Finite Element Approximations to the System
of Shallow Water Equations, Part I:
Continuous Time a Priori Error Estimates

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FINITE ELEMENT APPROXIMATIONS TO THE SYSTEM OF
SHALLOW WATER EQUATIONS, PART I: CONTINUOUS TIME A
PRIORI ERROR ESTIMATES *

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Abstract. Various sophisticated finite element models for surface water flow exist in the literature. Gray, Kolar, Luettich, Lynch and Westerink have developed a hydrodynamic model based on the generalized wave continuity equation (GWCE) formulation, and have formulated a Galerkin finite element procedure based on combining the GWCE with the nonconservative momentum equations. Numerical experiments suggest that this method is robust, accurate and suppresses spurious oscillations which plague other models. We analyze a slightly modified Galerkin model which uses the conservative momentum equations (CME). For this GWCE-CME system of equations, we present an a priori error estimate based on an $L^2$ projection.

Key words. shallow water equations, surface flow, mass conservation, momentum conservation, finite element model, error estimate, projection

AMS subject classifications. 35Q35, 35L65 65N30, 65N15

1. Introduction. In recent years, there has been much interest in the numerical solution to shallow water equations. Simulation of shallow water systems can serve numerous purposes. First, it can serve as means for modeling tidal fluctuations for those interested in capturing tidal energy for commercial purposes. Second, these simulations can be used to compute tidal ranges and surges such as tsunamis and hurricanes caused by extreme earthquake and storm events. This information can be used in the development planning of coastal areas. Finally, the shallow water hydrodynamic model can be coupled to a transport model in considering flow and transport phenomenon, thus making it possible to study remediation options for polluted bays and estuaries, to predict the impact of commercial projects on fisheries, to model freshwater-saltwater interactions, and to study allocation of allowable discharges by municipalities and by industry in meeting water quality controls.

The 2-dimensional shallow water equations are obtained by depth (or vertical) averaging of the continuum mass and momentum balances given by the 3-dimensional incompressible Navier-Stokes equations. Shallow water equations can be used to study flow in fluid domains whose bathymetric depth is much smaller than the characteristic length scale in the horizontal direction. We denote by $\xi(x, t)$ the free surface elevation over a reference plane and by $h_b(x)$ the bathymetric depth under that reference plane so that $H(x, t) = \xi + h_b$ is the total water column (see Figure 1). Also, we denote by $U(x, t), V(x, t)$ the depth-averaged horizontal velocities. Letting $\mathbf{v} = (UH, VH)'$, the 2-dimensional governing equations, in operator form [10], are the primitive continuity equation (CE)

$$\text{CE}(\xi, U, V; h_b) \equiv \frac{\partial \xi}{\partial t} + \nabla \cdot \mathbf{v} = 0,$$

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and the primitive non-conservative momentum equations (NCME), as derived by Westerink et al [24],

\[ \text{NCME}(\xi, U, V; \Phi) = \frac{\partial}{\partial t} \left( \frac{v}{H} \right) + \left( \frac{v}{H} \nabla \right) \frac{v}{H} + \tau_{bf} \left( \frac{v}{H} \right) + k \times f_c \left( \frac{v}{H} \right) + g \nabla \xi - \frac{1}{H} E_h \nabla \cdot \nabla v - \frac{1}{H} \tau_{us} + \nabla p_a - g \nabla \eta = 0, \]

where \( \Phi = (h_b, \tau_{bf}, f_c, g, E_h, \tau_{us}, p_a, \eta) \). In particular, \( \tau_{bf}(\xi, U, V) \) is a bottom friction function, \( k \) is a unit vector in the vertical direction, \( f_c \) is the Coriolis parameter, \( g \) is acceleration due to gravity, \( E_h \) is the horizontal eddy diffusion/dispersion (constant) coefficient, \( \tau_{us} \) is the applied free surface wind stress relative to the reference density of water, \( p_a(x, t) \) is the atmospheric pressure at the free surface relative to the reference density of water, and \( \eta(x, t) \) is the Newtonian equilibrium tide potential relative to the effective Earth elasticity factor. The primitive conservative momentum equations (CME) are derived from the NCME as

\[ \text{CME} \equiv H(\text{NCME}) + v(\text{CE})/H = 0. \]

The numerical procedure used to solve the shallow water equations must resolve the physics of the problem without introducing spurious oscillations or excessive numerical diffusion. Westerink et al [24] note a need for greater grid refinement near land boundaries to resolve important processes and to prevent energy from aliasing. Permitting a high degree of grid flexibility, the finite element method is a good candidate.

There has been substantial effort over the past two decades in applying finite element methods to the CE coupled with either the NCME or the CME. Early finite element simulations of shallow water systems were plagued by spurious oscillations. Various methods were introduced to eliminate these oscillations through artificial diffusion [15, 20]. These methods were generally unsuccessful due to excessive damping of physical components of the solution. Recently, Agoshkov et al [2, 4, 3] have investigated a finite element approximation, where the velocity field is approximated by piecewise linear polynomials and the elevation is approximated by the same functions plus some additional ones. They have studied the effects of various boundary conditions, and proven stability of various time discretization schemes for a linearized continuity equation-momentum equation system. In this paper, we will examine a finite element approximation to a modified shallow water model described below. Computational and experimental evidence in the literature suggest that this formulation leads
to approximate solutions with reduced oscillations. Moreover, these approximate solutions have accurately matched actual tidal data. This modified shallow water model is based on a reformulation of the CE, which we now describe.

1.1. Historical Development of the Wave Continuity and Generalized Wave Continuity Equations. In 1979, Lynch and Gray [12] derived the wave continuity equation (WCE) from the mass and momentum conservation equations,

\[
WCE(\xi, U, V; \Phi) \equiv \frac{\partial (CE)}{\partial t} - \nabla \cdot (CME) + \tau(CE) = 0,
\]

as a means to eliminate oscillations without resorting to numerical damping. Here, \(\tau(x, t)\) is a non-linear friction coefficient. In this shallow water formulation, the WCE is then coupled to either the CME or the NCME. The equivalence of this model to the more standard one based on the CE is discussed in [10].

This formulation has led to the development of robust finite element algorithms for depth-integrated coastal circulation models. The WCE approach has motivated a substantial computational and analytical effort [5, 6, 12, 16]. Using Fourier phase/space analysis of the linearized WCE-ME system of equations, Foreman [6] and Kininmark [10] prove that the WCE formulation suppresses spurious oscillations of the numerical solution, and is capable of capturing “2\Delta x” waves. The WCE formulation has also motivated substantial field applications; see [9], [7], [8], [13], [14], [17], [18], [22], [23], [19]. These studies have demonstrated the advantage of the WCE formulation for finite element applications in terms of achieving both a high level of computational accuracy and efficiency.

The generalized wave continuity equation (GWCE) [10] is essentially the same as the WCE except that multiplication of the continuity equation by \(\tau\) is replaced with multiplication by some general function that may be independent of time. Westerink and Luetttich [11] chose to replace \(\tau\) by a time-independent positive constant \(\tau_0\). Their version of the GWCE is given by

\[
(1) \quad \frac{\partial^2 \xi}{\partial t^2} + \tau_0 \frac{\partial \xi}{\partial t} - \nabla \cdot \left[ \nabla \cdot \left( \frac{1}{H} \nu \nu \right) + (k \times f_c \nu) \right] \\
-\tau_0 \nabla \xi + H g \nabla \xi + E_n \nabla \frac{\partial \xi}{\partial t} - \tau_{ws} + H \nabla p_a - H g \nabla \eta = 0.
\]

This choice of \(\tau_0\) yields a system of time-independent matrices when the GWCE is discretized in time using a three-level implicit scheme for linear terms. (Here and in the equations below we have used tensor notation, reviewed in Appendix A.)

The GWCE can be coupled to the CME, given by

\[
(2) \quad \frac{\partial \nu}{\partial t} + \nabla \cdot \left( \frac{1}{H} \nu \nu \right) + \tau_{s} \nu + (k \times f_c \nu) + H g \nabla \xi \\
-\tau_{ws} + H \nabla p_a - H g \nabla \eta = 0,
\]

or to the NCME. A finite element simulator based on the GWCE-NCME, which uses same-order polynomials to approximate elevation and velocity unknown, has been developed by Luetttich, et al. In [11], it was demonstrated that the approximations generated by this simulator accurately matched tidal data taken from the English Channel and southern North Sea.
To date, no formal convergence analysis of finite element approximations to the WCE or GWCE combined with either the NCME or the CME exists in the literature. In this paper, we analyze the coupled GWCE-CME system of equations for a Galerkin finite element approximation in space and continuous time.

The rest of this paper is outlined as follows. In section 2 we detail the assumptions we will need in our analysis. We also introduce the weak formulation associated with the GWCE-CME system of equations. In section 3, we introduce the finite element approximation to the weak solution. In section 4, we derive an \textit{a priori} error estimate based on an $L^2$ projection.

2. Preliminaries.

2.1. Notation and Definitions. For the purpose of our analysis, we define some notation used throughout the rest of this paper.

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

The $L^2$ inner product is denoted by

$$(\varphi, \psi) = \int_\Omega \varphi \cdot \psi \, dx, \quad \varphi, \psi \in [L^2(\Omega)]^n,$$

where "$\cdot$" here refers to either multiplication, dot product, or double dot product as appropriate. We denote the $L^2$ norm by $||| \cdot |||_{L^2(\Omega)} = \langle \cdot, \cdot \rangle^{1/2}$. In $\mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple with nonnegative integer components,

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},$$

and $|\alpha| = \sum_{i=1}^n \alpha_i$.

For $\ell$ any nonnegative integer, let

$$\mathcal{H}^\ell \equiv \{ \varphi \in L^2(\Omega) \mid D^\alpha \varphi \in L^2(\Omega) \text{ for } |\alpha| \leq \ell \}$$

be the Sobolev space with norm

$$||| \cdot |||_{\mathcal{H}^\ell(\Omega)} = \left( \sum_{|\alpha| \leq \ell} ||| D^\alpha \cdot |||_{L^2(\Omega)}^2 \right)^{1/2}.$$

For relevant properties of these spaces, please refer to [1]. \(C^\infty_0(\Omega)\) is the set of infinitely differentiable functions with compact support in $\Omega$. Also, $\mathcal{H}^1(\Omega)$ denotes the subspace of $\mathcal{H}^1(\Omega)$ obtained by completing $C^\infty_0(\Omega)$ with respect to the norm $||| \cdot |||_{\mathcal{H}^1(\Omega)}$.

Observe that $\mathcal{H}^\ell$ are spaces of $\mathbb{R}$-valued functions. Spaces of $\mathbb{R}^n$-valued functions will be denoted in boldface type, but their norms will not be distinguished. Thus, $L^2(\Omega) = [L^2(\Omega)]^n$ has norm $||| \varphi |||_2^2 = \sum_{i=1}^n ||| \varphi_i |||_2^2$; $\mathcal{H}^1(\Omega) = [\mathcal{H}^1(\Omega)]^n$ has norm $||| \varphi |||_{\mathcal{H}^1(\Omega)}^2 = \sum_{i=1}^n \sum_{|\alpha| \leq 1} ||| D^\alpha \varphi_i |||_2^2$; etc.

