On the Constants in Inverse Inequalities in $L_2$

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Abstract

In this paper we determine the constants in multivariate Markov inequalities in the $L_2$-norm on an interval, a triangle and a tetrahedron. Using orthonormal polynomials, we derive explicit expression for the constants on that given 1-simplex, 2-simplex and 3-simplex. Accurate values for the constants are crucial for the correct derivation of a priori and a posteriori error estimations in adaptive computation.

Keywords: multivariate Markov inequality, inverse inequality, finite element method

1 Introduction

Inverse inequalities (or Markov inequalities) play an important role in many areas of mathematical research. For instance they are commonly used in the error analysis of variational methods such as finite element methods and discontinuous Galerkin methods for solving partial differential equations. Explicit constants for some inverse inequalities can be found in [3].

The classical Markov inequality for univariate polynomials states that for any polynomial $u$ total degree $N$, the following inequality holds

$$\|u'\|_{L_\infty([a,b])} \leq \frac{2N^2}{|b-a|}\|u\|_{L_\infty([a,b])}.$$ 

A discussion of the exact constant in the univariate Markov inequality is given in [1]. It is proved that for a polynomial degree at most $N$ with real coefficients that have at most $m$ distinct complex zeros, the following inequality holds

$$\|u'\|_{L_\infty([-1,1])} \leq 32 \cdot 8^m N \|u\|_{L_\infty([-1,1])}.$$ 

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Another discussion of the exact constant in the univariate Markov inequality in $L_2$-norm is given in [6]. In that paper it is shown that for a polynomial $u$ total degree $N$, the following inequality holds

$$
\|u'\|_{L_2([-1,1])} \leq M_N \|u\|_{L_2([-1,1])},
$$

where $M_N$ coincides with the largest positive root of the following equation

$$
\sum_{k=0}^{N+1} (-1)^k x^{-2k} \frac{(N + 1 + 2k)!}{2^{2k} 2k! (N + 1 - 2k)!} = 0. \tag{1}
$$

In that paper a new simple elementary method is presented for finding $M_N$. The special case of the $L_2$-norm has been previously studied in [5] using spectral analysis methods. The authors show that $M_N$ is the solution of a certain equation which is equivalent to (1).

Schwab [9] obtains that for a polynomial $u$ total degree $N$ on a finite interval, the following inequality holds

$$
\|u'\|_{L_2([a,b])} \leq 2\sqrt{3} \frac{N^2}{|b-a|} \|u\|_{L_2([a,b])}.
$$

In our paper, we recover the same constant for a linear polynomial $u$ and we obtain a smaller constant for high order polynomial.

In the last thirty years possible extensions of the above estimates for multivariate polynomials have been widely investigated. In [2] the following result is proved: for a polynomial $u$ of total degree $N$ and a bounded convex set $K$

$$
\left\| \frac{\partial u}{\partial \xi} \right\|_{L_p(K)} \leq C N^2 \|u\|_{L_p(K)},
$$

for $0 < N \leq \infty$, $\frac{\partial}{\partial \xi}$ an arbitrary unit directional derivative, and $C$ a constant independent of $N$ and of $u$.

In [4], it is shown that certain directional derivatives of polynomials in two variables have unit bound at the Chebyshev nodes. Wilhelmsen [11] gives a Markov-type estimate for an arbitrary convex body $K \subset \mathbb{R}^m$. For a convex body $K \subset \mathbb{R}^m$, denote by $w(K)$ the minimal distance between two parallel supporting hyperplanes for $K$. Then it is shown in [11] that for a polynomial of total degree $N$, the following inequality holds

$$
\|\nabla u\|_{L_\infty(K)} \leq \frac{4N^2}{w(K)} \|u\|_{L_\infty(K)}.
$$

In [8] the above inequality is verified in the special case when $K$ is a triangle in $\mathbb{R}^2$. A similar result was shown in [7] improving the constant as $4N^2 - 2N$ instead of $4N^2$. 

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In addition, in [9] it is proved that for a polynomial $u$ total degree $N$, Markov inequality holds for a triangle $K$ or a quadrilateral $Q$ with an unknown constant $C$ in $L_2$ and $L_\infty$-norm,

$$
\|\nabla v\|_{L_\infty(K)} \leq C N^2 \|v\|_{L_\infty(K)} ,
$$

$$
\|\nabla v\|_{L_2(K)} \leq C N^2 \|v\|_{L_2(K)} ,
$$

In summary, we can say that using the above inequalities, one may get an exact constant for univariate Markov inequality in $L_\infty$ and $L_2$-norm and also one can estimate the exact constant for multivariate Markov inequalities in $L_\infty$-norm.

