Design Against Resonance

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by

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Abstract

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A method for maximizing the distance from the spectrum of an analytic, symmetric matrix with distinct eigenvalues from a given frequency is proposed. The method models the classical approach from optimization of finding the first derivative of the function to be maximized and setting it equal to zero. The function to be maximized is the norm of a resolvent in terms of the given perturbed matrix and the applied frequency. The problem of the function being nonsmooth is considered, and a solution is proposed. This solution entails using the concept of the generalized gradient. This method provides an efficient means for targeting the potential trouble spots for resonance and avoiding them with relative ease.
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Chapter 1

Introduction

When a force is applied to a body or mechanism at a frequency equal or very near to the natural frequency of the body or mechanism, the phenomena of resonance occurs. The result of resonance is that the amplitude of the forced oscillation by the applied force peaks, often beyond the strength or tolerance levels of the materials in the mechanism. As seen in the case of the Tacoma Narrows bridge, resonance can be incredibly destructive. Consequently engineers seek to design mechanical systems such as bridges, beams, shafts, etc., so to avoid resonance. Understanding the mathematics which governs the mechanics of the system is essential to designing mechanical systems to avoid resonance.

Mathematically speaking, the natural frequencies of a mechanism or body are its eigenvalues. Formulating a matrix to represent mathematically the motion or energy of the mechanism or body often involves finite elements or finite differences. Once the matrix is formulated, the problem of avoiding resonance becomes that of maximizing the distance from the spectrum, or set of eigenvalues, of the representative matrix from the applied frequency. In this project, the matrix is considered both analytic and symmetric, as well as having distinct eigenvalues and only one parameter. Restricting to such a case keeps the procedure introduced in this project at a simple level, but is still valid. As in the case of the Sturm-Liouville problem with separated boundary conditions, such a restriction does exist in "real life" problems.

Some proposed methods for this maximization problem are inefficient and cumbersome by the way in the distance from an external or applied frequency from the
spectrum of the matrix is measured. The motivation for the research in this thesis comes largely from observing these inefficiencies in other proposed techniques. Particularly in the method given by Bendsøe and Olhoff (1985), the given procedure is made inefficient by overlooking a way to directly pinpoint where the distance between the spectrum of the system they are working with and the driving frequency has a minimum. The problem they try to solve is maximizing the distance between the driving frequency and the natural frequency of a beam or shaft by using the cross-sectional area of the beam or shaft as the control variable. The equation they use to represent the vibration or whirling mode \( w(x) \) is

\[
(cE D^p w'')'' = \omega^2 \rho Dw
\]  

(1.1)

where

- \( \omega \) = natural frequency
- \( D(x) \) = cross-sectional area
- \( E \) = Young’s modulus
- \( \rho \) = mass density
- \( c, p \) : \( cD^p \) is the moment of interia of the cross-section

This model can be found in Love (1919), section 286. The objective they work towards in avoiding resonance is finding the best cross-sectional area distribution of the shaft or beam. The solution to the problem they formulate produces the maximum value for which the \( N-1 \) lower-order eigenfrequencies are less than or equal to the difference of the external excitation frequency and this maximum value. The value of \( N \) must be found by solving their formulated optimization problem, which utilizes a finite element method, for different values of \( N \) until a solution is found which corresponds
to the given values of $\Omega, W,$ and $l$, the external excitation frequency, total volume, and length, respectively. The system they use to find $N$ actually starts with $N = 1$ and increases $N$ as needed until the optimum value is found.

Clearly this method is inefficient, as the targeted value of $N$ could be as close as one or much further. This problem can be improved by subtracting the driving frequency from the linear system $A(t)$ and using the fact that the norm of $A(t) - \omega$, where $\omega$ is the driving frequency, is its maximum eigenvalue. Numerically this maximum can be found more efficiently than observing each eigenvalue one by one.

The maximization process presented in this thesis follows the classical approach of solving for the first derivative and setting it to zero. In keeping with the classical approach, several problems are encountered. One problem is finding the function in which to maximize to find the desired solution. Another problem is to find the first derivative of this function, for it is possible, and in fact true for the particular function used in this project, that the function is not smooth. The first problem is tackled by choosing to use an inner product function involving a resolvent $(A(t) - \omega)^{-1}$. The second problem is solved with the aid of Kato (1980) and the concept of generalized gradients as introduced in Clarke (1983). The full statement of the problem is as such:

Given

\[
\begin{align*}
S^n & = \text{the set of symmetric linear operators} \\
\tau & \mapsto A(t) \\
A & \in C^1(\mathbb{R}, S^n) \\
\sigma(t) & = \text{Spectrum of } A(t) \\
G(t) & = \text{dist}(\omega, \sigma(t)),
\end{align*}
\]
find

$$\max_t G(t)$$

(1.2)

To serve as a visual aid in understanding the nontriviality of this problem, the matrix

$$A(t) = \begin{bmatrix}
1 & 0 & te^{-t} \\
0 & 1 + 2te^{-t} & 0 \\
te^{-t} & 0 & 3
\end{bmatrix}$$

(1.3)

will be used as an example. Figure 1.1 shows a plot of the time versus the three

eigenvalues of A(t) and the given frequency ω. These eigenvalues are:

$$\lambda_1(t) = 1 + 2te^{-t}$$

(1.4)

$$\lambda_2(t) = \frac{3.75 + \sqrt{5.0625 + 4t^2e^{-2t}}}{2}$$

(1.5)

$$\lambda_3(t) = \frac{3.75 - \sqrt{5.0625 + 4t^2e^{-2t}}}{2}$$

(1.6)

The external frequency ω is equal to 2.5. Figure 1.2 is a plot of time versus the
distance function

$$\min(\text{abs}(\omega, \sigma(t))).$$

(1.7)

A point of special interest is approximately at the time $t = .55$. At this point the
second eigenvalue is no longer the closest to $\omega$; the first eigenvalue takes its place
as closest. This is of interest because $G(t)$ is not smooth here, and clearly by figure
1.2, at this $t$ there is a maximum. Since it is not smooth here, finding the classical
derivative at this point is not possible. Thus some alternative way to justify that $t$
is a maximizer must be used. This point will be expounded upon further later. This
matrix will be used later in an example to demonstrate the method given in this
thesis.
Figure 1.1: Plot of time versus the three eigenvalues of given $A(t)$ and the external frequency $\omega$.