For $X$, a normed space with norm $|| \cdot ||_X$ and a map $f: [0, T] \rightarrow X$, define

$$||f||_{L^2(0, T); X} = \int_0^T ||f(\cdot, t)||_X^2 \, dt,$$

$$||f||_{L^\infty(0, T); X} = \sup_{0 \leq t \leq T} ||f(\cdot, t)||_X.$$
2.2. **Variational Formulation.** We will consider the coupled system given by the GWCE-CME described in Section 1, with the following homogeneous Dirichlet boundary conditions for simplicity

\[
\begin{align*}
\xi(x, t) &= 0, \\
U(x, t) &= 0, \\
V(x, t) &= 0,
\end{align*}
\]

\(\{\begin{array}{c}
x \in \partial \Omega, \; t > 0,
\end{array}\}
\tag{3}
\]

and with the compatible initial conditions

\[
\begin{align*}
\xi(x, 0) &= \xi_0(x), \\
\frac{\partial \xi}{\partial t}(x, 0) &= \xi_1(x), \\
U(x, 0) &= U_0(x), \\
V(x, 0) &= V_0(x),
\end{align*}
\]

\(\{\begin{array}{c}
x \in \bar{\Omega},
\end{array}\}
\tag{4}
\]

where \(\partial \Omega\) is the boundary of \(\Omega \subset \mathbb{R}^2\) and \(\bar{\Omega} = \Omega \cup \partial \Omega\). Extensions to more general land and sea boundary conditions will be treated in a later paper. As noted in Kinmack [10], the condition necessary for the solution of the GWCE-CME system of equations to be the same as the solution of the primitive form is that

\[
\xi_1(x) = -\nabla \cdot v(x, 0).
\]

The weak form of this system is the following: For \(t \in (0, T]\), find \(\xi(x, t) \in \mathcal{H}_0^1(\Omega)\) and \(v(x, t) \in \mathcal{H}_0^1(\Omega)\) such that

\[
\begin{align*}
&\left(\frac{\partial^2 \xi}{\partial t^2}, v\right) + \tau_0 \left(\frac{\partial \xi}{\partial t}, v\right) + \left(\nabla \cdot \left\{ \frac{1}{H} v^2 \right\}, \nabla v\right) + \left((\tau_{nf} - \tau_o)v, \nabla v\right) \\
&\quad + (k \times f_n v, \nabla v) + (Hg \nabla \xi, \nabla v) + E_h \left(\nabla \frac{\partial \xi}{\partial t}, \nabla v\right) \\
&\quad - (\tau_{ws}, \nabla v) + (H \nabla p_a, \nabla v) - (Hg \nabla \eta, \nabla v) = 0, \quad \forall v \in \mathcal{H}_0^1(\Omega), t > 0,
\end{align*}
\]

\(\tag{5}\)

\[
\begin{align*}
&\left(\frac{\partial v}{\partial t}, w\right) + \left(\nabla \cdot \left\{ \frac{1}{H} v^2 \right\}, w\right) + (\tau_{nf} v, w) + (k \times f_n v, w) + (Hg \nabla \xi, w) \\
&\quad + E_h (\nabla v, \nabla w) - (\tau_{ws}, w) + (H \nabla p_a, w) - (Hg \nabla \eta, w) = 0, \quad \forall w \in \mathcal{H}_0^1(\Omega), t > 0,
\end{align*}
\]

\(\tag{6}\)

with initial conditions

\[
\begin{align*}
(\xi(x, 0), v) &= (\xi_0(x), v), \quad \forall v \in \mathcal{H}_0^1(\Omega), \\
(\frac{\partial \xi}{\partial t}(x, 0), v) &= (\xi_1(x), v), \quad \forall v \in \mathcal{H}_0^1(\Omega), \\
(v(x, 0), w) &= (v_0(x), w), \quad \forall w \in \mathcal{H}_0^1(\Omega).
\end{align*}
\]

\(\tag{7}\)

Here, we have set \(v_0 = (H_0 U_0, H_0 V_0)^T\).
2.3. Some Assumptions. Our analysis requires that we make certain physically reasonable assumptions about the solutions and the data. First, we assume for \((x, t) \in \bar{\Omega} \times (0, T)\)

- **A1** the solutions \((\xi, v)\) to (5)-(7) exist and are unique,
- **A2** \exists positive constants \(H_*\) and \(H^*\) such that \(H_* \leq H(x, t) \leq H^*\),
- **A3** the velocities \(U(x, t), V(x, t)\) are bounded,
- **A4** \(\frac{\partial}{\partial x_i} h_0(x)\) is bounded.

Assumption **A2** is obvious from Figure 1. Dimensional analysis as explained in [21] accounts for assumption **A3**. Second, we assume that for \((x, t) \in \bar{\Omega} \times (0, T)\)

- **A5** \exists positive constants \(\gamma_*\) and \(\gamma^*\) such that \(\gamma_* \leq gh_0(x) \leq \gamma^*\),
- **A6** \exists non-negative constants \(\tau_*\) and \(\tau^*\) such that \(\tau_* \leq \tau_f \leq \tau^*\),
- **A7** \(\tau_f - \tau_0\) is bounded,
- **A8** \exists non-negative constants \(f_*\) and \(f^*\) such that \(f_* \leq f_c \leq f^*\),
- **A9** \(E_h\) is a positive constant,
- **A10** \(p_a(x, t)\) and its first spatial derivatives are bounded,
- **A11** \(\eta(x, t)\) and its first spatial derivatives are bounded.

Finally, we make the following smoothness assumptions on the initial data and on the solutions:

- **A12** \(\xi_0(x) \in H_0^k(\Omega)\),
- **A13** \(\xi_1(x) \in H_0^k(\Omega)\),
- **A14** \(v_0(x) \in H_0^k(\Omega)\),
- **A15** \(H(x, \cdot) \in H_0^k(\Omega) \cap H^\ell(\Omega), \ t \in (0, T)\),
- **A16** \(v(x, \cdot) \in H_0^k(\Omega) \cap H^\ell(\Omega), \ t \in (0, T)\),

where \(k\) and \(\ell\) are defined in the next section.

3. Finite Element Approximation.

3.1. The Continuous-Time Galerkin Approximation. Let \(T\) be a triangulation of \(\Omega\) into elements \(E_i\), \(i = 1, \ldots, m\), with \(\text{diam}(E_i) = h_i\) and \(h = \max_i h_i\). Let \(S^h\) denote a finite dimensional subspace of \(H_0^k(\Omega)\) defined on this triangulation consisting of piecewise polynomials of degree \(k - 1\). Define \(\mathcal{H}(\Omega) = H_0^k(\Omega) \cap H^\ell(\Omega)\), and assume \(S^h\) satisfies the standard approximation property

\[
\inf_{\varphi \in S^h} \|v - \varphi\|_{H^s(\Omega)} \leq C h^{\ell-s} \|v\|_{H^\ell(\Omega)}, \quad v \in \mathcal{H}(\Omega),
\]

and the inverse assumption

\[
\|\varphi\|_{H^{\ell-s}(\Omega)} \leq C \|\varphi\|_{L^2(\Omega)} h^{-(\ell-s)}, \quad \varphi \in S^h(\Omega),
\]

for \(k, \ell \geq 2, 0 \leq s \leq \ell \leq k\), and where \(C\) is a constant independent of \(h\) and \(v\).

We define the continuous-time Galerkin approximations to \(\xi, v\) to be the mappings \(\Xi(x, t) \in S^h, \mathcal{Y}(x, t) \in S^h\) for each \(t > 0\) satisfying

\[
\left(\frac{\partial^2 \Xi}{\partial t^2}, v\right) + \tau_0 \left(\frac{\partial \Xi}{\partial t}, v\right) + \left(\nabla \cdot \left\{\frac{1}{2} \mathcal{Y}^2\right\}, \nabla v\right) + \left((\tau_f - \tau_0) \mathcal{Y}, \nabla v\right)
\]

\[
+ (k \times f, \mathcal{Y}, \nabla v) + (\Pi g \nabla \Xi, \nabla v) + E_h \left(\nabla \frac{\partial \Xi}{\partial t}, \nabla v\right)
\]

\[= -(\tau_{ws}, \nabla v) + (\Pi \nabla p_a, \nabla v) - (\Pi g \nabla \eta, \nabla v) = 0, \quad \forall v \in S^h(\Omega),
\]
(11) \[
\left( \frac{\partial \Xi}{\partial t}, w \right) + \left( \nabla \cdot \left( \frac{1}{\Pi} \mathbf{Y}^2 \right), w \right) + \left( \tau_{\mathbf{Y}}, \mathbf{w} \right) + \left( k \times f_c \mathbf{Y}, \mathbf{w} \right) + \left( \Pi g \nabla \zeta, w \right)
\]
\[+ E_h (\nabla \mathbf{Y}, \nabla w) - (\tau_{w,3}, w) + (\Pi \nabla p_a, w) - (\Pi g \nabla \eta, w) = 0, \quad \forall w \in \mathcal{S}^b(\Omega),\]

with boundary conditions
\[
\begin{align*}
\Xi(x, t) &= 0, \\
\mathbf{Y}(x, t) &= 0,
\end{align*}
\]
t > 0;

and with initial conditions
\[
\begin{align*}
(\Xi(x, 0), v) &= (\xi_0(x), v), \quad \forall v \in \mathcal{S}^h(\Omega), \\
\left( \frac{\partial \Xi}{\partial t}(x, 0), v \right) &= (\xi_1(x), v), \quad \forall v \in \mathcal{S}^h(\Omega), \\
(\mathbf{Y}(x, 0), w) &= (\mathbf{v}_0(x), w), \quad \forall w \in \mathcal{S}^h(\Omega).
\end{align*}
\]

Here, \( \Pi(x, t) = h_b(x) + \Xi(x, t) \).

We need to make certain assumptions about the Galerkin approximation similar to those made about the solution to (5)-(7). In particular, we assume that

\textbf{B1} \exists positive constants \( \Pi_* \) and \( \Pi^* \) such that

\[ \Pi_* \leq \Pi(x, t) \leq \Pi^*, \]

\textbf{B2} \( \mathbf{Y} \) is bounded.

4. \textit{A Priori Error Estimate}. To \( L^2 \) projections \( \tilde{\xi} \) and \( \tilde{v} \) satisfying

\[
\begin{align*}
\left( (\xi - \tilde{\xi})(\cdot, t), v \right) &= 0, \quad \forall v \in \mathcal{S}^h, t \geq 0, \\
\left( (\nu - \tilde{\nu})(\cdot, t), w \right) &= 0, \quad \forall w \in \mathcal{S}^h, t \geq 0,
\end{align*}
\]

we will compare our finite element approximations \( \Xi \) and \( \mathbf{Y} \) which satisfy (10)-(13).