In this current paper we shall present the exact constants in univariate and multivariate Markov inequality in $L_2$-norm. Using orthogonal polynomials, we reduce the problem to a simple eigenvalue problem and we establish bounds with known constants for Markov inequalities on an arbitrary 1-simplex, 2-simplex and 3-simplex. In [10], a similar technique is used to obtain the constants in the $hp$ trace inequalities.

The remainder of this paper is organized as follows. In section 2, the Markov inequality is proved for one dimensional domain and an explicit constant is found in $L_2$ norm. Section 3 gives us a Markov inequality constant for polynomial in two variables on triangular domain, while in Section 4, we give the result for a tetrahedron. Finally, in Section 5 we provide a brief conclusion.

## 2 One Dimensional Domain

In this section, we state and prove Markov inequality for a polynomial $u$ total degree $N$ on a finite interval and we give a closed form for the constants up to polynomial degree 4. For the case $5 \leq N \leq 10$, we give numerical values for this constant.

**Theorem 1** (Markov inequality on a finite interval) For an interval $K = [a, b]$ and for a polynomial $u$ of total degree $N = 1, 2, 3, 4$ the following result holds

$$
\|u'\|_{L_2(K)} \leq \frac{2 \sqrt{C_N}}{b - a} \|u\|_{L_2(K)},
$$

where $C_1 = 3$, $C_2 = 15$, $C_3 = \frac{45 + \sqrt{1005}}{2}$ and $C_4 = \frac{105 + 3 \sqrt{305}}{2}$.

**Proof.** Consider the reference interval $\hat{K} = [-1, 1]$ and associate $L_2$- orthonormal polynomial, namely the classical Legendre polynomial $\{\phi_n\}_{n=1}^N$. The reference interval $\hat{K} = [-1, 1]$ is mapped to the interval $K = [a, b]$ by the following transformation, which sends $-1$ to $a$ and 1 to $b$:

$$
x = \frac{(b - a)r}{2} + \frac{b + a}{2},
$$

where $r \in [-1, 1]$ and $x \in [a, b]$.

By chain rule and using a scaling argument, we obtain
\[\|u\|_{L^2(K)} = \left\| \frac{du}{dr} \right\|_{L^2(K)} = \frac{dr}{dx} \left\| \frac{du}{dr} \right\|_{L^2(K)} = \frac{2}{b-a} \left\| \frac{b-a}{2} \right\|_{L^2(K)}^{1/2} \left\| \frac{du}{dr} \right\|_{L^2(K)} \leq \frac{2}{b-a} \sqrt{C_N} \|u\|_{L^2(K)} \leq 2 |b-a| \sqrt{C_N} \|u\|_{L^2(K)},\]

where \(C_N\) can be determined for a given polynomial order \(N\) by solving the following eigenvalue problem for the maximum eigenvalue,

\[\left( \frac{d\phi_n}{dr}, \frac{d\phi_m}{dr} \right)_{\hat{K}} u_m = \lambda(\phi_n, \phi_m)_{\hat{K}} u_m.\]

Einstein summation is assumed for repeated indices. The \(L^2\) inner product on \(\hat{K}\) is denoted by \((\cdot, \cdot)_{\hat{K}}\). Defining \(S_{nm} = \left( \frac{d\phi_n}{dr}, \frac{d\phi_m}{dr} \right)_{\hat{K}}, M_{nm} = (\phi_n, \phi_m)_{\hat{K}}\) and using the orthonormality of the basis gives us \(M = I\) where \(I\) is the identity matrix. Then, the above problem reduces to a classical eigenvalue problem

\[S_{nm}u_m = \lambda u_m.\]

Let \(C_N\) be the maximum eigenvalue \(\lambda\), then we can write:

\[\left\| \frac{du}{dr} \right\|^2_{L^2(K)} \leq C_N \|u\|^2_{L^2(K)}.\]

For \(N = 1\), the orthonormal basis functions are simply

\[\phi_0(r) = \frac{\sqrt{2}}{2},\]

\[\phi_1(r) = \frac{\sqrt{6}}{2} r,\]

and the matrix \(S\) is

\[S = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix},\]

thus \(C_1 = 3\).