Figure 1.2: Graph of the minimum distance from the eigenvalues to the given external frequency $\omega = 2.5$. 
In this thesis, the mathematics involved in solving the proposed problem will be discussed and fully presented, as well as any background material in which the author feels will be an aid to understanding the mathematics involved in this project. The method of formulating the problem and the solving of it will be fully explained. Two examples will be given to aid in demonstrating the method.
Chapter 2

Summary of Important Fundamentals

This chapter will serve as a basic review of matrix theory. The concepts reviewed are essential to understanding the motivation behind the method given for avoiding resonance.

2.1 Spectral Representation

Any symmetric matrix $T$ has a spectral representation. That is, $T$ can be written, as shown in Kato (1980), as

\[ T = \sum_{i=1}^{n} \lambda_i P_i, \tag{2.1} \]

where $P_i$ is an eigenprojection of $T$ corresponding to the eigenvalue $\lambda_i$ of $T$ and satisfying the following conditions:

\[ P_i P_k = \delta_{ik} P_i, \tag{2.2} \]

\[ \sum_{i=1}^{n} P_i = I, \tag{2.3} \]

\[ P_i T = TP_i. \tag{2.4} \]

The spectral representation of a linear operator is useful in proving an important property about the norm of symmetric $A(t)$ to be seen in the next section. This section will focus on the norm of linear operators.
2.2 The Norm of the Linear Operator

Before defining the norm of the linear operator, it is useful to know the definition of the norm of a vector. For \( u \) a vector in \( \mathbb{R}^n \), the Euclidean norm of \( u \) is defined as

\[
\|u\| = \langle u, u \rangle^{\frac{1}{2}} = (u^T u)^{\frac{1}{2}}. \tag{2.5}
\]

Now, for a given \( n \times n \) matrix \( T \) and vector \( u \) the norm of \( T \) is defined as

\[
\|T\| = \sup_{\|u\|=1} \|Tu\|. \tag{2.6}
\]

For a symmetric matrix there is alternative definition of its norm. The following proposition and proof comes from Kato (1980):

**Proposition 2.1.** If \( T \) is symmetric,

\[
\|T\| = \max |\lambda_i|, \tag{2.7}
\]

where \( \lambda_i \) represents an eigenvalue of \( T \).

**Proof.** The norm of the vector \( Tu \) is

\[
\|Tu\| = \langle Tu, Tu \rangle^{\frac{1}{2}} = \langle T^T Tu, u \rangle^{\frac{1}{2}}. \tag{2.8}
\]

The spectral representation of \( T \) is

\[
T = \sum_{i=1}^{N} \lambda_i P_i \tag{2.9}
\]

and thus

\[
T^T T = TT = \sum_{i=1}^{N} \lambda_i^2 P_i. \tag{2.10}
\]

This gives

\[
\|Tu\|^2 = \sum_{i=1}^{N} \lambda_i^2 \langle P_i u, u \rangle \\
\leq \max |\lambda_i|^2 \sum_i \langle P_i u, u \rangle \\
= \max |\lambda_i|^2 \|u\|^2. \tag{2.11}
\]
Therefore from equation 2.8 it follows that $\|T\| \leq \max |\lambda_i|$. But if $u$ is an eigenvector of $T$, $\|Tu\| = |\lambda_i||u||$. It must therefore hold, by the definition of the norm of $T$, that $\|T\| = \max |\lambda_i|$. □

This result will now be used to derive the equivalent definition of $\|T\|$ which will be used in this project:

$$\|T\| = \sup_{\|u\|=1} |\langle Tu, u \rangle| \tag{2.12}$$

Now, $|\langle Tu, u \rangle| = |u^T Tu|$, which by Rayleigh's Principle is maximized over the set of vectors with norm of unity by the eigenvector $u^*$ corresponding to the largest eigenvalue $\lambda^*$. That is

$$\sup_{\|u\|=1} |\langle Tu, u \rangle| = |\langle Tu^*, u^* \rangle|$$
$$= |\langle \lambda^* u^*, u^* \rangle|$$
$$= |\lambda^*| |\langle u^*, u^* \rangle|$$
$$= |\lambda^*|$$
$$= \max |\lambda| = \|T\| \tag{2.13}$$

from equation 2.8 □

Moving on, another mathematical concept important to this thesis, the resolvent, will now be introduced.

### 2.3 The Resolvent

Given the matrix $A(t)$, assume $\omega$ is not in $\sigma(t)$, define the linear operator $R(\omega, t)$ as:

$$R(\omega, t) = (A(t) - \omega)^{-1} \tag{2.14}$$

This operator is called the resolvent. Assume also that $A(t) \in C(\mathbb{R}, L^n)$ and $A'(t) \in L^\infty(\mathbb{R}, S^n)$. Because $A(t)$ is continuous, its eigenvalues will be continuous as well.
Thus care must be taken in terms of $t$ to keep $R(t, \omega)$ defined. The following proposition asserts when $R(t, \omega)$ is continuous.

**Proposition 2.2** If $A(t) \in C^1(\mathbb{R}, S^n)$, $\omega$ is not in $\sigma(t_o)$ and $A'(t) \in L^\infty(\mathbb{R}, S^n)$ then there exists $\epsilon > 0$ such that $t \mapsto R(t, \omega)$ is continuous for $|t - t_o| < \epsilon$.

