1. Consider the map $f : [1, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{2} + \frac{1}{x}.$$ 

Prove that this map is contractive and conclude that it has unique fixed point. What is the fixed point?

**Proof.** First we need to show that $f([1, \infty)) \subset [1, \infty)$. So we will compute the minimum of $f$ over $[1, \infty)$. Since $f$ is smooth over $(1, \infty)$ and increasing as $x \to \infty$, then the minimum is attained either at $x = 1$ or where $f'(x) = 0$. We have $f(1) = 3/2 > 1$. The derivative $f' : (1, \infty) \to \mathbb{R}$ is

$$f'(x) = \frac{1}{2} - \frac{1}{x^2}. \quad (1)$$

So that $f'(x) = 0$ implies that $x = \sqrt{2}$ and we have $f(\sqrt{2}) = 2/\sqrt{2} > 1$. Hence in any case, we have that $f(x) > 1$ for all $x \in [1, \infty)$. Therefore, $f([1, \infty)) \subset [1, \infty)$ as desired.

Now we proceed to show that $f : [1, \infty) \to [1, \infty)$ is contractive. Again, we consider the derivative $f' : (1, \infty) \to \mathbb{R}$ given by (1). This implies that $|f'(x)| \leq 1/2$ for all $x \in (1, \infty)$. Now, from the mean value theorem, we have that given $x < y \in [1, \infty)$ we can find $c \in (x, y)$ such that

$$|f(x) - f(y)| = |f'(c)(x - y)| = |f'(c)||x - y| \leq \frac{1}{2}|x - y|.$$ 

Hence $f$ is a contraction mapping on $[1, \infty)$. Since $[1, \infty)$ equipped with the metric induced by the absolute value ($d(x, y) = |x - y|$) is a complete metric space, then the contraction mapping principle implies that $f$ has a unique fixed point on $[1, \infty)$. The fixed point satisfies

$$f(x) = \frac{x}{2} + \frac{1}{x} = x.$$ 

which implies that the unique fixed point is $x = \sqrt{2}$. 

2. Problem XVIII.1.3 in Lang page 504. Prove the following statement. Let $\overline{B}_r$ be the closed ball of radius $r$ centered at 0 in $E$. Let $f : \overline{B}_r \to E$ be a map such that:

(a) $|f(x) - f(y)| \leq b|x - y|$ with $0 < b < 1$.
(b) $|f(0)| \leq r(1 - b)$.

Show that there is a unique fixed point $x \in \overline{B}_r$ such that $f(x) = x$.

**Proof.** It only remains to show that $f(\overline{B}_r) \subset \overline{B}_r$. So let $x \in \overline{B}_r$ be arbitrary. We have that

$$|f(x)| \leq |f(x) - f(0)| + |f(0)| \leq b|x| + r(1 - b) \leq br + r(1 - b) = r.$$ 

Therefore, $f(x) \in \overline{B}_r$ for all $x \in \overline{B}_r$. Since $\overline{B}_r$ is a complete metric space and from assumption (a) we have that $f$ is contractive, the desired result follows from the contraction mapping principle. 


3. Problem XVIII.1.4 in Lang page 504. Notation as in previous problem. Let \( g \) be another map of \( B_r \) into \( E \) and let \( c > 0 \) be such that \( |g(x) - f(x)| \leq c \) for all \( x \). Assume that \( g \) has a fixed point \( x_2 \) and let \( x_1 \) be fixed point of \( f \). Show that \( |x_2 - x_1| \leq c/(1-b) \)

**Proof.**

\[
|x_2 - x_1| = |g(x_2) - f(x_1)| = |g(x_2) - f(x_2) + f(x_2) - f(x_1)| \leq |g(x_2) - f(x_2)| + |f(x_2) - f(x_1)| \leq c + b|x_2 - x_1| \\
i.e.,
\]

\[
(1 - b)|x_2 - x_1| \leq c \quad |x_2 - x_1| \leq c/(1 - b) \quad \text{since } 0 < b < 1
\]

\( \Box \)