We now consider the case \(N = 2\). We note that the basis functions are hierarchical, so we will only give the explicit expressions of the additional ones. Then,
we have

\[ \phi_0, \quad \phi_1, \quad \phi_2(r) = \frac{\sqrt{10}}{4}(3r^2 - 1), \]

and the matrix \( S \) is

\[
S = \begin{bmatrix}
0 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 15
\end{bmatrix}.
\]

So, in this case \( C_2 = 15 \). We note that the 2 \( \times \) 2 submatrix of \( S \) is the matrix that we get in the case of \( N = 1 \).

For \( N = 3 \), the orthonormal basis functions on the reference interval are given by

\[ \phi_0, \quad \phi_1, \quad \phi_2, \quad \phi_3(r) = \frac{\sqrt{14}}{4}(5r^3 - 3r), \]

and the matrix \( S \) becomes

\[
S = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 3 & 0 & \sqrt{21} \\
0 & 0 & 15 & 0 \\
0 & \sqrt{21} & 0 & 42
\end{bmatrix}.
\]

Thus, in this case \( C_3 = \frac{45 + \sqrt{1605}}{2} \).

Similarly for \( N = 4 \), the orthonormal basis functions are

\[ \phi_0, \quad \phi_1, \quad \phi_2, \quad \phi_3, \quad \phi_4(r) = \frac{3\sqrt{7}}{16}(35r^4 - 30r^2 + 3), \]

and the matrix \( S \) is

\[
S = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & \sqrt{21} & 0 \\
0 & 0 & 15 & 0 & 9\sqrt{5} \\
0 & \sqrt{21} & 0 & 42 & 0 \\
0 & 0 & 9\sqrt{5} & 0 & 0
\end{bmatrix}.
\]

Thus in this case \( C_4 = \frac{105 + 3\sqrt{805}}{2} \).

For higher order polynomials, a closed form bound on the eigenvalues is not obvious. However, it may be possible to use Gerschgorin’s theorem to localize the eigenvalues. Numerical values for \( C_N \) were computed as shown in Table 1.

In [9] it is in fact shown that the univariate Markov inequality is:

\[
u'\|L_2([a,b]) \leq \frac{2}{|b-a|} \sqrt{N^2(N + 1)(N + 1/2)}\|u\|L_2([a,b]) \quad \text{for} \quad N \geq 1.
\]
Our constant $C_N$ is consistent with this result. Indeed, we compute the ratio

$$\beta = \frac{C_{N_1}}{C_{N_2}} \frac{N_1^2(N_1 + 1)(N_1 + 1/2)}{N_2^2(N_2 + 1)(N_2 + 1/2)}$$

for two successive polynomial degrees $N_1$ and $N_2$. Table 1 shows that the ratio $\beta$ tends to 1. In addition, the fact that the ratio $C_N/(N^2(N + 1)(N + 1/2))$ decreases as $N$ is larger, shows that our constant is smaller than the one in (2).

Table 1: Experimentally determined constants in the discrete Markov inequality on an interval.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$C_N$</th>
<th>$C_N N^2(N+1)(N+1/2)$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.0000</td>
<td>1.0000</td>
<td>2.0000</td>
</tr>
<tr>
<td>2</td>
<td>15.0000</td>
<td>0.5000</td>
<td>1.4813</td>
</tr>
<tr>
<td>3</td>
<td>42.5312</td>
<td>0.3375</td>
<td>1.2783</td>
</tr>
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<td>4</td>
<td>95.0588</td>
<td>0.2641</td>
<td>1.1793</td>
</tr>
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<td>5</td>
<td>184.7262</td>
<td>0.2239</td>
<td>1.1245</td>
</tr>
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<td>6</td>
<td>326.1508</td>
<td>0.1991</td>
<td>1.0914</td>
</tr>
<tr>
<td>7</td>
<td>536.3742</td>
<td>0.1824</td>
<td>1.0699</td>
</tr>
<tr>
<td>8</td>
<td>834.8615</td>
<td>0.1705</td>
<td>1.0552</td>
</tr>
<tr>
<td>9</td>
<td>1243.5042</td>
<td>0.1616</td>
<td>1.0447</td>
</tr>
<tr>
<td>10</td>
<td>1786.6229</td>
<td>0.1547</td>
<td></td>
</tr>
</tbody>
</table>

3 Two Dimensional Domain

We discuss the Markov inequality for a 2-simplex and we give a closed form for the constant for polynomial degree 1 and 2. For $3 \leq N \leq 10$, numerical values are given.

