CAAM 402/502 Spring 2013
Homework 4
Solutions

1. Let \( f : V \to \mathbb{R}^n \) be a continuous function defined on the open set \( V \subset \mathbb{R}^n \). Suppose that \( f \) is injective on \( V \), has all the first order partial derivatives, and its Jacobian \( J_f(x) \) satisfies \( \det J_f(x) \neq 0 \) for all \( x \in V \). Prove that the inverse of \( f \), call it \( f^{-1} : f(V) \to V \) is continuous.

Proof. We will rely on the following theorem proved in class. Assuming the hypothesis above. Let \( a \in V \) and \( r > 0 \) be sufficiently small such that \( B_r(a) \subset V \). If \( f(a) \notin f(\partial B_r(a)) \) then there exists \( \delta > 0 \) such that \( B_\delta(f(a)) \subset f(B_r(a)) \).

Now we are ready to prove that \( f^{-1} : f(V) \to V \) is continuous. Let \( \epsilon > 0 \) and \( b \in f(V) \) be arbitrary. Since \( f \) is injective there exists unique \( a \in V \) such that \( f(a) = b \). We assume that \( \epsilon > 0 \) is sufficiently small so that \( B_\epsilon(a) \subset V \). Now since \( f \) is injective, we have that \( f(a) \notin f(\partial B_\epsilon(a)) \). From the theorem in class, then there exists \( \delta > 0 \) such that \( B_\delta(b) \subset f(B_\epsilon(f^{-1}(b))) \). In other words, if \( y \in B_\delta(b) \) then \( f^{-1}(y) \in B_r(f^{-1}(b)) \), which is the definition for the continuity of \( f^{-1} \) at \( b \in f(V) \). Since \( b \) is arbitrary, then we conclude that \( f^{-1} : f(V) \to V \) is continuous.

2. Suppose that \( f : V \to W \) is a bijection from \( V = B_r(a) \subset \mathbb{R}^n \) to \( W \subset \mathbb{R}^n \), and it is differentiable at \( a \). Suppose also that the inverse \( f^{-1} : W \to V \) is differentiable at \( f(a) \). Prove that \( \det J_f(a) \neq 0 \).

Proof. We have that \( f^{-1} \circ f = id \) on the set \( V \). Since \( f \) is differentiable at \( a \) and \( f^{-1} \) is differentiable at \( f(a) \), we can use the chain rule (pp. 471-472 in Lang) to obtain,

\[
I = J_{id(a)} = J_{f^{-1} \circ f}(a) = J_{f^{-1}}(f(a)) J_f(a).
\]

Therefore \( J_f(a) \) is invertible and hence \( \det J_f(a) \neq 0 \).

3. XVIII, 3.1 (Lang page 520).

Proof. The solution is from the book ‘Problems and Solutions for Undergraduate Analysis’ by Rami Shakarchi. Let \( y \in f(U) \) and select \( x \) such that \( f(x) = y \). By the inverse mapping theorem we know that \( f \) is locally \( C^1 \)-invertible at \( x \), so by definition there exists an open set \( U_1 \) such that \( x \in U_1 \) and \( f(U_1) \) is open. Since \( y \in f(U_1) \) we conclude that \( f(U) \) is open.

4. XVIII, 3.2 (Lang page 520).

Proof. The solution is from the book ‘Problems and Solutions for Undergraduate Analysis’ by Rami Shakarchi. The Jacobian of \( f \) at a point \((x, y)\) is

\[
J_f(x, y) = \begin{pmatrix} e^x & e^y \\ e^x & -e^y \end{pmatrix}
\]

whose determinant is \(-2e^x e^y \neq 0\), so \( f \) is locally invertible around every point of \( \mathbb{R}^2 \). Note that \( f \) is injective because if \( f(x_1, y_1) = f(x_2, y_2) \), then