For the purpose of succinctness in the rest of the paper, we define

\[
\begin{align*}
\theta &= \xi - \tilde{\xi}, \\
\psi &= \Xi - \tilde{\xi}, \\
\phi &= \nu - \tilde{\nu}, \\
\chi &= \mathbf{Y} - \tilde{\nu}.
\end{align*}
\]

Clearly, \( \xi - \Xi = \theta - \psi \) and \( \nu - \mathbf{Y} = \phi - \chi \). We shall call \( \theta \) and \( \phi \) the projection errors and we shall call \( \psi \) and \( \chi \) the affine errors.

The following results are standard.

\textbf{Lemma 4.1}. Let \( \xi \in L^2((0, T), \mathcal{H}(\Omega)) \) and \( \nu \in L^2((0, T), \mathcal{H}(\Omega)) \) and let \( (\tilde{\xi}, \tilde{\nu}) \) be the corresponding \( L^2 \) projections defined by (14). And let \( \theta \) and \( \phi \) be defined as above. \( \text{If for some integer } j \geq 0 \)

\[
\frac{\partial^j \xi}{\partial t^j} \in L^2((0, T); \mathcal{H}(\Omega)), \quad \frac{\partial^j \nu}{\partial t^j} \in L^2((0, T); \mathcal{H}(\Omega)),
\]

then

\[
\frac{\partial^2 \tilde{\xi}}{\partial t^2} \in L^2((0, T); \mathcal{S}^h(\Omega)), \quad \frac{\partial^2 \tilde{\nu}}{\partial t^2} \in L^2((0, T); \mathcal{S}^h(\Omega)),
\]

and

\[
\begin{align*}
\left\| \left( \frac{\partial}{\partial t} \right)^j \theta \right\|_{L^2((0, T); \mathcal{H}(\Omega))} &\leq Ch^{3-s} \left\| \left( \frac{\partial}{\partial t} \right)^j \xi \right\|_{L^2((0, T); \mathcal{H}(\Omega))}, \\
\left\| \left( \frac{\partial}{\partial t} \right)^j \phi \right\|_{L^2((0, T); \mathcal{H}(\Omega))} &\leq Ch^{3-s} \left\| \left( \frac{\partial}{\partial t} \right)^j \nu \right\|_{L^2((0, T); \mathcal{H}(\Omega))}
\end{align*}
\]
for some constant $C$ independent of $\xi, \nu, h, \ell$ and $q = \min(\ell, k)$.

The following lemma will be needed when we bound the right-hand sides of (18), (19), and (20).

**Lemma 4.2.** Let Assumptions A2, B1 hold. There exists constants $Q_1, Q_2$ such that

$$\left\| \frac{\nu}{H} - \frac{\nu}{\Pi} \right\| \leq Q_1 (||\theta|| + ||\psi||) + Q_2 (||\phi|| + ||\chi||).$$

**Proof.**

$$\left\| \frac{\nu}{H} - \frac{\nu}{\Pi} \right\| = \left\| \frac{\nu (\Pi - H) + (\nu - \nu H) H}{H \Pi} \right\|
\leq \left\| \frac{\nu}{H \Pi} \right\|_{L^\infty(\Omega)} ||\Pi - H|| + \left\| \frac{1}{\Pi} \right\|_{L^\infty(\Omega)} ||\nu - \nu H||
= Q_1 ||\Xi - \xi|| + Q_2 ||\nu - \nu H||
\leq Q_1 (||\theta|| + ||\psi||) + Q_2 (||\phi|| + ||\chi||).$$

Assumptions A2, B1 are used to get the first part of the inequality and assumption B1 is used to get the second part of the inequality. $\square$

We also need the following results.

**Lemma 4.3.** $\bar{\xi}, \bar{\nu}$ and their first-order spatial derivatives are bounded above in $L^\infty(\Omega)$.

**Proof.** Let $C$ denote a generic constant not necessarily the same at every occurrence. Let $\nu_I \in S^h$ denote the interpolant of $\nu$. Then, by inverse assumptions

$$||d^\alpha \bar{\nu}||_{L^\infty(\Omega)} \leq ||d^\alpha \bar{\nu} - d^\alpha \nu_I||_{L^\infty(\Omega)} + ||d^\alpha \nu_I||_{L^\infty(\Omega)}
\leq Ch^{-1} ||d^\alpha \bar{\nu} - d^\alpha \nu_I|| + C
\leq Ch^{-1} (||d^\alpha \bar{\nu} - d^\alpha \nu|| + ||d^\alpha \nu - d^\alpha \nu_I||) + C
\leq Ch^{-1} (Ch^{2-\alpha} ||\nu||_{H^2(\Omega)} + Ch^{2-\alpha} ||\nu||_{H^2(\Omega)}) + C
\leq Ch^{1-\alpha} + C \leq 2C$$

for $\ell \geq 2$ and $\alpha = 0, 1$. Thus $d^\alpha \bar{\nu}$ is bounded above in $L^\infty(\Omega)$. A similar argument holds in showing that $d^\alpha \bar{\xi}$ is bounded above in $L^\infty(\Omega)$. $\square$

Consequently, we can prove the following result.

**Theorem 4.4.** Let $0 \leq s \leq \ell \leq k, \ell, k \geq 2, H = \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^\ell(\Omega)$. Let $(\xi, \nu)$ be the solution to (5)-(7). Let $(\Xi, \Upsilon)$ be the Galerkin approximations to $(\xi, \nu)$. If $\xi(t) \in H(\Omega), \nu(t) \in H(\Omega)$ for each $t$; if $\Xi(t) \in S^h(\Omega), \Upsilon(t) \in S^h(\Omega)$ for each $t$; and suppose that assumptions A2-A11 hold; then, $\exists$ a constant $C = C(T, k, r)$ such that

$$\left\| \frac{\partial}{\partial t} (\xi - \Xi) \right\|_{L^2((0,T);L^2(\Omega))} + \||\xi - \Xi(\cdot, T)|| + ||\xi - \Xi||_{L^2((0,T);H^1(\Omega))}
+ \|\nu - \Upsilon(\cdot, T)|| + ||\nu - \Upsilon||_{L^2((0,T);H^1(\Omega))} \leq Ch^{\ell-1}.$$

In order to obtain an error estimate for $(\xi - \Xi)$ and $(\nu - \Upsilon)$, we must first obtain an estimate on affine error terms $(\Xi - \bar{\xi})$ and $(\Upsilon - \bar{\nu})$. Then, with the approximation result stated in Lemma 4.1 and with the estimate on the affine error to be obtained
in the proof of Theorem 4.4, an application of the triangle inequality will yield an estimate for \((\xi - \Xi)\) and \((u - \gamma)\).

**Proof.** It will be useful to employ the following expansion of terms (a)-(b) in (5)-(6):

\[
\left( \nabla \left( \frac{1}{H} \mathbf{v}^2 \right), \cdot \right) = \left( \left( \frac{\nabla \cdot \nabla}{H} \right) \mathbf{v}, \cdot \right) + \left( (\nabla \cdot \mathbf{v} \right) \frac{\mathbf{v}}{H}, \cdot \right) - \left( \mathbf{h} \cdot \mathbf{v} \right) \frac{\mathbf{v}}{H^2}, \cdot \) + \left( \nabla \xi \cdot \mathbf{v} \right) \frac{\mathbf{v}}{H^2}, \cdot \right)
\]

Similarly, the expansion of terms (a')-(b') in (10)-(11) gives

\[
\left( \nabla \cdot \left( \frac{1}{\Pi} \mathbf{r}^2 \right), \cdot \right) = \left( \left( \frac{\nabla \cdot \nabla}{\Pi} \right) \mathbf{r}, \cdot \right) + \left( (\nabla \cdot \mathbf{r} \right) \frac{\mathbf{r}}{\Pi}, \cdot \right) - \left( \mathbf{h} \cdot \mathbf{r} \right) \frac{\mathbf{r}}{\Pi^2}, \cdot \) - \left( \nabla \Xi \cdot \mathbf{r} \right) \frac{\mathbf{r}}{\Pi^2}, \cdot \right)
\]

Subtract (5) from (10) and (6) from (11), using the fact that we can write

\[
\left( \left( \frac{\nabla \cdot \nabla}{\Pi} \right) \mathbf{r} - \mathbf{v} \right) - \left( \left( \frac{\nabla \cdot \nabla}{H} \right) \mathbf{v} - \left( \frac{\nabla \cdot \nabla}{\Pi} \right) \mathbf{v} \right)
\]

\[
= \left( \left( \frac{\nabla \cdot \nabla}{\Pi} \right) \mathbf{v}, \cdot \right) - \left( \left( \frac{\nabla \cdot \nabla}{H} \right) \mathbf{v} - \left( \frac{\nabla \cdot \nabla}{\Pi} \right) \mathbf{v} \right)
\]

\[
= \left( \left( \frac{\nabla \cdot \nabla}{\Pi} \right) \mathbf{v}, \cdot \right) - \left( \left( \frac{\nabla \cdot \nabla}{H} \right) \mathbf{v} - \left( \frac{\nabla \cdot \nabla}{\Pi} \right) \mathbf{v} \right)
\]

and

\[
\left( \left( \nabla \Xi \cdot \mathbf{r} \right) \frac{\mathbf{r}}{\Pi^2}, \cdot \right) - \left( \left( \nabla \Xi \cdot \mathbf{v} \right) \frac{\mathbf{v}}{H^2} - \left( \nabla \Xi \cdot \mathbf{r} \right) \frac{\mathbf{r}}{\Pi^2}, \cdot \right)
\]

\[
= \left( \left( \nabla \Xi \cdot \mathbf{r} \right) \frac{\mathbf{r}}{\Pi^2}, \cdot \right) - \left( \left( \nabla \Xi \cdot \mathbf{v} \right) \frac{\mathbf{v}}{H^2} - \left( \nabla \Xi \cdot \mathbf{r} \right) \frac{\mathbf{r}}{\Pi^2}, \cdot \right)
\]