Theorem 2 (Markov inequality for a planar triangle) For a planar triangle $K$, let $|\partial K|$ be the perimeter length of $K$ and $|K|$ be the area of triangle $K$. Then, for a polynomial $u$ of total degree $N = 1, 2, 3, 4$ the following result holds

$$\|\nabla v\|_{L_2(K)} \leq \sqrt{\frac{C_N}{|K|}} \|v\|_{L_2(K)},$$

where $C_1 = 6$, $C_2 = \frac{45}{2}$, $C_3 = 56.8879$, $C_4 = 119.8047$.

Proof. Let $\hat{K}$ be the right angle reference triangle with

$$\hat{K} = \{(r, s) | -1 \leq r, s \leq 1; r + s \leq 0\}.$$

The reference triangle $\hat{K}$ is mapped to the triangle $K$ by the following transformation: $(r, s) \in \hat{K} \mapsto (x, y) \in K$, 

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Figure 1: Mapping from the reference triangle $\hat{K}$ to the physical triangle $K$.

\[
\begin{align*}
x &= r \left( \frac{x_2 - x_1}{2} + s \frac{x_3 - x_1}{2} + \frac{x_2 + x_3}{2} \right), \\
y &= r \left( \frac{y_2 - y_1}{2} + s \frac{y_3 - y_1}{2} + \frac{y_2 + y_3}{2} \right). 
\end{align*}
\]

(3)

Clearly, this mapping sends $(-1, -1)$ to $(x_1, y_1)$, $(1, -1)$ to $(x_2, y_2)$ and $(-1, 1)$ to $(x_3, y_3)$. In vector form, the transformation is

\[
\begin{bmatrix}
x \\ y
\end{bmatrix} = \begin{bmatrix}
\frac{x_2 - x_1}{2} & \frac{x_3 - x_1}{2} \\
\frac{y_2 - y_1}{2} & \frac{y_3 - y_1}{2}
\end{bmatrix}
\begin{bmatrix}
r \\ s
\end{bmatrix} + \begin{bmatrix}
\frac{x_2 + x_3}{2} \\
\frac{y_2 + y_3}{2}
\end{bmatrix}.
\]

Let $J$ denote the $2 \times 2$ matrix defined above.
The standard formula for a change of variables in a multiple integral gives

\[
\int_K f(x, y) \, dx \, dy = |\det(J)| \int_{\hat{K}} g(r, s) \, dr \, ds,
\]

where $g$ is defined by

\[
g(r, s) = f \left( r \left( \frac{x_2 - x_1}{2} + s \frac{x_3 - x_1}{2} + \frac{x_2 + x_3}{2} \right), r \left( \frac{y_2 - y_1}{2} + s \frac{y_3 - y_1}{2} + \frac{y_2 + y_3}{2} \right) \right).
\]

The Jacobian factor is constant:

\[
|\det(J)| = \frac{1}{4} |(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)|.
\]

Moreover

\[
\frac{|(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)|}{2} = |K|,
\]

which yields

\[
|\det(J)| = \frac{|K|}{2}.
\]

(4)
Chain rule, triangle inequality and a standard scaling argument give,

\[ \|\nabla u\|_{L^2(K)} = \|\nabla r \frac{\partial u}{\partial r} + \nabla s \frac{\partial u}{\partial s}\|_{L^2(K)} \]

\[ \leq |\nabla r| \left( |\det(J)|^{1/2} \left\| \frac{\partial u}{\partial r}\right\|_{L^2(\hat{K})} \right) + |\nabla s| \left( |\det(J)|^{1/2} \left\| \frac{\partial u}{\partial s}\right\|_{L^2(\hat{K})} \right) \]

\[ \leq (|\nabla r| + |\nabla s|) \sqrt{C_N |\det(J)|^{1/2} \|u\|_{L^2(\hat{K})}} \]

\[ = (|\nabla r| + |\nabla s|) \sqrt{C_N \|u\|_{L^2(\hat{K})}}, \]

where \(|\nabla r| = \sqrt{(\nabla r)^T (\nabla r)}\) and same for \(|\nabla s|\).

Similar to the one dimensional case, utilizing Einstein summation for repeated indices and using the orthonormality of the basis give us following eigenvalue problem

\[ S_{nm} u_n = \lambda u_m, \]

where \(S_{nm} = \left( \frac{\partial \phi_n}{\partial r}, \frac{\partial \phi_m}{\partial r} \right)_{\hat{K}}\), and \(\{\phi_n\}_{n=1}^{(N+1)(N+2)/2}\) is an orthonormal basis defined on the reference triangle \(\hat{K}\).