**Proof.** Without loss of generality, let $t_o = 0$. $A(t) - \omega$ can be written as

\begin{align*}
A(t) - \omega &= A(0) - \omega + A(t) - A(0) \\
&= [I - (A(0) - A(t))R(0, \omega)](A(0) - \omega). \\
\end{align*}

(2.15) \hspace{1cm} (2.16)

So

\[ R(t, \omega) = R(0, \omega)[I - (A(0) - A(t))R(0, \omega)]^{-1}. \]  

(2.17)

This exists when

\[ \|A(0) - A(t)\| < \frac{1}{\|R(0, \omega)\|}. \]  

(2.18)

Since derivatives of $A(t)$ are bounded, by the Mean-Value Theorem there exists some $K = \sup_t \|A'(t)\|$ such that

\[ \|A(0) - A(t)\| < K|t| \]

(2.19)

Restricting $t$ such that

\[ |t| < \frac{1}{K\|R(0, \omega)\|} \]

(2.20)

and letting $\epsilon = \frac{1}{K\|R(0, \omega)\|}$ assures that the inequality 2.19 holds for $|t| < \epsilon$. Now take the limit of both sides of equation 2.18 as $t$ goes to zero:

\[ \lim_{t \to 0} R(t, \omega) = \lim_{t \to 0} R(0, \omega)[I - (T(0) - T(t))R(0, \omega)]^{-1}. \]

(2.21)

By the restriction, $\|(A(0) - A(t))R(0, \omega)\|$ has a value less than one; thus

\[ [I - (A(0) - A(t))R(0, \omega)]^{-1} \]

(2.22)
can be expressed in the Neumann series
\[
\sum_{n=0}^{\infty} (A(0) - A(t))^n R(0, \omega)^n. \tag{2.23}
\]

Let
\[
f_n(t) = \sum_{k=0}^{n} (A(0) - A(t))^k R(0, \omega)^k \tag{2.24}
\]
and
\[
f(t) = \sum_{k=0}^{\infty} (A(0) - A(t))^k R(0, \omega)^k. \tag{2.25}
\]

Then \(f(t)\) converges since \(\|(A(0) - A(t))R(0, \omega)\| < 1\). In addition, \(f_n(t)\) converges to \(f(t)\) uniformly since there exists an \(N\) such that for all \(N > n\),
\[
|f_n(t) - f(t)| = \left| \sum_{k=n+1}^{\infty} (A(0) - A(t))^k R(0, \omega)^k \right| < \delta \tag{2.26}
\]
for all \(|t| < \epsilon\) and \(\delta > 0\), independent of \(n\), by the convergence of \(f(t)\). Since the limit of finite sums is the sum of the limits, define \(A_n\) such that
\[
\lim_{t \to 0} f_n(t) = A_n = \sum_{k=0}^{n} \lim_{t \to 0} (A(0) - A(t))^n R(0, \omega)^n, \tag{2.27}
\]
And it holds that
\[
\lim_{t \to 0} f(t) = \lim_{n \to \infty} A_n. \tag{2.28}
\]

By continuity of \(A(t)\), \(A_n = I\). Thus it must follow that
\[
\lim_{t \to 0} R(t, \omega) = \lim_{t \to 0} R(0, \omega)[I - (T(0) - T(t))R(0, \omega)]^{-1} = R(0, \omega) \tag{2.29}
\]
So \(R(t, \omega)\) is continuous for \(|t| < \epsilon\). \(\square\)

Now that it has been established when \(R(t, \omega)\) is continuous, consider the following claim.

**Claim.** If \(T \in C^1(\mathbb{R}, L^n)\) then \(\frac{d}{dt}[T(t)]^{-1} = -T(t)^{-1}\frac{d}{dt}[T(t)]T(t)^{-1}\).
Proof. Consider the identity

\[ S(t)^{-1} - T(t)^{-1} = -S(t)^{-1}(S(t) - T(t))T(t)^{-1}. \]  \hspace{1cm} (2.30)

Taking \( S(t) = 2T(t) \) gives

\[ S(t)^{-1} = \frac{1}{2} T(t)^{-1}. \]  \hspace{1cm} (2.31)

Substituting this result into equation 2.17 yields

\[ -\frac{1}{2} T(t)^{-1} = -\frac{1}{2} T(t)^{-1}(T(t))T(t)^{-1}. \]  \hspace{1cm} (2.32)

Now the derivative with respect to \( t \) of both sides gives the following:

\[ \frac{d}{dt} T(t)^{-1} = \frac{d}{dt} T(t)^{-1}T(t)T(t)^{-1} + T(t)^{-1} \frac{d}{dt} T(t)T(t)^{-1} \]
\[ + T(t)^{-1} T(t) \frac{d}{dt} T(t)^{-1} + \frac{d}{dt} T(t)^{-1}. \]  \hspace{1cm} (2.33)

Thus the result is

\[ \frac{d}{dt} T(t)^{-1} = -T(t)^{-1} \frac{d}{dt} T(t)T(t)^{-1}. \square \]  \hspace{1cm} (2.34)

Thus if \( R(t, \omega) \) is continuous, the derivative of \( R(t, \omega) \) can be found by using the above claim:

\[ \frac{\partial}{\partial t} R(t, \omega) = -R(t, \omega)A'(t)R(t, \omega). \]  \hspace{1cm} (2.35)

Thus it is seen for continuous \( R(t, \omega) \), continuous differentiability of \( R(t, \omega) \) also holds. This equation is an important part of the concepts essential in solving the problem of avoiding resonance. The formulation of the optimization problem for this purpose will be discussed in the next chapter.
Chapter 3

Development of the Optimization Problem

3.1 Formulation of the Optimization Problem

As discussed in the preceding chapter, the norm of a symmetric matrix $A(t) - \omega$ is the absolute value of its largest eigenvalue. Since the eigenvalues of the inverse of a matrix are simply the reciprocals of the eigenvalues of the original matrix, the norm of the resolvent $(A(t) - \omega)^{-1}$ is simply the maximum of the values $\frac{1}{\lambda_i(t) - \omega}$, where the $\lambda_i(t)$'s represent the eigenvalues of $A(t)$. This in essence will give the eigenvalue of $A(t)$ which is closest to $\omega$. That is, if $\lambda_*(t)$ is the closest eigenvalue to $\omega$, then the norm of $(A(t) - \omega)^{-1}$ will be $\left|\frac{1}{\lambda_*(t) - \omega}\right|$. Thus, in order to maximize the distance of $\lambda_*(t)$ from $\omega$, the objective would be to minimize the value of $\|\left((A(t) - \omega)^{-1}\right)\|$. And so the optimization problem becomes:

$$\min_{|t| << \|u\| = 1} \max \left|\left((A(t) - \omega)^{-1}u, u\right)\right|. \quad (3.1)$$

The immediate advantage of this method is that it avoids having to check every eigenvalue of the system one by one as the Bendsøe and Olhoff (1985) method does. Another advantage is that this problem is solvable by the classical approach of finding the first derivative of the function with respect to $t$

$$\max_{\|u\| = 1} \left|\left((A(t) - \omega)^{-1}u, u\right)\right|. \quad (3.2)$$

However, the max and absolute value functions are not smooth functions. Finding the derivative of this function, call it $F(t)$, is the subject of the next chapter. However,
the next sections in this chapter will be designated to discuss some fundamentals necessary to take on this endeavor.