Consequently, we obtain the following GWCE-CME error equations:

\[
(16) \quad \left( \frac{\partial^2 \psi}{\partial t^2}, \mathbf{v} \right) + \tau \left( \frac{\partial \psi}{\partial t}, \mathbf{v} \right) + \left( \left( \frac{\nabla \cdot \nabla}{\Pi} \right) \mathbf{v}, \mathbf{v} \right) + \left( \left( \nabla \cdot \mathbf{r} \right) \frac{\mathbf{r}}{\Pi}, \nabla \mathbf{v} \right)
\]
\[-\left((\nabla \psi \cdot \mathbf{r}) \frac{\mathbf{r}}{\Pi^2} \cdot \nabla v\right) + ((\tau_f - \tau_0) \chi, \nabla v) + (k \times f_c \chi, \nabla v) + (\Pi g \nabla \psi, \nabla v) + E_h \left(\nabla \frac{\partial \psi}{\partial t}, \nabla v\right) + (\psi \nabla p_a, \nabla v) - (\psi g \nabla \eta, \nabla v)\]

\[= \left(\left(\frac{v}{H} \cdot \nabla\right) \phi, \nabla v\right) + \left(\left(\frac{v}{H} - \frac{\mathbf{r}}{\Pi}\right) \cdot \nabla \tilde{v}, \nabla v\right) + \left(\nabla \cdot \phi \frac{v}{H}, \nabla v\right) + \left(\nabla \cdot \tilde{v} \left[\frac{v}{H} - \frac{\mathbf{r}}{\Pi}\right], \nabla v\right) - \left(\left\{(\frac{v}{H})^2 - \left(\frac{\mathbf{r}}{\Pi}\right)^2\right\} \nabla h_b, \nabla v\right)\]

\[+ \left(\nabla \cdot \mathbf{v} \left(\frac{\mathbf{v}}{H^2}, \nabla v\right) - \left(\left\{(\frac{v}{H})^2 - \left(\frac{\mathbf{r}}{\Pi}\right)^2\right\} \nabla \tilde{\xi}, \nabla v\right) - (\tau_f - \tau_0) \phi, \nabla v\right) + (k \times f_c \phi, \nabla v) + (H g \nabla \theta, \nabla v) + \left(\theta g \nabla \tilde{\xi}, \nabla v\right) - (\psi g \nabla \tilde{\xi}, \nabla v)\]

\[+ E_h \left(\nabla \frac{\partial \theta}{\partial t}, \nabla v\right) + (\theta \nabla p_a, \nabla v) - (\theta g \nabla \eta, \nabla v), \quad \forall v \in \mathcal{S}^h(\Omega), t > 0.\]

\[(17) \quad \frac{\partial \chi}{\partial t}, w) + \left(\left(\frac{\mathbf{r}}{\Pi} \cdot \nabla\right) \chi, w\right) + \left(\nabla \cdot \chi \frac{\mathbf{r}}{\Pi}, w\right) - \left((\nabla \psi \cdot \mathbf{r}) \frac{\mathbf{r}}{\Pi^2}, w\right)\]

\[+ (\tau_f \chi, w) + (k \times f_c \chi, w) + (\Pi g \nabla \psi, w) + E_h (\nabla \chi, \nabla w) + (\psi \nabla p_a, w) - (\psi g \nabla \eta, w)\]

\[= \left(\left(\frac{v}{H} \cdot \nabla\right) \phi, w\right) + \left(\left(\frac{v}{H} - \frac{\mathbf{r}}{\Pi}\right) \cdot \nabla \tilde{v}, w\right) + \left(\nabla \cdot \phi \frac{v}{H}, w\right) + \left(\nabla \cdot \tilde{v} \left[\frac{v}{H} - \frac{\mathbf{r}}{\Pi}\right], w\right) - \left(\left\{(\frac{v}{H})^2 - \left(\frac{\mathbf{r}}{\Pi}\right)^2\right\} \nabla h_b, w\right)\]

\[\left(\nabla \cdot \mathbf{v} \left(\frac{\mathbf{v}}{H^2}, w\right) - \left(\left\{(\frac{v}{H})^2 - \left(\frac{\mathbf{r}}{\Pi}\right)^2\right\} \nabla \tilde{\xi}, w\right) + (\tau_f \phi, w)\right) + (k \times f_c \phi, w) + (H g \nabla \theta, w) + \left(\theta g \nabla \tilde{\xi}, w\right) - (\psi g \nabla \tilde{\xi}, w)\]

\[+ E_h (\nabla \phi, \nabla w) + (\theta \nabla p_a, w) - (\theta g \nabla \eta, w), \quad \forall w \in \mathcal{S}^h(\Omega), t > 0.\]

We now choose the test functions employed to obtain the affine error estimate. Let $\tau$ be a positive constant to be chosen. We let $v_1(\cdot, t) = \int_t^T e^{-\tau s} \psi(\cdot, s) \, ds$ and $v_2(\cdot, t) = \int_t^T e^{-\tau s} \frac{\partial \psi}{\partial t}(\cdot, s) \, ds$ be the test functions in (16). And, we let $w = \chi$ be the test function in (17).

First, integrate (16)-(17) in time over $(0, T]$ using the test functions above. We will see that our choice of test functions for $v$ in (16) allow us to circumvent the use of Gronwall's Lemma. In using $w = \chi$ as the test function in (17), we obtain, however, a relation to which we apply Gronwall's Lemma. Summing these equations and then taking bounds above and below to the result will yield a relation giving an estimate of the affine error.

Thus, we will first investigate the use of $v_1$ and $v_2$ as the test functions in (16) followed by temporally integrating over $(0, T]$. Note that $v_1(\cdot, T) = 0$, $v_2(\cdot, T) = 0$. Also recall that given $\zeta$, the following relations hold: $\left(\frac{2\zeta}{\mathbf{r}}, \zeta\right) = \frac{1}{2} \frac{d}{dt} \left(||\zeta||^2\right)$ and $\frac{1}{2} \frac{d}{dt} \left(e^{-\tau t} ||\zeta||^2\right) = \frac{1}{2} e^{-\tau t} \frac{d}{dt} \left(||\zeta||^2\right) - \frac{\tau}{2} e^{-\tau t} ||\zeta||^2$. 

Now, consider the first two terms of (16). When \( v = v_1 \), we obtain

\[
\int_0^T \left( \frac{\partial^2 \psi}{\partial t^2}, v_1 \right) \, dt = - \int_0^T \left( \frac{\partial \psi}{\partial t}, \frac{\partial v_1}{\partial t} \right) \, dt + \left( \frac{\partial \psi}{\partial t}, v_1 \right) \bigg|_0^T = \int_0^T e^{-rt} \left( \frac{\partial \psi}{\partial t}, \psi \right) \, dt \\
= \frac{1}{2} \int_0^T \frac{d}{dt} \left( e^{-rt} ||\psi||^2 \right) \, dt + \frac{r}{2} \int_0^T e^{-rt} ||\psi||^2 \, dt \\
= \frac{1}{2} e^{-rT} ||\psi(., T)||^2 + \frac{r}{2} \int_0^T e^{-rt} ||\psi||^2 \, dt;
\]

\[
\tau \int_0^T \left( \frac{\partial \psi}{\partial t}, v_1 \right) \, dt = -\tau \int_0^T \left( \psi, \frac{\partial v_1}{\partial t} \right) \, dt = \tau \int_0^T e^{-rt} ||\psi||^2 \, dt.
\]

The first equalities above result from temporal integration by parts. We also have from the diffusion term upon integrating by parts in time:

\[
E_h \int_0^T \left( \nabla \frac{\partial \psi}{\partial t}, \nabla v_1 \right) \, dt = E_h \int_0^T e^{-rt} ||\nabla \psi||^2 \, dt.
\]

We are also able to manipulate the eighth term in (16) by using the definition of \( v_1 \) as follows:

\[
\int_0^T (gh_b \nabla \psi, \nabla v_1) \, dt = -\frac{1}{2} \int_0^T e^{rt} \frac{d}{dt} (gh_b \nabla v_1, \nabla v_1) \, dt \\
= -\frac{1}{2} \int_0^T \frac{d}{dt} \left( e^{rt} \sum_{i=1}^2 \left( \sqrt{g_h} \int_t^T e^{-rs} \frac{\partial \psi}{\partial x_i} \, ds \right)^2 \right) \, dt \\
+ \frac{r}{2} \sum_{i=1}^2 \int_0^T e^{rt} \left( \sqrt{g_h} \int_t^T e^{-rs} \frac{\partial \psi}{\partial x_i} \, ds \right)^2 \, dt \\
\geq \frac{\tau}{2} \sum_{i=1}^2 \left( \int_0^T e^{-rs} \frac{\partial \psi}{\partial x_i} \, ds \right)^2 \int_0^T e^{rt} \left( \int_t^T e^{-rs} \frac{\partial \psi}{\partial x_i} \, ds \right)^2 \, dt.
\]

Similarly, using \( v = v_2 \) as the test function in (16) followed by temporally integrating over \((0, T)\) yields

\[
\int_0^T \left( \frac{\partial^2 \psi}{\partial t^2}, v_2 \right) \, dt = -\int_0^T \left( \frac{\partial \psi}{\partial t}, \frac{\partial v_2}{\partial t} \right) \, dt = \int_0^T e^{-rt} \left| \frac{\partial \psi}{\partial t} \right|^2 \, dt;
\]

\[
\tau \int_0^T \left( \frac{\partial \psi}{\partial t}, v_2 \right) \, dt = -\tau \int_0^T \left( \psi, \frac{\partial v_2}{\partial t} \right) \, dt = \tau \int_0^T e^{-rt} \left( \psi, \frac{\partial \psi}{\partial t} \right) \, dt \\
= \frac{\tau}{2} e^{-rT} ||\psi(., T)||^2 + \tau \frac{r}{2} \int_0^T e^{-rt} ||\psi||^2 \, dt.
\]

The first equality below follows from the definition of \( v_2 \):

\[
E_h \int_0^T \left( \nabla \frac{\partial \psi}{\partial t}, \nabla v_2 \right) \, dt = -\frac{E_h}{2} \int_0^T e^{rt} \frac{d}{dt} (\nabla v_2, \nabla v_2) \, dt \\
= -\frac{E_h}{2} \int_0^T \frac{d}{dt} \left( e^{rt} \sum_{i=1}^2 \left( \int_t^T e^{-rs} \frac{\partial^2 \psi}{\partial t \partial x_i} \, ds \right)^2 \right) \, dt.
\]
\[ + \frac{\tau E_h}{2} \sum_{i=1}^{2} \int_0^T e^{-r_s} \left( \int_0^T e^{-r_s} \left( \frac{\partial^2 \psi}{\partial t \partial x_i} \right) \right)^2 \, dt \]
\[ = \frac{E_h}{2} \sum_{i=1}^{2} \left( \int_0^T e^{-r_s} \frac{\partial^2 \psi}{\partial t \partial x_i} \, ds \right)^2 \]
\[ + \frac{\tau E_h}{2} \sum_{i=1}^{2} \int_0^T e^{-r_s} \left( \int_0^T e^{-r_s} \frac{\partial^2 \psi}{\partial t \partial x_i} \, ds \right)^2 \, dt. \]

The temporal integration of terms \( \left( \frac{\partial \chi}{\partial t}, \chi \right), (\tau f \chi, \chi), E_h (\nabla \chi, \nabla \chi) \) in (17) are straightforward.