Defining \(C_N\) as the maximum eigenvalue \(\lambda\) allows us to state:

\[ \left\| \frac{\partial u}{\partial r}\right\|_{L^2(\hat{K})}^2 \leq C_N \|u\|_{L^2(\hat{K})}^2, \]

and by symmetry the same constant applies for the norm of the \(s\)-derivative of \(u\).

For \(N = 1\), the orthonormal basis functions are:

\[ \phi_{0,0}(r, s) = \frac{\sqrt{2}}{2}, \]
\[ \phi_{0,1}(r, s) = \frac{1 + 3s}{2}, \]
\[ \phi_{1,0}(r, s) = \frac{\sqrt{3}(1 + 2r + s)}{2}, \]

and the matrix \(S\) is

\[ S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \]

thus \(C_1 = 6\).

We now consider the case \(N = 2\). As in the one dimensional problem, since the basis functions are hierarchical, the first three basis functions are the same as
above. The orthonormal basis functions are:

\[
\begin{align*}
\phi_{0,0}, & \quad \phi_{0,1}, \quad \phi_{1,0}, \\
\phi_{0,2}(r, s) &= \frac{\sqrt{6}(2s + 5s^2 - 1)}{4}, \\
\phi_{1,1}(r, s) &= \frac{3\sqrt{2}(3 + 5s)(1 + 2r + s)}{8}, \\
\phi_{2,0}(r, s) &= \frac{\sqrt{30}(1 + 6r + 4s + 6r^2 + 6rs + 6s^2)}{8},
\end{align*}
\]

and the stiffness matrix is given by

\[
S = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 2\sqrt{6} & 3\sqrt{2} & 0 & 0 & \sqrt{42} \\
0 & 0 & 2\sqrt{6} & \frac{3}{2} & 10\sqrt{3} & 0 & 0 & 0 \\
0 & 0 & 3\sqrt{2} & 10\sqrt{3} & 36 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{45}{2} & 6\sqrt{3} & 0 \\
0 & 0 & 0 & \sqrt{42} & 0 & 0 & 0 & 56 \\
0 & 0 & 0 & 0 & 0 & 0 & 6\sqrt{3} & 44 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 56
\end{bmatrix}.
\]

Thus, in this case \( C_2 = \frac{45}{2} \).

For \( N = 3 \), the matrix \( S \) becomes

\[
S = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 2\sqrt{6} & 3\sqrt{2} & 0 & 0 & \sqrt{42} \\
0 & 0 & 2\sqrt{6} & \frac{3}{2} & 10\sqrt{3} & 0 & 0 & 0 \\
0 & 0 & 3\sqrt{2} & 10\sqrt{3} & 36 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{45}{2} & 6\sqrt{3} & 0 \\
0 & 0 & 0 & \sqrt{42} & 0 & 0 & 0 & 56 \\
0 & 0 & 0 & 0 & 0 & 0 & 6\sqrt{3} & 44 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 56
\end{bmatrix}.
\]

and in this case an upper bound for the maximum eigenvalue is \( C_3 = 56.8879 \).

By a similar argument, for \( N = 4 \) we obtain the numerical value for \( C_4 = 119.8047 \). Table 2 shows the constants \( C_N \) for \( 1 \leq N \leq 10 \). We also see that the eigenvalues scale asymptotically as \( N^4 \).

In Table 2, numerical values for \( C_N \) are computed.
Table 2: Experimentally determined constants in the discrete Markov inequality on a triangle.

<table>
<thead>
<tr>
<th>N</th>
<th>(C_N)</th>
<th>(\frac{C_N}{N})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.0000</td>
<td>6.0000</td>
</tr>
<tr>
<td>2</td>
<td>22.5000</td>
<td>1.4063</td>
</tr>
<tr>
<td>3</td>
<td>56.8879</td>
<td>0.7023</td>
</tr>
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<td>4</td>
<td>119.8047</td>
<td>0.4680</td>
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<tr>
<td>5</td>
<td>224.1195</td>
<td>0.3586</td>
</tr>
<tr>
<td>6</td>
<td>385.2210</td>
<td>0.2972</td>
</tr>
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<td>7</td>
<td>620.8674</td>
<td>0.2586</td>
</tr>
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<td>8</td>
<td>951.2557</td>
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</tr>
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<td>9</td>
<td>1399.0115</td>
<td>0.2132</td>
</tr>
<tr>
<td>10</td>
<td>1989.1818</td>
<td>0.1989</td>
</tr>
</tbody>
</table>