3.2 The Generalized Gradient

Clarke (1983) approaches the prevalent problem of trying to optimize nonsmooth functions. The generalized gradient is defined, which is considered as a replacement for the classical first derivative for nonsmooth functions. The presentation of this concept begins with the definition of the generalized directional derivative. If $f$ is Lipschitz near some point $x$, for $v$ an arbitrary vector in $\mathbb{R}^n$, the generalized directional derivative of a function $f$, denoted $f^\circ$ is:

$$f^\circ(x; v) = \lim_{y \to x, t \to 0} \sup f(y + tv) - f(y) \quad \frac{t}{t}.$$  \hspace{1cm} (3.3)

Now consider the following Proposition.

**Proposition.** Let $f$ be Lipschitz of rank $K$ near $x$. Then the function $v \mapsto F^\circ(x; v)$ is finite, positively homogeneous, and subadditive on $\mathbb{R}^n$, and satisfies

$$f^\circ(x; v) \leq K\|v\|.$$  \hspace{1cm} (3.4)

As Clarke (1983) states, this Proposition is what makes the concept of $f^\circ$ very useful.

The fact that it is positively homogeneous and subadditive validates the following definition of the generalized gradient:

**Definition** The *generalized gradient* of $f$ at $x$, $\partial f(x)$, is the subset of $\mathbb{R}$,

$$\{\zeta \in \mathbb{R}^n : f^\circ \geq \langle \zeta, v \rangle, \quad \forall v \in \mathbb{R}^n\}.$$  \hspace{1cm} (3.5)
If $f$ happens to be smooth, $\partial f$ reduces to $\nabla f$.

Now the question arises about how to apply the generalized gradient to the function of interest,

$$F(t) = \sup_{\|u\|=1} |\langle (A(t) - \omega)^{-1}u, u \rangle|.$$ \hfill (3.6)

Before beginning, it must be known that $F(t)$ is a Lipschitz function. It is well-known that the absolute value function is Lipschitz. The following will show that $\langle (A(t) - \omega)^{-1}u, u \rangle$ is also Lipschitz for properly restricted $t$.

**Proposition.** If $\omega$ is not in $\sigma(A(t))$ and $\sigma(A(t_o))$, then if $A(t)$ is Lipschitz, $\langle (A(t) - \omega)^{-1}u, u \rangle$ is Lipschitz.

**Proof.** Let $|t - t_o| < \epsilon$ for some $\epsilon > 0$.

$$|\langle (A(t) - \omega)^{-1}u, u \rangle - \langle (A(t_o) - \omega)^{-1}u, u \rangle| = |\langle (A(t) - \omega)^{-1} - (A(t_o) - \omega)^{-1}u, u \rangle|.$$ \hfill (3.7)

From equation 2.29 this gives

$$|\langle (A(t) - \omega)^{-1} - (A(t_o) - \omega)^{-1}u, u \rangle|$$

$$= |\langle (A(t) - \omega)^{-1}(A(t_o) - A(t))(A(t_o) - \omega)^{-1}u, u \rangle|$$

$$\leq |u^T(A(t) - \omega)^{-1}(A(t_o) - A(t))(A(t_o) - \omega)^{-1}u|$$

$$\leq \|((A(t) - \omega)^{-1}(A(t_o) - A(t))(A(t_o) - \omega)^{-1}\|$$

$$\leq \|((A(t) - \omega)^{-1}\|\|((A(t_o) - A(t))\|\|(A(t_o) - \omega)^{-1}\|$$

by definition and properties of the matrix norm. Since $A(t)$ is Lipschitz, it follows that there exists a $K$ such that

$$\|((A(t) - \omega)^{-1}\|\|((A(t_o) - A(t))\|\|(A(t_o) - \omega)^{-1}\|$$

$$\leq \|A(t) - \omega\| K|t - t_o| \|(A(t_o) - \omega)\|^{-1}\|$$

$$= \hat{K}|t - t_o|$$
since \( \|(A(t) - \omega)^{-1}\| \) and \( \|(A(t_0) - \omega)^{-1}\| \) must be bounded for \( \omega \) outside of the respective spectrums. Thus the Lipschitz condition holds for \( \langle (A(t) - \omega)^{-1}u, u \rangle \). □

Now consider the following proposition.

**Proposition.** If \( x \mapsto \Phi(x, y) \) is Lipschitz with respect to \( x \) uniformly for \( y \in M \), then \( x \mapsto \sup_{y \in M} \Phi(x, y) \) is Lipschitz.

**Proof.** Consider \( \sup_{y \in M} \Phi(x, y) > \sup_{y \in M} \Phi(z, y) \) so

\[
| \sup_{y \in M} \Phi(x, y) - \sup_{y \in M} \Phi(z, y) | = \sup_{y \in M} \Phi(x, y) - \sup_{y \in M} \Phi(z, y),
\]

where for some \( \epsilon > 0 \), \( |x - z| < \epsilon \). Suppose that \( \sup_{y \in M} \Phi(x, y) \) is attained at \( y_x \).

Then

\[
\sup_{y \in M} \Phi(x, y) - \sup_{y \in M} \Phi(z, y) \leq \Phi(x, y_x) - \Phi(z, y_x).
\]

But by the Lipschitz condition of \( \Phi(x, y) \) with respect to \( x \),

\[
\Phi(x, y_x) - \Phi(z, y_x) < K(y_x)|x - z|
\]

which implies that

\[
\sup_{y \in M} \Phi(x, y) - \sup_{y \in M} \Phi(z, y) < K(y_x)|x - z|.
\]

Similarly, consider \( \sup_{y \in M} \Phi(z, y) > \sup_{y \in M} \Phi(x, y) \) to give the result

\[
|\Phi(x, y_x) - \Phi(z, y_x)| < K(y_x)|x - z|.
\]

Since the Lipschitz condition is uniform over \( M \), this \( K(y_x) \) can be any finite constant such that the inequality is valid for all \( y \in M \). Thus the Lipschitz condition with respect to \( y \) holds. □
Providing the continuous differentiability of \( A(t) \), it follows that \( F \) is Lipschitz and the concepts of Clarke (1983) can be applied.