Using \( v_1(\cdot, t) = \int_0^T e^{-r_s} \psi(\cdot, s) \, ds \) as the test function in (16), integrating in time over \( (0, T] \), and using the relations above yields

\[ 2 \frac{1}{2} e^{-r_T} ||\psi(T)||^2 + \frac{(r + 2\tau_o)}{2} \int_0^T e^{-r_T} ||\psi||^2 \, dt + \frac{\tau_s}{2} \sum_{i=1}^{2} \left| \int_0^T e^{-r_T} \frac{\partial \psi}{\partial x_i} \, ds \right|^2 \]
\[ + \frac{\tau_f}{2} \sum_{i=1}^{2} \int_0^T e^{-r_T} \left| \int_0^T e^{-r_T} \frac{\partial \psi}{\partial x_i} \, ds \right|^2 \, dt + E_h \int_0^T e^{-r_T} ||\nabla \psi||^2 \, dt \]
\[ \leq - \int_0^T \left( \left( \frac{\mathcal{Y}}{\Pi} \right) \chi, \nabla v_1 \right) \, dt - \int_0^T \left( \left( \nabla \chi \right) \frac{\mathcal{Y}}{\Pi}, \nabla v_1 \right) \, dt + \int_0^T \left( \left( \nabla \psi \cdot \mathcal{X} \right) \frac{\mathcal{Y}}{\Pi^2}, \nabla v_1 \right) \, dt \]
\[ - \int_0^T (\tau f_0 - \tau_0) \chi, \nabla v_1 \, dt - \int_0^T (k \times f_0 \chi, \nabla v_1) \, dt - \int_0^T (\Xi g \nabla \psi, \nabla v_1) \, dt \]
\[ - \int_0^T (\psi \nabla p_a, \nabla v_1) \, dt + \int_0^T (\psi g \nabla \eta, \nabla v_1) \, dt \]
\[ + \int_0^T \left( \left( \frac{\mathcal{V}}{\mathcal{H}} \mathcal{V} \right) \phi, \nabla v_1 \right) \, dt + \int_0^T \left( \left[ \frac{\mathcal{V}}{\mathcal{H}} - \frac{\mathcal{Y}}{\Pi} \right] \cdot \nabla \phi, \nabla v_1 \right) \, dt + \int_0^T \left( \left( \nabla \psi \right) \frac{\mathcal{V}}{\mathcal{H}}, \nabla v_1 \right) \, dt \]
\[ + \int_0^T \left( \nabla \phi \cdot \mathcal{V} \right) \frac{\mathcal{V}}{\mathcal{H}}, \nabla v_1 \right) \, dt - \int_0^T \left( \left[ \left( \frac{\mathcal{V}}{\mathcal{H}} \right)^2 - \left[ \frac{\mathcal{Y}}{\Pi} \right]^2 \right] \cdot \nabla h_b, \nabla v_1 \right) \, dt \]
\[ + \int_0^T \left( \nabla \theta \cdot \mathcal{V} \right) \frac{\mathcal{V}}{\mathcal{H}}, \nabla v_1 \right) \, dt - \int_0^T \left( \left[ \left( \frac{\mathcal{V}}{\mathcal{H}} \right)^2 - \left[ \frac{\mathcal{Y}}{\Pi} \right]^2 \right] \cdot \nabla \xi, \nabla v_1 \right) \, dt \]
\[ + \int_0^T \left( \left( \tau f_0 - \tau_0 \right) \phi, \nabla v_1 \right) \, dt + \int_0^T \left( k \times f_0 \phi, \nabla v_1 \right) \, dt \]
\[ + \int_0^T (H g \nabla \theta, \nabla v_1) \, dt + \int_0^T \left( \theta g \nabla \xi, \nabla v_1 \right) \, dt - \int_0^T (\psi g \nabla \xi, \nabla v_1) \, dt \]
\[ + E_h \int_0^T \left( \nabla \psi \cdot \frac{\partial \theta}{\partial t}, \nabla v_1 \right) \, dt + \int_0^T \left( \theta \nabla p_a, \nabla v_1 \right) \, dt - \int_0^T (\theta g \nabla \eta, \nabla v_1) \, dt \]
\[ = (A_1 + \cdots + A_8) + (P_1 + \cdots + P_{15}). \]

Here, \( A \) denotes an affine error term and \( P \) denotes a projection error term.

We will explore the nonlinear terms appearing in the right-hand side of (18) in more detail. We will make exhaustive use of the AGMI.

The bounds on \( A_1, A_2 \) and on \( A_3 \) result from using the Hölder Inequality (HI),
the AGM as well as Assumptions A9, B1, B2:

\[ A_1 \leq \epsilon_1 \int_0^T e^{-rt} \| \nabla \chi \|^2 dt + \frac{1}{4 \epsilon_1} \left\| \frac{R}{\Pi} \right\|_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}^2 \int_0^T e^{rt} \| \nabla v_1 \|^2 dt \]

\[ \leq \frac{11 E_h}{192} \| \nabla \chi \|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + \frac{48}{11 E_h} \left\| \frac{R}{\Pi} \right\|_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}^2 \int_0^T e^{rt} \| \nabla v_1 \|^2 dt; \]

\[ A_2 \leq 2 \epsilon_2 \int_0^T e^{-rt} \| \nabla \chi \|^2 dt + \frac{1}{4 \epsilon_2} \left\| \frac{R}{\Pi} \right\|_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}^2 \int_0^T e^{rt} \| \nabla v_1 \|^2 dt \]

\[ \leq \frac{22 E_h}{192} \| \nabla \chi \|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + \frac{48}{11 E_h} \left\| \frac{R}{\Pi} \right\|_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}^2 \int_0^T e^{rt} \| \nabla v_1 \|^2 dt; \]

\[ A_3 \leq \epsilon_3 \int_0^T e^{-rt} \| \nabla \psi \|^2 dt + \frac{1}{4 \epsilon_3} \left\| \frac{R^2}{\Pi^2} \right\|_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}^2 \int_0^T e^{rt} \| \nabla v_1 \|^2 dt \]

\[ = \frac{E_h}{12} \int_0^T e^{-rt} \| \nabla \psi \|^2 dt + \frac{3}{E_h} \left\| \frac{R^2}{\Pi^2} \right\|_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|^2 dt. \]

The bound on \( A_4 \) results from the HI and Assumption A7:

\[ A_4 \leq \epsilon_4 \int_0^T e^{-rt} \| \chi \|^2 dt + \frac{1}{4 \epsilon_4} \| \tau_f - \tau_0 \|^2_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|^2 dt \]

\[ \leq \frac{1}{4} \| \chi \|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + \| \tau_f - \tau_0 \|^2_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|^2 dt. \]

The bound on \( A_5 \) results from the HI and Assumption A8:

\[ A_5 \leq \epsilon_5 \int_0^T e^{-rt} \| \chi \|^2 dt + \frac{1}{4 \epsilon_5} (f^*)^2 \int_0^T e^{rt} \| \nabla v_1 \|^2 dt \]

\[ \leq \frac{1}{4} \| \chi \|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + (f^*)^2 \int_0^T e^{rt} \| \nabla v_1 \|^2 dt. \]

The bound for \( A_6 \) follows from the HI, the AGMI, and Assumptions A9, B1:

\[ A_6 \leq \epsilon_6 \int_0^T e^{-rt} \| \nabla \psi \|^2 dt + \frac{1}{4 \epsilon_6} \| \Xi g \|^2_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|^2 dt \]

\[ = \frac{E_h}{12} \int_0^T e^{-rt} \| \nabla \psi \|^2 dt + \frac{3}{E_h} \| \Xi g \|^2_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|^2 dt. \]

The bounds for \( A_7 \) and for \( A_8 \) follow from the HI, the AGMI, and Assumptions A10, A11:

\[ A_7 \leq \epsilon_7 \int_0^T e^{-rt} \| \psi \|^2 dt + \frac{1}{4 \epsilon_7} \| \nabla p_a \|^2_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|^2 dt \]

\[ = \frac{\tau_0}{21} \int_0^T e^{-rt} \| \psi \|^2 dt + \frac{21}{4 \tau_0} \| \nabla p_a \|^2_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|^2 dt; \]

\[ A_8 \leq \epsilon_8 \int_0^T e^{-rt} \| \psi \|^2 dt + \frac{1}{4 \epsilon_8} \| g \nabla \eta \|^2_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|^2 dt \]

\[ = \frac{\tau_0}{21} \int_0^T e^{-rt} \| \psi \|^2 dt + \frac{21}{4 \tau_0} \| g \nabla \eta \|^2_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|^2 dt. \]
From the HI, the AGMI, and Assumptions A2, A3, we obtain the following upper bound on the projection error terms $P_1$ and $P_3$.

\[ P_1 = \int_0^T \left( \left( \frac{v}{H} \cdot \nabla \right) \phi, \nabla v_1 \right) dt \]

\[ \leq \frac{\epsilon_1}{2^2} \| \nabla \phi \|_{L^2((0,T);L^2(\Omega))}^2 + \frac{1}{4 \epsilon_1} \left\| \frac{v}{H} \right\|_{L^\infty((0,T);L^\infty(\Omega))}^2 \int_0^T e^{rt} \| \nabla v_1 \|_{L^2(\Omega)}^2 dt \]

\[ \equiv \frac{1}{18} \| \nabla \phi \|_{L^2((0,T);L^2(\Omega))}^2 + \frac{9}{2} \left\| \frac{v}{H} \right\|_{L^\infty((0,T);L^\infty(\Omega))}^2 \int_0^T e^{rt} \| \nabla v_1 \|_{L^2(\Omega)}^2 dt; \]

\[ P_3 = \int_0^T \left( \left( \nabla \cdot \frac{v}{H} \right), \nabla v_1 \right) dt \]

\[ \leq \frac{2 \epsilon_3}{2} \| \nabla \phi \|_{L^2((0,T);L^2(\Omega))}^2 + \frac{1}{4 \epsilon_3} \left\| \frac{v}{H} \right\|_{L^\infty((0,T);L^\infty(\Omega))}^2 \int_0^T e^{rt} \| \nabla v_1 \|_{L^2(\Omega)}^2 dt \]

\[ \equiv \frac{1}{9} \| \nabla \phi \|_{L^2((0,T);L^2(\Omega))}^2 + \frac{9}{2} \left\| \frac{v}{H} \right\|_{L^\infty((0,T);L^\infty(\Omega))}^2 \int_0^T e^{rt} \| \nabla v_1 \|_{L^2(\Omega)}^2 dt. \]