\[
\begin{cases} e^{x_1} + e^{y_1} = e^{x_2} + e^{y_2}, \\ e^{x_1} - e^{y_1} = e^{x_2} - e^{y_2}, \end{cases}
\]
so adding the two equations we get \( x_1 = x_2 \) and subtracting the two equations we see that \( y_1 = y_2 \). This shows that\( f : R^2 \to f(R^2) \) has a set inverse. Note that \( f(R^2) = \{(x, y) \in R^2 : x > y\} = V \) because \( e^x + e^y > e^x - e^y \) and if \( (a, b) \in V \), then
\[
 f(\log \frac{a + b}{2}, \log \frac{a - b}{2}) = (a, b)
\]
From this analysis, we also see that the map \( g : V \to R^2 \) defined by
\[
 g(x, y) = (\log \frac{x + y}{2}, \log \frac{x - y}{2})
\]
is a \( C^1 \)-inverse for \( f \) because \( g(f(x, y)) = (x, y) \) and all the partial derivatives of \( g \) exist and are continuous on \( V \).

5. XVIII, 3.3 (Lang page 520). Let \( f : R^2 \to R^2 \) be given by \( f(x, y) = (e^x \cos y, e^x \sin y) \). Show that \( Df(x, y) \) is invertible for all \( (x, y) \in R^2 \), that \( f \) is locally invertible at every point, but does not have an inverse defined on all of \( R^2 \).

**Proof.** Given \( f(x, y) = (f_1, f_2) = (e^x \cos y, e^x \sin y) \), Jacobian \( J_f \) can be computed as,
\[
 J_f(x, y) = \begin{bmatrix}
 \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
 \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{bmatrix} = \begin{bmatrix}
 e^x \cos y & -e^x \sin y \\
 e^x \sin y & e^x \cos y
\end{bmatrix}
\]
and \( \det J_f(x, y) = e^{2x} \neq 0 \) for any \( (x, y) \in R^2 \). Hence \( J_f \) is invertible at every point, so that \( f \) is invertible at every point \( (x, y) \in R^2 \) from inverse mapping theorem. Observe that \( f(x, y) = f(x, y + 2\pi) \). i.e., \( f \) is not one to one. Hence \( f \) does not have an inverse defined on all of \( R^2 \)

6. XVIII, 3.4 (Lang page 520). Let \( f : R^2 \to R^2 \) be given by \( f(x, y) = (x^2 - y^2, 2xy) \). Determine the points of \( R^2 \) at which \( f \) is locally invertible, and determine whether \( f \) has an inverse defined on all of \( R^2 \).

**Proof.** Given \( f(x, y) = (f_1, f_2) = (x^2 - y^2, 2xy) \), Jacobian \( J_f \) can be computed as,
\[
 J_f(x, y) = \begin{bmatrix}
 \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
 \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{bmatrix} = \begin{bmatrix}
 2x & -2y \\
 2y & 2x
\end{bmatrix}
\]
and \( \det J_f(x, y) = 4(x^2 + y^2) \neq 0 \) for any \( (x, y) \neq (0, 0) \). Hence \( J_f \) is invertible at every point \( (x, y) \in R^2 \setminus \{(0, 0)\} \), so that \( f \) is invertible at every point \( (x, y) \in R^2 \setminus \{(0, 0)\} \) from inverse mapping theorem. At \((0, 0)\) in any neighborhood \( B_r((0,0)) \), if \((x, y) \in B_r((0,0)) \), then \((-x, -y) \in B_r((0,0)) \) and \( f(x, y) = f(-x, -y) \). Hence \( f \) is not locally invertible at \((0, 0)\) and thus does not have a global inverse on all of \( R^2 \).

**Note:** Converse of inverse mapping theorem is not true. i.e., Jacobian is not invertible does not imply that local inverse does not exist. For example consider \( f : R \to R \) and \( f(x) = x^3 \). We can observe that \( f'(x) = 3x^2 \) and \( f'(0) = 0 \), that is derivative is not invertible at \( x = 0 \) but this function has a global inverse defined as \( f^{-1}(x) = x^{1/3} \).