Homework 3 Posted on Tuesday 02/12/2013 and due on Thursday 03/04/2013.

• Problems for 402 and 502

1. Consider the map $f : [1, \infty) \to \mathbb{R}$ defined by
   
   $$ f(x) = \frac{x}{2} + \frac{1}{x}, $$

   Prove that this map is contractive and conclude that it has a unique fixed point. What is the fixed point?

2. Problem XVIII.1.4, Lang page 504.

3. Problem XVIII.1.5, Lang page 504.

4. Let $f : S \to S$, where $S$ is a Banach space. Suppose $\{\alpha_n\}_{n \geq 1}$ is a nonnegative sequence in $\mathbb{R}$ converging to 0. Suppose also that $f$ satisfies
   
   $$ \|f^{(n)}(x) - f^{(n)}(y)\| \leq \alpha_n \|x - y\|, \quad \text{for all } x, y \in A, \text{ and all } n \geq 1, $$

   where $f^{(n)}(x) = f(f^{(n-1)}(x)) = f \circ f \circ \ldots \circ f(x)$. Prove that $f$ has a unique fixed point in $S$.

• Additional problems for 502

1. Let $T : C([0, 1]) \to C([0, 1])$ be defined by
   
   $$ Tf(x) = x + \int_0^x tf(t)dt, $$

   where $C([0, 1])$ denotes the set of continuous functions on $[0, 1]$, with values in $\mathbb{R}$. Prove that $T$ has a unique fixed point which satisfies the differential equation
   
   $$ f'(x) = xf(x) + 1. $$

2. Problem XVIII.1.8, Lang page 505.