The bound for $P_2$ and $P_4$ results from the HI, the AGMI, Lemma 4.2, and Lemma 4.3:

\[ P_2 = \int_0^T \left( \left[ \frac{v}{H} - \frac{Y}{\Pi} \right] \cdot \nabla \tilde{u}, \nabla v_1 \right) dt \]

\[ \leq \epsilon_2a \int_0^T e^{-rt} \| \theta \|_2^2 dt + \epsilon_2b \int_0^T e^{-rt} \| \psi \|_2^2 dt + \epsilon_2c \int_0^T e^{-rt} \| \phi \|_2^2 dt + \epsilon_2d \int_0^T e^{-rt} \| \chi \|_2^2 dt \]

\[ + \frac{1}{4} \left( \frac{Q_1^2}{\epsilon_2a} + \frac{Q_1^2}{\epsilon_2b} + \frac{Q_2^2}{\epsilon_2c} + \frac{Q_2^2}{\epsilon_2d} \right) \| \nabla \tilde{u} \|_{L^\infty((0,T);L^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|_{L^2(\Omega)}^2 dt \]

\[ \leq \frac{1}{42} \| \theta \|_{L^2((0,T);L^2(\Omega))}^2 + \frac{\tau_0}{21} \int_0^T e^{-rt} \| \psi \|_2^2 dt + \frac{1}{82} \| \phi \|_{L^2((0,T);L^2(\Omega))}^2 + \frac{1}{4} \| \chi \|_{L^2((0,T);L^2(\Omega))}^2 \]

\[ + \frac{1}{4} \left( 42Q_1^2 + \frac{21}{\tau_0} Q_1^2 + 82Q_2^2 + 4Q_2^2 \right) \| \nabla \tilde{u} \|_{L^\infty((0,T);L^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|_{L^2(\Omega)}^2 dt; \]

\[ P_4 = \int_0^T \left( \left( - \frac{v}{H} + \frac{Y}{\Pi} \right), \nabla v_1 \right) dt \]

\[ \leq \epsilon_4a \int_0^T e^{-rt} \| \theta \|_2^2 dt + \epsilon_4b \int_0^T e^{-rt} \| \psi \|_2^2 dt + \epsilon_4c \int_0^T e^{-rt} \| \phi \|_2^2 dt + \epsilon_4d \int_0^T e^{-rt} \| \chi \|_2^2 dt \]

\[ + \frac{1}{4} \left( \frac{Q_1^2}{\epsilon_4a} + \frac{Q_1^2}{\epsilon_4b} + \frac{Q_2^2}{\epsilon_4c} + \frac{Q_2^2}{\epsilon_4d} \right) 2 \| \nabla \tilde{u} \|_{L^\infty((0,T);L^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|_{L^2(\Omega)}^2 dt \]

\[ \leq \frac{1}{42} \| \theta \|_{L^2((0,T);L^2(\Omega))}^2 + \frac{\tau_0}{21} \int_0^T e^{-rt} \| \psi \|_2^2 dt + \frac{1}{82} \| \phi \|_{L^2((0,T);L^2(\Omega))}^2 + \frac{1}{4} \| \chi \|_{L^2((0,T);L^2(\Omega))}^2 \]

\[ + \frac{1}{2} \left( 42Q_1^2 + \frac{21}{\tau_0} Q_1^2 + 82Q_2^2 + 4Q_2^2 \right) \| \nabla \tilde{u} \|_{L^\infty((0,T);L^\infty(\Omega))} \int_0^T e^{rt} \| \nabla v_1 \|_{L^2(\Omega)}^2 dt. \]

The bound for $P_5$ results from the HI, the AGMI, Lemma 4.2, Lemma 4.3 and Assumptions A2, A3, A4, B1, B2:

\[ P_5 = \int_0^T \left( \left[ \frac{v}{H} \right]^2 - \left[ \frac{Y}{\Pi} \right]^2 \right), \nabla h_5, \nabla v_1 \right) dt
\[
- \int_0^T \left( \left\{ \left[ \frac{v}{H} - \frac{X}{\Pi} \right] \left[ \frac{v}{H} + \frac{Y}{\Pi} \right] + \left[ \frac{Yv - vY}{H\Pi} \right] \right\} \cdot \nabla h_b, \nabla v_1 \right) dt \\
\leq \epsilon'_5 a \int_0^T e^{-rt} ||\theta||^2 dt + \epsilon'_5 b \int_0^T e^{-rt} ||\psi||^2 dt + \epsilon'_5 c \int_0^T e^{-rt} ||\phi||^2 dt + \epsilon'_5 d \int_0^T e^{-rt} ||\chi||^2 dt \\
+ 8 \epsilon'_5 e \int_0^T e^{-rt} ||\chi||^2 dt + 4 \epsilon'_5 f \int_0^T e^{-rt} ||\phi||^2 dt \\
+ \frac{1}{4} \left( \frac{Q_1^2}{\epsilon'_5 a} + \frac{Q_1^2}{\epsilon'_5 b} + \frac{Q_2^2}{\epsilon'_5 c} + \frac{Q_2^2}{\epsilon'_5 d} \right) \left\| \frac{v}{H} + \frac{X}{\Pi} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \int_0^T e^{rt} \left\| \nabla v_1 \right\|^2 dt \\
+ \frac{2}{\epsilon'_5 e} \left\| \frac{\nabla h_b}{H\Pi} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \left\| \phi \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 + \frac{1}{\epsilon'_5 f} \left\| \frac{\nabla h_b}{H\Pi} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \left\| \vec{\sigma} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \\
\leq \frac{1}{42} ||\theta||^2_{C^2((0, T); C^2(\Omega))} + \frac{\tau_0}{21} \int_0^T e^{-rt} ||\phi||^2 dt + \frac{1}{82} \left\| \phi \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 + \frac{1}{4} \left\| \chi \right\|_{C^2((0, T); C^2(\Omega))}^2 \\
+ 2 \left\| \chi \right\|_{C^2((0, T); C^2(\Omega))}^2 + \frac{2}{41} \left\| \phi \right\|_{C^2((0, T); C^2(\Omega))}^2 \\
+ \frac{1}{4} \left( 42Q_1^2 + 21\tau_0^2 + 82Q_2^2 + 4Q_3^2 \right) \left\| \frac{v}{H} + \frac{X}{\Pi} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \int_0^T e^{rt} \left\| \nabla v_1 \right\|^2 dt \\
+ 8 \left\| \frac{\nabla h_b}{H\Pi} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \left\| \phi \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 + 82 \left\| \frac{\nabla h_b}{H\Pi} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \left\| \vec{\sigma} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2.
\]

From the HI, the AGMI, and Assumption A2, A3, we obtain the following upper bound on the projection error term \( P_6 \):

\[
P_6 = \int_0^T \left( (\nabla \theta \cdot v) \frac{v}{H^2}, \nabla v_1 \right) dt \\
\leq \epsilon'_6 \left\| \nabla \theta \right\|_{C^2((0, T); C^2(\Omega))}^2 + \frac{1}{4} \epsilon'_6 \left\| \frac{v^2}{H^2} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \int_0^T e^{rt} \left\| \nabla v_1 \right\|_{C^2(\Omega)}^2 dt \\
= \frac{1}{12} \left\| \nabla \theta \right\|_{C^2((0, T); C^2(\Omega))}^2 + \frac{3}{4} \left\| \frac{v^2}{H^2} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \int_0^T e^{rt} \left\| \nabla v_1 \right\|^2_{C^2(\Omega)} dt.
\]

The bound for \( P_7 \) results from the HI, the AGMI, Lemma 4.2, Lemma 4.3, and Assumptions A2, A3, B1, B2:

\[
P_7 = - \int_0^T \left( \left\{ \left[ \frac{v}{H} \right]^2 - \left[ \frac{X}{\Pi} \right]^2 \right\} \cdot \nabla \vec{\xi}, \nabla v_1 \right) dt \\
= - \int_0^T \left( \left\{ \left[ \frac{v}{H} - \frac{X}{\Pi} \right] \left[ \frac{v}{H} + \frac{X}{\Pi} \right] + \left[ \frac{Yv - vY}{H\Pi} \right] \right\} \cdot \nabla \vec{\xi}, \nabla v_1 \right) dt \\
\leq \epsilon'_7 a \int_0^T e^{-rt} ||\theta||^2 dt + \epsilon'_7 b \int_0^T e^{-rt} ||\psi||^2 dt + \epsilon'_7 c \int_0^T e^{-rt} ||\phi||^2 dt + \epsilon'_7 d \int_0^T e^{-rt} ||\chi||^2 dt \\
+ 8 \epsilon'_7 e \int_0^T e^{-rt} ||\chi||^2 dt + 4 \epsilon'_7 f \int_0^T e^{-rt} ||\phi||^2 dt \\
+ \frac{1}{4} \left( \frac{Q_1^2}{\epsilon'_7 a} + \frac{Q_1^2}{\epsilon'_7 b} + \frac{Q_2^2}{\epsilon'_7 c} + \frac{Q_2^2}{\epsilon'_7 d} \right) \left\| \frac{v}{H} + \frac{X}{\Pi} \right\|_{L^\infty((0, T); L^\infty(\Omega))}^2 \int_0^T e^{rt} \left\| \nabla v_1 \right\|^2 dt.
\[ + \frac{2}{\epsilon t_e} \left| \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \quad \frac{1}{\epsilon t_f} \left| \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \quad \left| \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \].

\[ = \frac{1}{42} \int_0^T e^{-rt} ||| \theta |||^2 dt + \frac{\tau_0}{21} \int_0^T e^{-rt} ||| \psi |||^2 dt + \frac{1}{82} \int_0^T e^{-rt} ||| \phi |||^2 dt + \frac{1}{4} ||| \chi |||^2_{L^2((0,T),C^2(\Omega))} \]

\[ + 2 \left( 42Q_2^2 + 21Q_1^2 + 82Q_2^2 + 4Q_2^2 \right) \left[ \left| \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \right] \int_0^T e^{-rt} ||| \nabla v_1 |||^2 dt \]

\[ + \frac{1}{4} \left( 42Q_2^2 + 21Q_1^2 + 82Q_2^2 + 4Q_2^2 \right) \left[ \left| \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \right] \int_0^T e^{-rt} ||| \nabla v_1 |||^2 dt \]

Term \( P_8 \) is treated similarly to \( A_4 \) and term \( P_9 \) is treated similarly to \( A_5 \) if, for both, we treat \( \phi \) like \( \chi \), and let \( \epsilon_8 = \epsilon_9 = \frac{1}{82} \).

From the HI, the AGMI, and Assumption \( A_2 \) we obtain the the following upper bound on the projection error term \( P_{10} \):

\[ P_{10} = \int_0^T (Hg \nabla \theta, \nabla v_1) dt \]

\[ \leq \epsilon_{10} \left| \nabla \theta \right|^2_{L^2((0,T),C^2(\Omega))} + \frac{1}{4\epsilon_{10}} \left| \nabla \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \int_0^T e^{-rt} \left| \nabla v_1 \right|^2 dt \]

\[ = \frac{1}{12} \left| \nabla \theta \right|^2_{L^2((0,T),C^2(\Omega))} + 3 \left| Hg \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \int_0^T e^{-rt} \left| \nabla v_1 \right|^2 dt. \]

Obtaining bounds on \( P_{11} \), on \( P_{12} \), and on \( P_{13} \) are straightforward, upon applying HI, AGMI, and additionally applying Lemma 4.3 to \( P_{11} \) and \( P_{12} \):

\[ P_{11} = \int_0^T (g \nabla \xi, \nabla v_1) dt \]

\[ \leq \epsilon_{11} \int_0^T e^{-rt} \left| \theta ||^2 dt + \frac{1}{4\epsilon_{11}} \left| g \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \int_0^T e^{-rt} \left| \nabla v_1 \right|^2 dt \]

\[ \leq \frac{1}{42} \left| \theta ||^2_{L^2((0,T),C^2(\Omega))} + \frac{21}{2} \left| g \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \right] \int_0^T e^{-rt} \left| \nabla v_1 \right|^2 dt ; \]

\[ P_{12} = - \int_0^T (g \nabla \xi, \nabla v_1) dt \]

\[ \leq \epsilon_{12} \int_0^T e^{-rt} \left| \psi ||^2 dt + \frac{1}{4\epsilon_{12}} \left| g \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \int_0^T e^{-rt} \left| \nabla v_1 \right|^2 dt \]

\[ \equiv \frac{\tau_0}{21} \int_0^T e^{-rt} \left| \psi ||^2 dt + \frac{21}{4\epsilon_{12}} \left| g \nabla \xi \right|^2_{L^\infty((0,T),C^\infty(\Omega))} \right] \int_0^T e^{-rt} \left| \nabla v_1 \right|^2 dt ; \]

\[ P_{13} = E_h \int_0^T \left( \nabla \frac{\partial \theta}{\partial t}, \nabla v_1 \right) dt \]

\[ \leq E_h^2 \epsilon_{13} \left| \nabla \frac{\partial \theta}{\partial t} \right|^2_{L^2((0,T),C^2(\Omega))} + \frac{1}{4\epsilon_{13}} \int_0^T e^{-rt} \left| \nabla v_1 \right|^2 dt \]

\[ \equiv \frac{E_h^2}{4} \left| \nabla \frac{\partial \theta}{\partial t} \right|^2_{L^2((0,T),C^2(\Omega))} + \int_0^T e^{-rt} \left| \nabla v_1 \right|^2 dt. \]
Term $\mathcal{P}_{14}$ is treated similarly to $\mathcal{A}_7$ and term $\mathcal{P}_{15}$ is treated similarly to $\mathcal{A}_6$ if, for both, we treat $\theta$ like $\psi$ and let $\epsilon'_{15} = \epsilon'_{16} = \frac{1}{42}$.