Now let us compute \(|\nabla r| + |\nabla s|\):
\[
|\nabla r| + |\nabla s| = (r_x^2 + r_y^2)^{1/2} + (s_x^2 + s_y^2)^{1/2}.
\]
By (3) we have
\[
J = \frac{\partial (x, y)}{\partial (r, s)} = \begin{bmatrix} x_r & x_s \\ y_r & y_s \end{bmatrix} = \begin{bmatrix} \frac{(x_2-x_1)}{2} & \frac{(x_3-x_1)}{2} \\ \frac{(y_2-y_1)}{2} & \frac{(y_3-y_1)}{2} \end{bmatrix}.
\]
Then
\[
J^{-1} = \frac{\partial (r, s)}{\partial (x, y)} = \begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix} = \frac{1}{\det(J)} \begin{bmatrix} \frac{(y_3-y_1)}{2} & \frac{(x_3-x_1)}{2} \\ -\frac{(y_2-y_1)}{2} & \frac{(x_2-x_1)}{2} \end{bmatrix}.
\] (5)
Therefore (4) and (5) allow us to say,
\[
(r_x^2 + r_y^2)^{1/2} + (s_x^2 + s_y^2)^{1/2} = \frac{[(y_3-y_1)^2 + (x_3-x_1)^2]^{1/2} + [(y_2-y_1)^2 + (x_2-x_1)^2]^{1/2}}{2|\det(J)|}
= \frac{\text{dist}((x_1,y_1),(x_3,y_3)) + \text{dist}((x_1,y_1),(x_2,y_2))}{|K|}
\leq \frac{|\partial K|}{|K|},
\]
where \(\text{dist}(\cdot,\cdot)\) denote the distance between two points.
We conclude that
\[|\nabla r| + |\nabla s| \leq \frac{|\partial K|}{|K|}.
\]
We note in particular that as
\[
\frac{|\partial K|}{|K|} \approx h_K^{-1},
\]
where \(h_K\) is the longest edge of the element \(K\). □
4 Three Dimensional Domain

In this section, we consider a tetrahedron and we find the closed form for the constant up to polynomial degree 3. In the case of polynomial degree between 4 and 10, we give numerical values of the constant.

**Theorem 3** (Markov inequality for a tetrahedron) For a tetrahedral element $K$, let $|\partial K|$ denote the surface area of the $K$ and $|K|$ denote the volume of the $K$. Then for a polynomial $u$ of total degree $N = 1, 2, 3, 4$ the following result holds

$$\|\nabla v\|_{L^2(K)} \leq 2\sqrt{C_N|\partial K|} \|v\|_{L^2(K)},$$

where $C_1 = 10$, $C_2 = \frac{63}{2}$, $C_3 = 42 + 12\sqrt{7}$, $C_4 = 148.4089$.

**Proof.** Let $\hat{K}$ be the standard tetrahedron with

$$\hat{K} = \{(r, s, t) \mid -1 \leq r, s, t \leq 1; r + s + t \leq -1\}.$$

The standard tetrahedron $\hat{K}$ is mapped to the physical tetrahedron $K$ by an affine mapping, which sends $(-1, -1, -1)$ to $(x_1, y_1, z_1)$, $(1, -1, -1)$ to $(x_2, y_2, z_2)$, $(-1, 1, -1)$ to $(x_3, y_3, z_3)$, and $(-1, -1, 1)$ to $(x_4, y_4, z_4)$: $(r, s, t) \in \hat{K} \mapsto (x, y, z) \in K$.

\begin{align*}
x &= \frac{r}{2}(x_2 - x_1) + \frac{s}{2}(x_3 - x_1) + \frac{t}{2}(x_4 - x_1) + \frac{1}{2}(x_2 + x_3 + x_4 - x_1) \\
y &= \frac{r}{2}(y_2 - y_1) + \frac{s}{2}(y_3 - y_1) + \frac{t}{2}(y_4 - y_1) + \frac{1}{2}(y_2 + y_3 + y_4 - y_1) \\
z &= \frac{r}{2}(z_2 - z_1) + \frac{s}{2}(z_3 - z_1) + \frac{t}{2}(z_4 - z_1) + \frac{1}{2}(z_2 + z_3 + z_4 - z_1).
\end{align*}

**Figure 2:** Mapping from the reference tetrahedron $\hat{K}$ to the physical tetrahedron $K$ where $F1, F2$ and $F3$ denote faces of the physical tetrahedron $K$. 