Enough has been covered now to determine the generalized gradient of the function \( F(t) \). Since finding this generalized gradient is the climax of this thesis, it is appropriate to designate the next chapter for this purpose.
Chapter 4

Solving for the Generalized Gradient

At this point the equation to minimize has been found:

\[ F(t) = \sup_{\|u\|=1} |\langle (A(t) - \omega)^{-1}u, u \rangle|. \] (4.1)

As noted in Chapter three, the sup and absolute value functions are not smooth. Thus the temptation to immediately invoke the classical chain rule must be overcome. The absolute value function is smooth everywhere but zero; hence the values of \( u \) where \( |\langle (A(t) - \omega)^{-1}u, u \rangle| \) is zero are the only points of nondifferentiability. But these values correspond to where \((A(t) - \omega)^{-1}\) has a zero eigenvalue. This, in effect, is equivalent to where \( A(t) - \omega \) has an eigenvalue of infinity. For the purposes of this project, such a case is not considered. Only bounded operators are studied. Thus the absolute value is considered smooth everywhere in the domain for which the resolvent is defined.

The sup function, however, is nonsmooth when the value of the supremum is attained at more than one point. This circumstance is a likely situation for this problem, as keeping the applied frequency at an optimum distance from any one eigenvalue in the spectrum of \( A(t) \) most often entails keeping it at an equal distance between two eigenvalues of \( A(t) \). One must then resort to another means than the classical approach for finding the first derivative of this function. The method of choice for this task is finding the generalized gradient, as introduced in Chapter three.
4.1 Application of the Generalized Gradient to the Problem of Interest

Using these ideas, the task now is to find the generalized gradient of the equation 4.1. To begin, certain variables will be defined to simplify the notation for the task. Let

\[ \Phi(t, u) = |(A(t) - \omega)^{-1}u, u)|, \]  \hspace{1cm} (4.2)

\[ \text{Argmax}(t) = \{u \in \mathbb{R}^n : \|u\| = 1 \text{ and } \Phi(t, u) = F(t)\}. \]  \hspace{1cm} (4.3)

The following claim for the form of \( \partial F(t) \) is now made:

\[ \partial F(t) = \text{co}\{\partial \Phi(t, u) : u \in \text{Argmax}(t)\}. \]  \hspace{1cm} (4.4)

4.1.1 Step One: Finding the Derivative of \( \Phi \)

Before proving the above claim it is important to know what the form of \( \partial \Phi \) is. As discussed above, \( \Phi \) is analytic because \( A(t) \) is analytic and its eigenvalues are bounded. Thus all that is entailed in finding \( \partial \Phi \) is utilization of the classical chain rule. That is,

\[ \partial \Phi = \frac{d}{dt} \Phi = \text{sgn} \cdot \frac{d}{dt} ((A(t) - \omega)^{-1}u, u). \]  \hspace{1cm} (4.5)

where

\[ \text{sgn} = \begin{cases} 
-1 & \text{if } ((A(t) - \omega)^{-1}u, u) < 0 \\
1 & \text{if } ((A(t) - \omega)^{-1}u, u) > 0.
\end{cases} \]  \hspace{1cm} (4.6)

In chapter two it was proven that

\[ \frac{d}{dt}(A(t) - \omega)^{-1} = (A(t) - \omega)^{-1} \frac{d}{dt} A(t)(A(t) - \omega)^{-1}. \]  \hspace{1cm} (4.7)

Thus it is easily seen that for any \( u \in \mathbb{R}^n \),

\[ \frac{d}{dt} \Phi(u, t) = \text{sgn} \cdot -u^T(A(t) - \omega)^{-1}\left[\frac{d}{dt} A(t)\right](A(t) - \omega)^{-1}u. \]  \hspace{1cm} (4.8)
4.1.2 Step Two: Proof of the Generalized Gradient for $F(t)$

In this section equation 4.1 will be proven for the form of the generalized gradient of $F(t)$. The basic structure of the proof comes from Clarke (1983).

**Proof.** Define $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ as:

$$g(t; v) = \max\{\langle \zeta, v \rangle : \zeta \in \partial \Phi(t, u), u \in \text{Argmax}(t)\}.$$ (4.9)

**Lemma.** $F^*(t; v) \leq g(t; v)$.

**Proof.** Define

$$\Delta_i = \frac{F(t_i + \lambda_i v) - F(t_i)}{\lambda_i}$$ (4.10)

where as $i \rightarrow 0$, $t_i \rightarrow t$, and $\lambda_i \downarrow 0$, $\Delta \rightarrow F^*$

Let $u_i \in \text{Argmax}(t_i + \lambda_i v)$ Then

$$\Delta_i \leq \frac{\Phi(u_i, t_i + \lambda_i v) - \Phi(t_i)}{\lambda_i}.$$ (4.11)

Since $\Phi$ is continuously differentiable in the domains considered, the Mean-Value theorem states that there exists $\zeta_i \in \frac{\partial}{\partial t} \Phi(u_i, t_i^*)$, where $t_i^*$ is between $t_i$ and $t_i + \lambda_i v$ such that

$$\frac{\Phi(u_i, t_i + \lambda_i v) - \Phi(u_i, t_i)}{\lambda_i} = \langle \zeta_i, v \rangle.$$ (4.12)

As seen by equation 4.5, each $\frac{\partial}{\partial t} \Phi(.,.)$ is a singleton. Thus suppose $u_i \rightarrow u$, and $\zeta_i \rightarrow \zeta$. Then

$$F^*(t; v) = \lim_{i \rightarrow \infty} \Delta_i \leq \langle \zeta, v \rangle$$ (4.13)

from equations 3.3 and 4.10. For any $\tilde{u}$ on the unit circle in $\mathbb{R}^n$,

$$\Phi(u_i, t_i + \lambda_i v) \geq \Phi(\tilde{u}, t_i + \lambda_i v).$$ (4.14)
since \( u_i \in \text{Argmax}(t_i + \lambda_i v) \).