Using $v_2(\cdot, t) = \int_t^T e^{-r t} \frac{\partial}{\partial t} \psi(\cdot, s) \, ds$ as the test function in (16), integrating in time over $(0, T)$, and using the relations above yields

\begin{equation}
(19) \quad \int_0^T e^{-rt} \left( \left< \frac{\partial \psi}{\partial t} \right>_{\mathcal{T}} \right|^2 \, dt + e^{-rT} \frac{\partial \psi}{\partial t}(\cdot, T) + \frac{r}{2} \int_0^T e^{-rt} \left< \frac{\partial \psi}{\partial t} \right>^2 \, dt \\
+ \frac{E_h}{2} \sum_{i=1}^2 \int_0^T e^{-rt} \left. \frac{\partial^2 \psi}{\partial t \partial x_i} \right|_{\mathcal{T}} \, dt \right|^2 \, dt \\
+ \frac{r}{2} \int_0^T e^{-rt} \left. \frac{\partial^2 \psi}{\partial t \partial x_i} \right|_{\mathcal{T}} \, dt \right|^2 \, dt = - \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
- \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
- \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
+ \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
+ \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
- \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
- \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
- \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
- \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
- \int_0^T \left( \left( \frac{\nabla \cdot \psi}{\nabla} \right)_{\mathcal{T}} \cdot \nabla v_2 \right) \, dt \, dt \\
= \left( \mathcal{A}_1 + \ldots + \mathcal{A}_8 \right) + \left( \mathcal{P}_1 + \ldots + \mathcal{P}_{15} \right). 
\end{equation}

The terms on the right-hand side of the inequality are handled as in (18). For affine error terms we have $\hat{\epsilon}_{1} = \hat{\epsilon}_{2} = \frac{11 E_h}{192}$, $\hat{\epsilon}_{3} = \hat{\epsilon}_{6} = \frac{E_h}{12}$, $\hat{\epsilon}_{4} = \hat{\epsilon}_{5} = \frac{1}{4}$, $\hat{\epsilon}_{7} = \hat{\epsilon}_{8} = \frac{1}{42}$. Note that term $\mathcal{A}_6$ differs from $\mathcal{A}_6$ by one term. The upper bound on $\mathcal{A}_6$ follows from Assumptions A2, A9:

$$
\mathcal{A}_6 \leq \hat{\epsilon}_{6} \int_0^T e^{-rt} ||\nabla \psi||^2 \, dt + \frac{1}{4 \hat{\epsilon}_{6}} ||\Pi g||^2_{L^2((0,T);L^\infty(\Omega)))} \int_0^T e^{rt} ||\nabla v_2||^2 \, dt \\
\leq \frac{E_h}{12} \int_0^T e^{-rt} ||\nabla \psi||^2 \, dt + \frac{3}{E_h} (\Pi g)^2 \int_0^T e^{rt} ||\nabla v_2||^2 \, dt.
$$

For projection error terms we have $\hat{\epsilon}_{1} = \hat{\epsilon}_{2} = \frac{1}{4}$, $\hat{\epsilon}_{2a} = \hat{\epsilon}_{4a} = \hat{\epsilon}_{5a} = \hat{\epsilon}_{7a} = \frac{1}{42}$, $\hat{\epsilon}_{2b} = \hat{\epsilon}_{5b} = \frac{7}{21}$, $\hat{\epsilon}_{6} = \hat{\epsilon}_{7} = \frac{1}{12}$, $\hat{\epsilon}_{8} = \frac{1}{21}$, $\hat{\epsilon}_{9} = \frac{1}{4}$, and $\hat{\epsilon}_{10} = \frac{1}{4}$.
Using $w = \chi$ as the test function in (17) followed by integration in time over $(0, T]$, yields

$$
\frac{1}{2} ||\chi(\cdot, T)||^2 + E_h ||\nabla \chi||^2_{L^2((0, T); L^2(\Omega))} + \tau_\ast ||\chi||^2_{L^2((0, T); L^2(\Omega))}
$$

$$
\leq - \int_0^T \left( \left( \frac{\rho}{\pi} \right) \nabla \chi, \chi \right) dt - \int_0^T \left( \nabla \chi, \frac{\rho}{\pi} \right) dt + \int_0^T \left( \nabla \psi \cdot \frac{\rho}{\pi} \right) dt
$$

$$
- \int_0^T \left( \psi \nabla \phi_a, \chi \right) dt + \int_0^T \left( \psi \nabla \eta, \chi \right) dt + \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \phi, \chi \right) dt + \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \eta, \chi \right) dt
$$

$$
+ \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \phi, \chi \right) dt + \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \eta, \chi \right) dt
$$

$$
- \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \phi, \chi \right) dt + \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \eta, \chi \right) dt
$$

$$
- \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \phi, \chi \right) dt + \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \eta, \chi \right) dt
$$

$$
- \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \phi, \chi \right) dt + \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \eta, \chi \right) dt
$$

$$
- \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \phi, \chi \right) dt + \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \eta, \chi \right) dt
$$

$$
- \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \phi, \chi \right) dt + \int_0^T \left( \left( \frac{\psi}{H} \right) \nabla \eta, \chi \right) dt
$$

$$
= (\vec{A}_1 + \vec{A}_2 + \vec{A}_3) + (\vec{A}_6 + \vec{A}_7 + \vec{A}_8) + (\vec{P}_1 + \ldots + \vec{P}_7) + (\vec{P}_9 + \ldots + \vec{P}_{15}).
$$

The terms on the right-hand side of the inequality are handled as in (18). Specifically, for affine error terms, we have $\vec{c}_1 = \vec{c}_2 = \frac{5}{4}E_h, \vec{c}_3 = \vec{c}_5 = \frac{E_h}{12},$ and $\vec{c}_7 = \vec{c}_8 = \frac{1}{2}$. For projection error terms, we have $\vec{c}_1 = \vec{c}_3 = \frac{1}{18}, \vec{c}_2 = \vec{c}_4 = \vec{c}_6 = \vec{c}_7 = \vec{c}_8 = \frac{1}{42}, \vec{c}_9 = \vec{c}_10 = \vec{c}_11 = \vec{c}_12 = \vec{c}_13 = \vec{c}_14 = \vec{c}_15 = \frac{1}{42}$. Again, observe that term $\vec{A}_6$ differs from $A_6$ by one component and is thus treated similarly to $\vec{A}_6$ in (19). Also, the treatment of term $\vec{P}_{13}$ differs slightly from the treatment of the corresponding terms in the previous equations:

$$
\vec{P}_{13} = E_h \int_0^T (\nabla \phi, \nabla \chi) dt
$$

$$
\leq E_h \frac{1}{4 \vec{c}_{13}} ||\nabla \phi||^2_{L^2((0, T); L^2(\Omega))} + \vec{c}_{13} E_h ||\nabla \chi||^2_{L^2((0, T); L^2(\Omega))}
$$

$$
\leq \frac{32}{5} E_h ||\nabla \phi||^2_{L^2((0, T); L^2(\Omega))} + \frac{5}{128} E_h ||\nabla \chi||^2_{L^2((0, T); L^2(\Omega))}.
$$

We now define some constants that succinctly emphasize the need for assumptions A2–A11 and B1, B2 in taking upper bounds of (18), (19), and (20). Let

$$
M_1 = \left( 21 + \frac{21}{2\gamma} \right)
$$

$$
M_2 = \frac{3}{E_h} ||\chi||^2_{L^\infty((0, T); L^\infty(\Omega))}
$$
\[ M_3 = 9 \left\| \frac{\partial}{\partial t} g \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 + 3 \left\| \frac{\partial}{\partial x} \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 + 4 \left\| H g \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 \]
\[ M_4 = 1 + \frac{3}{2} \left\| \tau_\sigma - \tau_\sigma f \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 + (f^*)^2 + \frac{96}{11 E_h} \left\| \tilde{\tau}_h \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 \]
\[ M_5 = \frac{3}{2} E_h^2 \left( \Pi g \right)^2 \]
\[ M_7 = \frac{3}{2} (M_1 Q^2_1 + 43 Q^2_2) \left( \left\| \nabla \tilde{\phi} \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 + \left\| \nabla \tilde{q} \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 \right) \]
\[ M_8 = \frac{1}{2} (M_1 Q^2_1 + 43 Q^2_2) \left( \left\| \nabla h \cdot \left[ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right] \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 + \left\| \nabla \tilde{\xi} \cdot \left[ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right] \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 \right) \]
\[ M_9 = \frac{1}{2} M_1 \left( \left\| \nabla \phi \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 + \left\| \nabla \theta \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 \right) \]
\[ M_{10} = \left( \left\| \frac{\partial}{\partial t} \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 + \left\| \frac{\partial}{\partial y} \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 \right) \left( 8 \left\| \phi \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 + 82 \left\| \tilde{\phi} \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 \right) \]
\[ M_{11} = \frac{94}{5 E_h} \left\| \tilde{\tau}_h \right\|_{L^\infty((0,T); L^\infty(\Omega))}^2 \]

Consequently, we observe that the right-hand-side of (18) can be bounded by

\[ \left( 21 \right) \quad \frac{\tau_\sigma}{3} \int_0^T e^{-\tau_\sigma t} \left\| \psi \right\|^2 dt + \frac{E_h}{6} \int_0^T e^{-\tau_\sigma t} \left\| \nabla \psi \right\|^2 dt \]
\[ + C_{v1} \sum_{i=1}^2 \int_0^T e^{-\tau_\sigma t} \left\| \frac{\partial}{\partial t} \phi \right\|^2 dt + \frac{11}{2} \left\| \chi \right\|_{L^2((0,T); L^2(\Omega))}^2 \]
\[ + \frac{11 E_h}{64} \left\| \nabla \chi \right\|_{L^2((0,T); L^2(\Omega))}^2 + \frac{1}{6} \left\| \theta \right\|_{L^2((0,T); L^2(\Omega))}^2 + \frac{1}{6} \left\| \nabla \theta \right\|_{L^2((0,T); L^2(\Omega))}^2 \]
\[ + \frac{E_h}{4} \left\| \nabla \theta \right\|^2_{L^2((0,T); L^2(\Omega))} + \frac{7}{41} \left\| \phi \right\|_{L^2((0,T); L^2(\Omega))}^2 + \frac{1}{6} \left\| \nabla \phi \right\|_{L^2((0,T); L^2(\Omega))}^2 \]

where \( C_{v1} = (M_2 + \cdots + M_5) + (M_7 + \cdots + M_{10}) \).