4. Problem XVIII.1.5 in Lang page 504. Let \( K \) be a continuous function of two variables, defined for \( (x, y) \) in the square \( a \leq x \leq b \) and \( a \leq y \leq b \). Assume that \( \|K\| \leq C \) for some constant \( C > 0 \). Let \( f \) be a continuous function on \([a, b]\) and let \( r \) be a real number satisfying the inequality

\[
|r| < \frac{1}{C(b - a)}
\]

Show that there is one and only one function \( g \) continuous on \([a, b]\) such that

\[
f(x) = g(x) + r \int_a^b K(t, x)g(t)\,dt
\]

**Proof.** Let \( \mathcal{L} : C([a, b]) \to C([a, b]) \) and \( \mathcal{L}(g)(x) = f(x) - r \int_a^b K(t, x)g(t)\,dt \). Here \( C([a, b]) \) is a set of all continuous functions defined on \([a, b]\), and we equip this space with the sup-norm \( \| \cdot \| \). Let \( g_1, g_2 \in C([a, b]) \) be arbitrary. We have,

\[
\|\mathcal{L}g_1 - \mathcal{L}g_2\| = \left\| \left( f(x) - r \int_a^b K(t, x)g_1(t)\,dt \right) - \left( f(x) - r \int_a^b K(t, x)g_2(t)\,dt \right) \right\|
\]

\[
= |r| \left\| \int_a^b K(t, x)(g_2(t) - g_1(t))\,dt \right\|
\]

\[
\leq |r|C(b - a)\|g_2 - g_1\|
\]

\[
= \alpha\|g_1 - g_2\|
\]

Hence \( \|\mathcal{L}g_1 - \mathcal{L}g_2\| \leq \alpha\|g_1 - g_2\| \) with \( \alpha = |r|C(b - a) < 1 \).

As \( C([a, b]) \) is complete and there exists a number \( 0 < \alpha < 1 \) such that \( \|\mathcal{L}(g_1) - \mathcal{L}(g_2)\| \leq \alpha\|g_1 - g_2\| \), from shrinking lemma \( \mathcal{L} \) has a unique fixed point, i.e., there exists a unique continuous function \( g \) such that \( \mathcal{L}(g) = g \), in other words, \( f(x) = g(x) + r \int_a^b K(t, x)g(t)\,dt \) \( \Box \)
5. Let $f : S \to S$, where $S$ is a Banach space. Suppose that $\{\alpha_n\}_{n \geq 1}$ is a nonnegative sequence in $\mathbb{R}$ converging to 0. Suppose also that $f$ satisfies

$$\|f^n(x) - f^n(y)\| \leq \alpha_n \|x - y\|, \quad \text{for all } x, y \in S, \quad \text{and all } n \geq 1,$$

where $f^n = f \circ f \circ \cdots \circ f$. Prove that $f$ has a unique fixed point.

Proof. Since $\{\alpha_n\}$ is a nonnegative sequence in $\mathbb{R}$ converging to 0, then there exists $N \in \mathbb{N}$, such that $\alpha_n \leq \frac{1}{2}$ for all $n \geq N$. Then, in particular, we have that $\|f^N(x) - f^N(y)\| \leq \frac{1}{2} \|x - y\|$, for all $x, y \in S$. By the contraction mapping theorem, $f^N$ has a unique fixed point in $S$; call it $x_0$, i.e. $f^N(x_0) = x_0$. Then $f^N(f(x_0)) = f^{N+1}(x_0) = f(f^N(x_0)) = f(x_0)$. We get that $f(x_0)$ is also a fixed point of $f^N$. By uniqueness, it follows that $f(x_0) = x_0$.

So far we have proven that $f$ has a fixed point in $S$. Now we prove uniqueness. Assume $x_1 \in S$ is also a fixed point of $f$, i.e., $f(x_1) = x_1$. Then $f^N(x_1) = f^{N-1}(x_1) = \cdots = f(x_1) = x_1$. This implies that $x_1$ is the unique fixed point of $f^N$. Hence, it follows that $x_0 = x_1$. \qed
502 problem 1