\[ x = \frac{r}{2}(x_2 - x_1) + \frac{s}{2}(x_3 - x_1) + \frac{t}{2}(x_4 - x_1) + \frac{1}{2}(x_2 + x_3 + x_4 - x_1) \\
y = \frac{r}{2}(y_2 - y_1) + \frac{s}{2}(y_3 - y_1) + \frac{t}{2}(y_4 - y_1) + \frac{1}{2}(y_2 + y_3 + y_4 - y_1) \\
z = \frac{r}{2}(z_2 - z_1) + \frac{s}{2}(z_3 - z_1) + \frac{t}{2}(z_4 - z_1) + \frac{1}{2}(z_2 + z_3 + z_4 - z_1). \]
Moreover, the Jacobian factor is constant:

\[ \frac{1}{\det(J)} = \frac{1}{8} \left| (x_2 - x_1)(y_3 - y_1)(z_4 - z_1) + (y_2 - y_1)(z_3 - z_1)(x_4 - x_1) + (z_2 - z_1)(x_3 - x_1)(y_4 - y_1) \right. \]
\[ \left. - (x_4 - x_1)(y_3 - y_1)(z_2 - z_1) - (y_4 - y_1)(z_3 - z_1)(x_2 - x_1) - (z_4 - z_1)(x_3 - x_1)(y_2 - y_1) \right|. \]

Let \( J \) denote the \( 3 \times 3 \) matrix defined above.

The Jacobian factor is constant:

\[ \det(J) = \frac{3}{4} |K|. \]  

(7)

Chain rule, triangle inequality and a standard scaling argument give,

\[ \| \nabla u \|_{L^2(K)} = \left\| \nabla r \frac{\partial u}{\partial r} + \nabla s \frac{\partial u}{\partial s} + \nabla t \frac{\partial u}{\partial t} \right\|_{L^2(K)} \]
\[ \leq \| \nabla r \left| \frac{\partial u}{\partial r} \right|_{L^2(K)} + |\nabla s| \left| \frac{\partial u}{\partial s} \right|_{L^2(K)} + |\nabla t| \left| \frac{\partial u}{\partial t} \right|_{L^2(K)} \]
\[ = \| \nabla r \left( |\det(J)|^{1/2} \left| \frac{\partial u}{\partial r} \right|_{L^2(K)} \right) + \| \nabla s \left( |\det(J)|^{1/2} \left| \frac{\partial u}{\partial s} \right|_{L^2(K)} \right) \]
\[ + \| \nabla t \left( |\det(J)|^{1/2} \left| \frac{\partial u}{\partial t} \right|_{L^2(K)} \right) \]
\[ \leq (|\nabla r| + |\nabla s| + |\nabla t|) \sqrt{C_N} |\det(J)|^{1/2} \left\| u \right\|_{L^2(K)} \]
\[ = (|\nabla r| + |\nabla s| + |\nabla t|) \sqrt{C_N} \left\| u \right\|_{L^2(K)}. \]

The constant \( C_N \) can be determined by solving the following classical eigenvalue problem for the maximum eigenvalue.

\[ \left( \frac{\partial \phi_n}{\partial r} - \frac{\partial \phi_m}{\partial r} \right) K \begin{pmatrix} u_m \\ \phi_n \end{pmatrix} = \lambda \begin{pmatrix} u_m \\ \phi_n \end{pmatrix}. \]
where \( \{ \phi_n \}_{n=1}^{(N+1)(N+2)(N+3)/6} \) is an orthonormal basis defined on the reference tetrahedron \( \hat{K} \).

Defining \( C_N \) as the maximum eigenvalue \( \lambda \) allows us to state:

\[
\left\| \frac{\partial u}{\partial r} \right\|_{L^2(\hat{K})}^2 \leq C_N \left\| u \right\|_{L^2(\hat{K})}^2 ,
\]

and by symmetry the same constant applies for the norm of the partial derivative of \( u \) with respect to \( s \) and \( t \).

For \( N = 1 \), the orthonormal basis functions on the reference tetrahedron \( \hat{K} \) are:

\[
\phi_{0,0,0}(r, s, t) = \frac{\sqrt{3}}{2} , \quad \phi_{1,0,0}(r, s, t) = \frac{\sqrt{30}}{4} (2 + 2r + s + t) ,
\]

\[
\phi_{0,1,0}(r, s, t) = \frac{\sqrt{10}}{4} (2 + 3r + s) , \quad \phi_{0,0,1}(r, s, t) = \frac{\sqrt{5}}{2} (1 + 2t) ,
\]

and the matrix \( S \) becomes

\[
S = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10
\end{bmatrix} ,
\]

thus \( C_1 = 10 \).