Since it is known from chapter three that each \( \Phi(\cdot , \cdot) \) is Lipschitz with respect to \( t \). Using this fact the result is

\[
\Phi(u_i, t_i + \lambda_i v) \geq \Phi(\hat{u}, t_i) - K\|t_i + \lambda_i v - t_i\|. \tag{4.15}
\]

Taking the limit as \( i \to \infty \) of both sides gives

\[
\Phi(u, t) \geq \Phi(\hat{u}, t). \tag{4.16}
\]

Since \( \hat{u} \) is arbitrary, it follows that \( u \in \text{Argmax}(t) \). Thus is follows by definition of \( g \) and equation 4.13 that

\[
F^o(t; v) \leq \langle \zeta, v \rangle \leq g(t; v). \square \tag{4.17}
\]

Now equation 4.17 and the lemma give for \( \zeta \in \partial F(t) \),

\[
\min_{v \in \mathbb{R}} \max_{w \in Z} \{ \langle w, v \rangle - \langle \zeta, v \rangle \} = 0, \tag{4.18}
\]

where \( Z = \text{co}\{ \frac{\partial}{\partial t} \Phi(u, t) : u \in \text{Argmax}(t) \} \). Applying Aubin's lop-sided Minimax theorem asserts the existence of \( w \in Z \) such that for all \( v \in \mathbb{R} \),

\[
\langle w - \zeta, v \rangle \geq 0. \tag{4.19}
\]

Since \( v \) is arbitrary, \( \zeta = w \). But this implies that

\[
\partial F(t) \subset \text{co}\{ \frac{d}{dt} \Phi(u, t) : u \in \text{Argmax}(t) \}. \tag{4.20}
\]

Now, for all \( u \in \text{Argmax}(t) \), by definition of the norm of \( (A(t) - \omega)^{-1} \) given in equation 2.14, \( u \) must be the eigenvector corresponding to the largest eigenvalue of
\((A(t) - \omega)^{-1}\). This eigenvalue is of the form \(\frac{1}{\lambda - \omega}\), where \(\lambda\) is the eigenvalue of \(A(t)\) corresponding to the vector \(u\). If \(\omega\) is situated at equal distance between two distinct eigenvalues \(\lambda_1\) and \(\lambda_2\) of \(A(t)\), \(\text{Argmax}(t)\) will consist of four unit vectors, namely \(u_1, -u_1, u_2, -u_2\), where \(u_1\) is the eigenvector corresponding to \(\lambda_1\), and \(u_2\) to \(\lambda_2\). Thus
\[
\text{co}\{\frac{\partial}{\partial t} \Phi(u, t) : u \in \text{Argmax}(t)\}
\] (4.21)
is the convex combination of the four terms \(\frac{\partial}{\partial t} \Phi(u_1, t), \frac{\partial}{\partial t} \Phi(-u_1, t), \frac{\partial}{\partial t} \Phi(u_2, t), \frac{\partial}{\partial t} \Phi(-u_2, t)\). That is, given any \(c_1, c_2, c_3, c_4\), such that \(\sum_{i=1}^{4} c_i = 1\) and \(c_i > 0\) for all \(i = 1, 2, 3, 4\), then \(u \in \text{co}\{\frac{\partial}{\partial t} \Phi(u, t) : u \in \text{Argmax}(t)\}\) implies
\[
u = c_1 \frac{\partial}{\partial t} \Phi(u_1, t) + c_2 \frac{\partial}{\partial t} \Phi(-u_1, t) + c_3 \frac{\partial}{\partial t} \Phi(u_2, t) + c_4 \frac{\partial}{\partial t} \Phi(-u_2, t).\] (4.22)
From equation 4.5 this gives
\[
(c_1 + c_2) \frac{\partial}{\partial t} \Phi(u_1, t) + (c_3 + c_4) \frac{\partial}{\partial t} \Phi(u_2, t).\] (4.23)

It is important now to show that \(\Phi(t, u)\) is regular. The following Proposition taken justifies this fact.

**Proposition.** Let \(f\) be Lipschitz near \(x\); then if \(f\) is strictly differentiable at \(x\), and \(f\) is regular at \(x\)

Now, for \(\zeta \in Z\),
\[
\langle \zeta, v \rangle \leq (c_1 + c_2) \Phi^\circ(u_1, t; v) + (c_3 + c_4) \Phi^\circ(u_2, t; v)
\] (4.24)
\[
= (c_1 + c_2) \Phi'(u_1, t; v) + (c_3 + c_4) \Phi'(u_2, t; v),
\] (4.25)
where \(\Phi'(u, t; v)\) is the usual one-sided derivative with respect to \(t\), by regularity of \(\Phi(t, u)\). Now it is needed to show that \(F\) is regular. Let
\[
\alpha = \liminf_{\lambda \to 0} \frac{F(t + \lambda v) - F(t)}{\lambda}.
\] (4.26)
It will suffice to show $F^\circ \leq \alpha$. Let $u \in \text{Argmax}(t)$.

$$\frac{F(t + \lambda v) - F(t)}{\lambda} \geq \frac{\Phi(u, t + \lambda v) - \Phi(u, t)}{\lambda}$$

(4.27)

Taking the limit infimum as lambda decreases to zero of both sides gives

$$\alpha \geq \Phi'(u, t; v) = \Phi^\circ(u, t; v)$$

(4.28)

since $\Phi$ is regular. Thus

$$\alpha \geq \max\{\Phi^\circ(u, t; v) : u \in \text{Argmax}(t)\}$$

(4.29)

$$= g(t; v).$$

(4.30)

Therefore

$$\alpha \geq g \geq F^\circ$$

(4.31)

by the lemma. Thus $F^\circ(t) \leq \alpha \Rightarrow F^\circ$ is regular. Now

$$\langle \zeta, v \rangle = (c_1 + c_2) \lim_{\lambda \to 0} \frac{\Phi(u_1, t + \lambda v) - \Phi(u_1, t)}{\lambda}
+ (c_3 + c_4) \lim_{\lambda \to 0} \frac{\Phi(u_2, t + \lambda v) - \Phi(u_2, t)}{\lambda}
\leq (c_1 + c_2) \limsup_{\lambda \to 0} \frac{\Phi(u_1, t + \lambda v) - \Phi(u_1, t)}{\lambda}
+ (c_3 + c_4) \limsup_{\lambda \to 0} \frac{\Phi(u_2, t + \lambda v) - \Phi(u_2, t)}{\lambda}
= (c_1 + c_2) \limsup_{\lambda \to 0} \frac{F(t + \lambda v) - f(t)}{\lambda}
+ (c_3 + c_4) \limsup_{\lambda \to 0} \frac{F(t + \lambda v) - F(t)}{\lambda}.