For any \( f, g \in C([0,1]) \)

\[
\|Tf - Tg\|_{\infty} = \sup_{x} \left| x + \int_{0}^{x} tf(t)dt - \left( x + \int_{0}^{x} tg(t)dt \right) \right|
\]

\[
= \sup_{x} \left| \int_{0}^{x} (f(t) - g(t))dt \right|
\]

\[
\leq \sup_{x} \left| \int_{0}^{x} t|f(t) - g(t)|dt \right|
\]

\[
\leq \sup_{x} \left| \int_{0}^{x} t \sup_{s} |f(s) - g(s)|dt \right|
\]

\[
= \sup_{s} |f(s) - g(s)| \sup_{x} \int_{0}^{x} |t|dt
\]

\[
= \|f - g\|_{\infty} \int_{0}^{1} |t|dt
\]

\[
= \frac{1}{2} \|f - g\|_{\infty}
\]

So \( T \) is a contraction on \( (C([0,1]), \| \cdot \|_{\infty}) \). Since \( (C([0,1]), \| \cdot \|_{\infty}) \) is a complete space, the shrinking lemma (also known as the contraction mapping principle and Banach fixed point theorem) shows that \( T \) has a unique fixed point, call it \( f \).

Since \( f(x) = (Tf)(x) = x + \int_{0}^{x} tf(t)dt \), upon differentiating both sides, the fundamental theorem of calculus gives us

\[
f'(x) = 1 + xf(x)
\]

2. (a) Note: the process of part (a) is similar to part (b).

First, let \( M = \{ \alpha : R \to R \mid \alpha \text{ is continuous and increasing function on } R, \text{ and } \alpha(x + 1) = \alpha(x) + 1, \forall x \in R \} \). Prove \( M \) is complete.

Secondly, define \( T : M \to M \), s.t. \( (T\alpha)(x) = g(\alpha(nx)) \), where \( g \) is the inverse function of \( f \). Prove the range of \( T \) is contained in \( M \).

Additionally, prove \( T \) is a shrinking map. Because \( g'(y) = \frac{1}{f'(g(y))} \) and \( f'(x) > 1 \). Then by MNT

\[
| g(\alpha(nx)) - g(\beta(nx)) | < | \alpha(nx) - \beta(nx) |, \forall x \in R
\]

(b) By assumption, \( f \) is continuous, strictly increasing, and therefore \( f \) has an inverse \( g \).

We contend that \( T \) maps \( M \) into \( M \). Clearly, \( T\alpha \) is continuous and increasing because \( g \) and \( \alpha \) are continuous and increasing. By induction, we find

\[
\alpha(n(x + 1)) = \alpha(nx) + n
\]

so \( f((T\alpha)(x + 1)) = \alpha(nx) + n \), and

\[
f((T\alpha)(x) + 1) = f((T\alpha)(x)) + n = \alpha(nx) + n
\]

The function \( f \) is injective so \( (T\alpha)(x + 1) = (T\alpha)(x) + 1 \) which proves our contention. The map \( T \) is a shrinking map because the condition on \( f \) implies

\[
r_1(g(x) - g(y)) \leq f(g(x)) - f(g(y)) \leq r_2(g(x) - g(y))
\]
thus
\[ \frac{x-y}{r_2} \leq g(x) - g(y) \leq \frac{x-y}{r_1} \]

so there exists a constant 0 < K < 1 such that \(|g(x) - g(y)| \leq K \ |x - y|\).

Finally we show that M is complete. Let \( \{\alpha_n\} \) be a Cauchy sequence in M. By induction, we see that \( \alpha(x + j) = \alpha(x) + j \) for all integers \( j \), so if \( \|\alpha_n - \alpha_m\| < \varepsilon \), then \( |\alpha_n(x) - \alpha_m(x)| < \varepsilon \) for all \( x \). Use an argument as in Theorem 3.1 of chapter 12, and the fact that the limit of a uniformly convergent sequence of continuous functions is continuous, to show that there exists a continuous function \( \alpha \) such that \( \alpha_n \to \alpha \) as \( n \to \infty \). Since \( \alpha_n(x + 1) = \alpha_n(x) + 1 \) in the limit we have \( \alpha(x + 1) = \alpha(x) + 1 \) and \( \alpha \) is increasing, whence M is complete.

The shrinking lemma implies that there exists a map \( \alpha_0 \) such that \( T\alpha_0 = \alpha_0 \) or equivalently
\[ g(\alpha_0(nx)) = \alpha_0(x) \]

thus \( \alpha_0(nx) = f(\alpha_0(x)) \).