For \( N = 2, 3, 4 \) using a similar argument, we obtain \( C_2 = \frac{63}{2} \), \( C_3 = 42 + 12\sqrt{7} \), and \( C_4 = 148.4089 \) respectively.

Numerical values for \( C_N \) are computed in Table 3. The eigenvalues clearly scale asymptotically as \( N^4 \).

Now let us compute \( |\nabla r| + |\nabla s| + |\nabla t| \):

\[
|\nabla r| + |\nabla s| + |\nabla t| = (r_x^2 + r_y^2 + r_z^2)^{1/2} + (s_x^2 + s_y^2 + s_z^2)^{1/2} + (t_x^2 + t_y^2 + t_z^2)^{1/2}.
\]

By (6) we have

\[
J = \frac{\partial(x, y, z)}{\partial(r, s, t)} = \begin{bmatrix}
x_r & x_s & x_t \\
y_r & y_s & y_t \\
z_r & z_s & z_t
\end{bmatrix} = \begin{bmatrix}
\frac{x_2-x_1}{2} & \frac{x_3-x_1}{2} & \frac{x_4-x_1}{2} \\
\frac{y_2-y_1}{2} & \frac{y_3-y_1}{2} & \frac{y_4-y_1}{2} \\
\frac{z_2-z_1}{2} & \frac{z_3-z_1}{2} & \frac{z_4-z_1}{2}
\end{bmatrix} .
\]

Then we can write
Table 3: Experimentally determined constants in the discrete Markov inequality.

<table>
<thead>
<tr>
<th>N</th>
<th>( C_N )</th>
<th>( \frac{C_N}{N^4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.0000</td>
<td>10.0000</td>
</tr>
<tr>
<td>2</td>
<td>31.5000</td>
<td>1.9688</td>
</tr>
<tr>
<td>3</td>
<td>73.7490</td>
<td>0.9105</td>
</tr>
<tr>
<td>4</td>
<td>148.4089</td>
<td>0.5797</td>
</tr>
<tr>
<td>5</td>
<td>269.5513</td>
<td>0.4313</td>
</tr>
<tr>
<td>6</td>
<td>452.0694</td>
<td>0.3488</td>
</tr>
<tr>
<td>7</td>
<td>717.7792</td>
<td>0.2990</td>
</tr>
<tr>
<td>8</td>
<td>1085.8205</td>
<td>0.2651</td>
</tr>
<tr>
<td>9</td>
<td>1587.8353</td>
<td>0.2420</td>
</tr>
<tr>
<td>10</td>
<td>2245.8720</td>
<td>0.2246</td>
</tr>
</tbody>
</table>

\[
| J^{-1} | = \left| \frac{\partial (r, s, t)}{\partial (x, y, z)} \right| = \left| \begin{array}{ccc} |r_x| & |r_y| & |r_z| \\ |s_x| & |s_y| & |s_z| \\ |t_x| & |t_y| & |t_z| \end{array} \right| \\
= \frac{1}{|\det(J)|} \left[ \frac{|\partial F_1|}{2} \frac{|\partial F_2|}{2} \frac{|\partial F_3|}{2} \right], \quad (8)
\]

where \( |\partial F_1|, |\partial F_2| \) and \( |\partial F_3| \) denote the area of the faces \( F_1, F_2, F_3 \) of the element \( K \), respectively.

Therefore (7) and (8) allow us to say,

\[
|\nabla r| + |\nabla s| + |\nabla t| \leq 2 \left| \frac{|\partial K|}{|K|} \right|.
\]

We conclude that

\[
|\nabla r| + |\nabla s| + |\nabla t| \leq 2 \left| \frac{|\partial K|}{|K|} \right|.
\]

Note that

\[
\frac{|\partial K|}{|K|} \approx h_K^{-1},
\]

where \( h_K \) is the longest edge of the element \( K \). □
5 Conclusion

In that paper, we have obtained explicit expressions for the Markov inequality constants on 1-simplex, 2-simplex and 3-simplex in $L_2$-norm. From this work, computable constants in some inverse inequalities are obtained. One can effectively utilize these results to give guaranteed computable upper bounds of a priori and posteriori error estimation of finite element solutions.

References