(since $u_1, u_2 \in \text{Argmax}(t)$)

$$= (c_1 + c_2 + c_3 + c_4) F'(t; v)$$

$$= F'(t; v)$$

$$= F^\circ(t; v)$$

(4.32)
by regularity of $F$. Therefore

$$
\langle \zeta, v \rangle \leq F^0(t; v) \Rightarrow \zeta \in \partial F(t).
$$

(4.33)

And equations 4.33 and 4.20 give equality. □.

To demonstrate the use of this method, the next section will show this technique for two example matrices.

### 4.2 Two Examples

#### 4.2.1 Example One

One matrix used to test the proposed maximization process is

$$
A(t) = \begin{bmatrix}
1 & te^{-t} \\
\frac{1}{t}e^{-t} & 2
\end{bmatrix}.
$$

(4.34)

The actual solution to finding the value of $t$ such the the eigenvalues of $A(t) - \omega$ are maximized can be found simply by differentiating the distance from eigenvalues to $\omega$ with respect to $t$. Doing this shows that at $t = 1$, $|\langle(A(t) - \omega)^{-1}u, u\rangle|$ is maximized.

Following the proposed method, the value of the resolvent $(A(t) - \omega)^{-1}$, its eigenvalues, and its derivative $(A(t) - \omega)^{-1}A'(t)(A(t) - \omega)^{-1}$ must be found. The resolvent is found to be

$$
\frac{1}{(1-\omega)(2-\omega) - t^2 e^{-2t}}
\begin{bmatrix}
2 - \omega & -te^{-t} \\
-te^{-t} & 1 - \omega
\end{bmatrix}
$$

(4.35)

and its derivative is

$$
\begin{bmatrix}
\frac{-2te^{-2t}(t-1)(\omega-2)}{(1-\omega)(2-\omega) - t^2 e^{-2t})^2} & \frac{-e^{-t}(t-1)(t^2 e^{-2t} + (1-\omega)(2-\omega))}{((1-\omega)(2-\omega) - t^2 e^{-2t})^2} \\
\frac{-e^{-t}(t-1)(t^2 e^{-2t} + (1-\omega)(2-\omega))}{((1-\omega)(2-\omega) - t^2 e^{-2t})^2} & \frac{-2e^{-2t}(t-1)(\omega-1)}{((1-\omega)(2-\omega) - t^2 e^{-2t})^2}
\end{bmatrix}
$$

(4.36)
The eigenvalues $\beta_1, \beta_2$ are

$$
\beta_1 = \frac{(3 - 2\omega) + \sqrt{1 + 4t^2e^{-2t}}}{2((1 - \omega)(2 - \omega) - t^2e^{-2t})}
$$
(4.37)

$$
\beta_2 = \frac{(3 - 2\omega) - \sqrt{1 + 4t^2e^{-2t}}}{2((1 - \omega)(2 - \omega) - t^2e^{-2t})}
$$
(4.38)

Since the object is to find

$$
\partial F(t) = \text{cof}\left[\frac{\partial}{\partial t}|(A(t) - \omega)^{-1}u, u_1| : u \in \text{Argmax}(t)\right],
$$
(4.40)

what is needed are the eigenvectors of $(A(t) - \omega)^{-1}$ which correspond to the maximum eigenvalues. As argued previously, it must be considered that $\omega$ is situated between two adjacent eigenvalues, or in this case, $|\beta_1| = |\beta_2|$. Without loss of generality, assume $\beta_1 > 0$. Then $|\beta_2| = \beta_1$, which yields

$$
|\beta_2| = \frac{2\omega + \sqrt{1 + 4t^2e^{-2t}} - 3}{2((1 - \omega)(2 - \omega) - t^2e^{-2t})}
$$
(4.41)

for $\omega = \frac{3}{2}$. The corresponding eigenvectors can be expressed as

$$
u_1 = \left[ \begin{array}{c} 
\frac{1 - \sqrt{1 + 4t^2e^{-2t}}}{2te^{-t}} \\
1 
\end{array} \right],
\quad
u_2 = \left[ \begin{array}{c} 
\frac{1 + \sqrt{1 + 4t^2e^{-2t}}}{2te^{-t}} \\
1 
\end{array} \right].
$$
(4.42)

The objective is to find the $t$ where the elements of the convex hull

$$
(c_1 + c_2)\frac{\partial}{\partial t}\Phi(u_1, t) + (c_3 + c_4)\frac{\partial}{\partial t}\Phi(u_2, t)
$$
(4.43)

is zero for all $c_1, ..., c_4$, where as before,

$$
\Phi(u, t) = |(A(t) - \omega)^{-1}u, u)|.
$$
(4.44)

For this example equation 4.43 yields

$$
\frac{(c_1 + c_2)(1 - t)(-2\omega + (-2\omega + t^2e^{-2t} + \omega^2)\sqrt{1 + 4t^2e^{-2t}})}{t((1 - \omega)(2 - \omega) - t^2e^{-2t})^2}
$$
(4.45)
\[ + \frac{(c_1 + c_2)(1 - t)((4\omega - 5)t^2e^{-2t} + \omega^2)}{t((1 - \omega)(2 - \omega) - t^2e^{-2t})^2} \]  
\[ - \frac{(c_3 + c_4)(t - 1)(2\omega + (-2\omega + t^2e^{-2t} + \omega^2)\sqrt{1 + 4t^2e^{-2t}})}{t((1 - \omega)(2 - \omega) - t^2e^{-2t})^2} \]  
\[ - \frac{(c_3 + c_4)(t - 1)((4\omega - 5)t^2e^{-2t} - \omega^2)}{t((1 - \omega)(2 - \omega) - t^2e^{-2t})^2} \]

Clearly for any \(c_1, ..., c_4\) equation 4.45 is zero for \(t = 1\). This agrees with the result found from differentiating the distance of the eigenvalues of \(A(t)\) from \(\omega\), as discussed previously. Thus the method proved to be accurate for this example. However, this example is actually very simplistic in the sense that for \(\omega = 3/2\), all the eigenvalues of \(A(t)\) are a maximum distance from \(\omega\), as seen in figure 4.1. Thus, as was predicted theoretically, the generalized gradient \(\partial F(t)\) reduces to the singleton 0. Figure 4.2 demonstrates the smoothness of \(F(t)\), and the maximum of \(F(t)\) is clear.