The right-hand-side of (19) can be bounded by

\[ \left( 22 \right) \quad \frac{\tau_\sigma}{3} \int_0^T e^{-\tau_\sigma t} \left\| \psi \right\|^2 dt + \frac{E_h}{6} \int_0^T e^{-\tau_\sigma t} \left\| \nabla \psi \right\|^2 dt \]
\[ + C_{v2} \sum_{i=1}^2 \int_0^T e^{-\tau_\sigma t} \left\| \frac{\partial}{\partial t} \phi \right\|^2 dt + \frac{11}{2} \left\| \chi \right\|_{L^2((0,T); L^2(\Omega))}^2 \]
\[ + \frac{11 E_h}{64} \left\| \nabla \chi \right\|_{L^2((0,T); L^2(\Omega))}^2 + \frac{1}{6} \left\| \theta \right\|_{L^2((0,T); L^2(\Omega))}^2 + \frac{1}{6} \left\| \nabla \theta \right\|_{L^2((0,T); L^2(\Omega))}^2 \]
\[ + \frac{E_h}{4} \left\| \nabla \theta \right\|^2_{L^2((0,T); L^2(\Omega))} + \frac{7}{41} \left\| \phi \right\|_{L^2((0,T); L^2(\Omega))}^2 + \frac{1}{6} \left\| \nabla \phi \right\|_{L^2((0,T); L^2(\Omega))}^2 \]

where \( C_{v2} = (M_2 + \cdots + M_4) + (M_6 + \cdots + M_{10}) \).

Finally, the right-hand-side of (20) can be bounded by

\[ \left( 23 \right) \quad \frac{\tau_\sigma}{3} \int_0^T e^{-\tau_\sigma t} \left\| \psi \right\|^2 dt + \frac{E_h}{6} \int_0^T e^{-\tau_\sigma t} \left\| \nabla \psi \right\|^2 dt + C_w \int_0^T e^{-\tau_\sigma t} \left\| \chi \right\|^2 dt \]
\[ + 5 \left\| \chi \right\|_{L^2((0,T); L^2(\Omega))}^2 + \frac{10 E_h}{64} \left\| \nabla \chi \right\|_{L^2((0,T); L^2(\Omega))}^2 \]
5. Conclusions. We have analyzed a full nonlinear coupled GWCE-CME system of equations. Making physically-realistic assumptions, we derived an a priori error estimate of the Galerkin finite element approximations to the solutions of GWCE-CME system of equations, in weak formulation, by using an $L^2$ projection. This led to a suboptimal estimate. That is, if we use continuous, piecewise polynomials of degree $2q - 1$ to approximate the elevation and velocity unknowns on a mesh with grid-spacing $h$, then the approximations tend to the solutions of the weak form like $h^{2q-1}$. To our knowledge, our error analysis of a system of shallow water equations is the first of its kind.

6. Future Work. We attempted to analyze the coupled GWCE-NCME system of equations, but technical difficulties impeded this task. Future work will attempt to analyze this particular system of equations.

Future work will also include analysis of boundary conditions such as land and sea boundary conditions. We will also analyze various temporal discretization schemes. And, we will attempt to extend the results in this paper to optimality.

7. Acknowledgments. The authors thank Martha Carey and Amr Elbakry for their valuable feedback.
\[
\begin{align*}
+ \| \chi(\cdot, T) \|^2 + \beta_2 \| \chi \|^2_{L^2((0,T); H^1(\Omega))} \\
\leq \frac{1}{2} \left[ \| \theta \|^2_{L^2((0,T); H^1(\Omega))} + E_h^2 \left\| \frac{\partial \theta}{\partial t} \right\|^2_{L^2((0,T); L^2(\Omega))} \\
+ \left( 1 + \frac{64}{5} E_h \right) \| \phi \|^2_{L^2((0,T); H^1(\Omega))} + \left( \tilde{C} w + 32 \right) \| \chi \|^2_{L^2((0,T); L^2(\Omega))} \right]
\end{align*}
\]

where \( \tilde{C} w = 2C w e^{rT} \).

Multiply (25) by \( 2e^{rT} \) and follow with an application of Gronwall’s Lemma to obtain
\[
(\tau_0 + 1) \left\| \psi(\cdot, T) \right\|^2 + 2 \left\| \frac{\partial \psi}{\partial t} \right\|^2_{L^2((0,T); L^2(\Omega))} + \beta_1 \| \psi \|^2_{L^2((0,T); H^1(\Omega))} \\
+ \| \chi(\cdot, T) \|^2 + \beta_2 \| \chi \|^2_{L^2((0,T); H^1(\Omega))} \\
\leq \tilde{C} \left\{ \| \theta \|^2_{L^2((0,T); H^1(\Omega))} + E_h^2 \left\| \frac{\partial \theta}{\partial t} \right\|^2_{L^2((0,T); L^2(\Omega))} + \left( 1 + \frac{64}{5} E_h \right) \| \phi \|^2_{L^2((0,T); H^1(\Omega))} \right\}
\]

where \( \tilde{C} = e^{(r+\alpha_1)T}, \alpha_1 = e^{rT} (\tilde{C} w + 32) \).

Letting
\[
\kappa_1 = \min \left\{ (\tau_0 + 1), 2, \beta_1, 1, \beta_2 \right\}, \kappa_2 = \max \left\{ 1, E_h^2, \left( 1 + \frac{64}{5} E_h \right) \right\}
\]
yields
\[
(27) \quad \left\| \psi(\cdot, T) \right\|^2 + \left\| \frac{\partial \psi}{\partial t} \right\|^2_{L^2((0,T); L^2(\Omega))} + \| \psi \|^2_{L^2((0,T); H^1(\Omega))} \\
+ \| \chi(\cdot, T) \|^2 + \| \chi \|^2_{L^2((0,T); H^1(\Omega))} \\
\leq \tilde{C} \left\{ \| \theta \|^2_{L^2((0,T); H^1(\Omega))} + \left\| \frac{\partial \theta}{\partial t} \right\|^2_{L^2((0,T); L^2(\Omega))} + \| \phi \|^2_{L^2((0,T); H^1(\Omega))} \right\}
\]

with \( \tilde{C} = \frac{\kappa_2}{\kappa_1} \tilde{C} \). Observe that \( \tilde{C} \) depends on \( r, k \) and \( T \).

Use the approximation result stated in Lemma 4.1,
\[
\| \theta \|^2_{L^2((0,T); H^1(\Omega))}, \left\| \frac{\partial \theta}{\partial t} \right\|^2_{L^2((0,T); L^2(\Omega))}, \| \phi \|^2_{L^2((0,T); H^1(\Omega))} \leq C h^{k-1},
\]
to obtain
\[
(28) \quad \left\| \psi(\cdot, T) \right\| + \left\| \frac{\partial \psi}{\partial t} \right\|_{L^2((0,T); L^2(\Omega))} + \| \psi \|^2_{L^2((0,T); H^1(\Omega))} \\
+ \| \chi(\cdot, T) \| + \| \chi \|^2_{L^2((0,T); H^1(\Omega))} \leq C h^{k-1}
\]

where \( \tilde{C} \approx C \sqrt{\tilde{C}} \).

The result of the theorem now follows by an application of the triangle inequality to the projection error and to the affine error (28). \( \square \)

Remark: It is clear that stability can be proven using arguments similar to those used in the proof of the error estimate.
5. Conclusions. We have analyzed a full nonlinear coupled GWCE-CME system of equations. Making physically-realistic assumptions, we derived an a priori error estimate of the Galerkin finite element approximations to the solutions of GWCE-CME system of equations, in weak formulation, by using an $L^2$ projection. This led to a suboptimal estimate. That is, if we use continuous, piecewise polynomials of degree $2q - 1$ to approximate the elevation and velocity unknowns on a mesh with grid-spacing $h$, then the approximations tend to the solutions of the weak form like $h^{2q-1}$. To our knowledge, our error analysis of a system of shallow water equations is the first of its kind.

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A. Review of Tensor Notation. Let $\varphi, \psi \in \mathbb{R}^2$.
The dyadic product is defined as $(\varphi \psi)_{ij} = \varphi_i \psi_j$. Thus,

$$\varphi^2 = \begin{pmatrix}
\varphi_1^2 & \varphi_1 \varphi_2 \\
\varphi_2 \varphi_1 & \varphi_2^2
\end{pmatrix}.$$

The dot product of a tensor with a vector is the usual matrix-vector multiplication result $[M \cdot \varpi]_i = \sum_j M_{ij} \varpi_j$.
The scalar product (or double-dot product) of two tensors is defined as $S:T = \sum_{i,j} S_{ij} T_{ij}$.

The gradient of a vector is defined as $\{\nabla \varphi\}_{ij} = \frac{\partial \varphi_j}{\partial x_i}$. For example,

$$\nabla \varphi = \begin{pmatrix}
\frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_1} \\
\frac{\partial \varphi_1}{\partial x_2} & \frac{\partial \varphi_2}{\partial x_2}
\end{pmatrix}.$$

The divergence of a tensor is defined as $[\nabla \cdot S]_i = \sum_j \frac{\partial}{\partial x_j} S_{ij}$. Thus,

$$\nabla \cdot \nabla \varphi = \begin{pmatrix}
\frac{\partial^2 \varphi_1}{\partial x_1^2} + \frac{\partial^2 \varphi_1}{\partial x_2^2} \\
\frac{\partial^2 \varphi_2}{\partial x_1^2} + \frac{\partial^2 \varphi_2}{\partial x_2^2}
\end{pmatrix} = \begin{pmatrix}
\Delta \varphi_1 \\
\Delta \varphi_2
\end{pmatrix}.$$

Observe that $$(\nabla \varphi ; \nabla \varphi) = \left( \frac{\partial}{\partial x_1} \varphi_1 \right)^2 + \left( \frac{\partial}{\partial x_2} \varphi_1 \right)^2 + \left( \frac{\partial}{\partial x_1} \varphi_2 \right)^2 + \left( \frac{\partial}{\partial x_2} \varphi_2 \right)^2$$
and that

$$\nabla \cdot (\nabla \varphi) = \frac{\partial}{\partial x_1} \varphi_1^2 + \frac{\partial}{\partial x_2} \varphi_1 \varphi_2 + \frac{\partial}{\partial x_1} \varphi_1 \varphi_2 + \frac{\partial}{\partial x_2} \varphi_2^2.$$

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