabla_h w_{0,h}\right) &= \sum_{r=1}^{R} (F(\rho, c, q_p, q_c)(x_r, z_r, t; x_s, z_s) - F_{data}(t)) \delta(x - x_s, z - z_s) \\
\frac{\partial w_{0,h}}{\partial t}(x, z, T) &= 0 \\
w_{0,h}(0, z, t) &= w_{0,h}(X, z, t) = w_{0,h}(x, 0, t) = w_{0,h}(x, Z, t) = 0
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 w_h}{\partial t^2} - \nabla_h\left(\frac{1}{\rho} \nabla_h w_h\right) &= \nabla_h\left(\frac{1}{\rho} \nabla_h (q_p + 2q_c) w_{0,h}\right) - \nabla_h\left(\frac{q_p}{\rho} \nabla_h w_{0,h}\right) \\
\frac{\partial w_h}{\partial t}(x, z, T) &= 0 \\
w_h(0, z, t) &= w_h(X, z, t) = w_h(x, 0, t) = w_h(x, Z, t) = 0
\end{align*}
\]

The two gradients are now given by

\[
\nabla_{\rho,h} \mathcal{J} = \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{w_{0,h}}{\rho} \left( \nabla_h\left(\frac{1}{\rho} \nabla_h u_h\right) + (q_p + 2q_c) \nabla_h\left(\frac{1}{\rho} \nabla_h u_{0,h}\right) - \nabla_h\left(\frac{q_p}{\rho} \nabla_h u_{0,h}\right) \right) \Delta t \\
+ \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{1}{\rho^2} \left( \nabla_h w_{0,h} \nabla_h u_h + \nabla_h((q_p + 2q_c) w_{0,h}) \nabla_h u_{0,h} - q_p \nabla_h w_{0,h} \nabla_h u_{0,h} \right) \Delta t \\
+ \sum_{s=1}^{S} \sum_{n=1}^{N} \left( \frac{w_h}{\rho} \nabla_h\left(\frac{1}{\rho} \nabla_h u_{0,h}\right) + \frac{1}{\rho^2} \nabla_h w_h \nabla_h u_{0,h} \right) \Delta t
\]

\[
\nabla_{c,h} \mathcal{J} = \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{2w_{0,h}}{c} \left( \nabla_h\left(\frac{1}{\rho} \nabla_h u_h\right) + (q_p + 2q_c) \nabla_h\left(\frac{1}{\rho} \nabla_h u_{0,h}\right) - 2 \frac{w_h}{c} \nabla_h\left(\frac{q_p}{\rho} \nabla_h u_{0,h}\right) \right) \Delta t \\
+ \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{2w_h}{c} \nabla_h\left(\frac{1}{\rho} \nabla_h u_{0,h}\right) \Delta t
\]
References