The next subsection will manipulate a less simplistic example to demonstrate this method.

### 4.2.2 Example Two

This subsection will demonstrate the method on the familiar \(3 \times 3\) matrix

\[
A(t) = \begin{bmatrix}
1 & 0 & te^{-t} \\
0 & 1 + 2te^{-\frac{1}{4}t} & 0 \\
0 & 3
\end{bmatrix}. \tag{4.52}
\]

As was seen in figures 1.1 and 1.2, for \(\omega = 2.5\), there is a "trade-off" between eigenvalues

\[
\lambda_1(t) = 1 + 2te^{-\frac{1}{4}t} \tag{4.53}
\]
**Figure 4.1:** Centered external frequency between the two eigenvalues of the matrix $A(t)$.

**Figure 4.2:** Graph of the minimum distance from the eigenvalues to the given external frequency $\omega = 3/2$. 
\[ \lambda_2(t) = \frac{3.75 + \sqrt{5.0625 + 4t^2e^{-2t}}}{2} \] (4.54)

as to which eigenvalue is actually closest to \( \omega \). It is for this "trade-off" that this example is interesting. As will be seen below, the generalized gradient \( \partial F(t) \) will actually consist of an interval which must contain zero, as at this point \( t \) there is a maximum.

Numerically the "trade-off" point was found to occur at approximately the time \( t = .5485 \). Before this \( t \) value, the \textbf{Argmax}(\( t \)) consists of the unit eigenvector corresponding to the eigenvalue \( \lambda_2(t) \). After this \( t \) it consists of the unit eigenvector corresponding to \( \lambda_2(t) \). Thus \( \partial F(t) \) will be, for \( t < .5485 \), the singleton of the inner product \( \Phi(t, u_2) \), for \( u_2 \) the eigenvector corresponding to \( \lambda_2 \), and for \( t > .5485 \), \( \partial F(t) \) is likewise the singleton \( \Phi(t, u_1) \) for \( u_1 \) the corresponding eigenvector of \( \lambda_1 \). \( \partial F(t) \) at \( t = .5485 \) must now be considered. Intuitively from the definition of the generalized gradient, the result should be as follows. Following the curve from figure 1.2, the equation for \( F(t) \) before this value of \( t \) is \( \lambda_2(t) \) as defined at equation 4.54. Thus the tangent at \( t = .5485 \) from the left direction will be

\[ u_2^T (A(t) - 2.5)^{-1} A'(t) (A(t) - 2.5)^{-1} u_2 = .2392 \] (4.55)
evaluated at \( t = .5485 \). Likewise, from the left of this value of \( t \), the tangent direction will be

\[ u_1^T (A(t) - 2.5)^{-1} A'(t) (A(t) - 2.5)^{-1} u_1 = -5.0923 \] (4.56)
evaluated at this \( t \). By definition of the generalized gradient, \( \partial F(t) \) is the interval whose endpoints are the values of expressions 4.55 and 4.56 evaluated at \( t = .5485 \), i.e.,

\[ \partial F(t) = [-5.0923, .2392] \] (4.57)
This is consistent with equation 4.1, as any member of the convex hull

\[ \text{co}\{\frac{d}{dt}\Phi(u, t) : u \in \text{Argmax}(t)\} \]  

(4.58)

will actually represent a line between the two endpoints of the interval (*). That is, any member of expression 4.57 is also in the interval. Likewise, any member in the interval is also in the convex hull 4.57. Thus they are the same.

These two examples give one an idea of the results one will receive using this method of optimization. Both of these examples, however, were worked primarily by hand. This approach will be impossible in more realistic problems. For this reason a numerical optimization code to implement this method is ideal, but pursuing the creation of such a code would be for more advanced research. This chapter concludes the presentation of the material of this research.
Chapter 5

Concluding Remarks

The objective of this thesis is to produce a more efficient method of maximizing the distance of an external frequency from the spectrum of a linear system. An efficient method for such an endeavor would be valuable to engineers and mathematicians alike. Specifically, it would be of crucial aid in designing linear systems against resonance.

The first chapter serves as an introduction to the thesis and a motivation for researching this problem. Most of this motivation comes from expounding slightly on the weakness of another method for solving the problem given by Bendsoe and Olhoff (1985). In addition, this chapter discusses the problems that are faced in attempting to solve the problem and gives a visual aid for these problems by using an example matrix.

The second chapter serves as a review for the mathematics fundamental to both understanding what is involved in solving this problem and for aid in proving propositions needed to do so. Conditions for continuity of the resolvent, as well as differentiability, are part of the focus. In addition, the derivation of the matrix norm used is included.

The third chapter introduces the concepts of the generalized gradient and the generalized directional derivative from Clarke (1990). These are needed in working with the potential nondifferentiability of the function $F(t)$ to be maximized. In addition, the Lipschitz condition needed for $F(t)$ to use these concepts is proven.
The fourth chapter is the climax of the thesis. It gives the generalized gradient of $F(t)$ and gives a proof for this form. Most importantly it offers two examples for understanding this method of solving the problem of avoiding resonance.

This thesis leaves an open end to research for utilizing this method. One problem to be considered is the adaption of the method for solving the generalized eigenvalue problem associated with finite element analysis. Another possibility would be to produce a numerical algorithm for this method for solving “real life” problems whose analytical solutions are difficult to obtain. These two applications should serve as motivation for continuing research on this problem and this method of solving it.
Bibliography


