By considering the identity
\[(\lambda I - A)(\lambda I + A) = \lambda^2 I - A^2\]
for any \(\lambda \in \mathbb{C}\) and \(A \in \mathcal{L}(E)\), \(E\) a Banach space, show that \(\sigma(A^2) = \{\lambda^2 : \lambda \in \sigma(A)\}\). Deduce that, for any \(\lambda \in \sigma(A)\),
\[|\lambda| \leq \|A^2\|^{1/2}.
\]

Can you replace 2 by 3 in the previous problem? By \(n \in \mathbb{N}\)? By \(-1\)?

Hint: consider principal roots of unity, i.e., \(e^{2k\pi i/n}, k = 0, \ldots, n - 1\).

(Spectral Mapping Theorem) Let \(p\) be a degree \(n\) polynomial, and suppose \(A \in \mathcal{L}(E)\) for a Banach space \(E\). If \(p(z) = c_0 + c_1 z + \cdots + c_n z^n\), we define \(p(A)\) via
\[p(A) = c_0 I + c_1 A + \cdots + c_n A^n.
\]
Prove that \(\sigma(p(A)) = p(\sigma(A))\), where
\[p(\sigma(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}.
\]
Now let \(r\) be a rational function of degree \((m, n)\), i.e., \(r(z) = p(z)/q(z)\) with \(p\) and \(q\) polynomials of degree \(m\) and \(n\). Suppose that the poles of \(r\) are disjoint from \(\sigma(A)\) (why?), and define \(r(A) = p(A)q(A)^{-1}\). Prove that \(\sigma(r(A)) = r(\sigma(A))\).

Banded Laurent operators correspond to doubly-infinite matrices acting on \(\ell^p(\mathbb{Z})\) with the constant \(a_k\) on the \(k\)th diagonal for \(-m \leq k \leq n\), with \(0 \leq m, n < \infty\) and zero elsewhere. One such example is the doubly-infinite shift operator, which is zero everywhere except for ones on the first superdiagonal,
\[
C = \begin{pmatrix}
\vdots & \vdots & \vdots \\
0 & 1 & \ddots \\
0 & 0 & 1 \\
\vdots & \vdots & \ddots \\
\end{pmatrix},
\]
where we can take \(m = 0\) and \(n = 1\) with \(a_0 = 0\), \(a_1 = 1\). If \(x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \ell^p(\mathbb{Z})\), then
\[Cx = C(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, x_0, x_1, x_2, \ldots).
\]
(a) Show that for \(C\) acting on \(\ell^\infty(\mathbb{Z})\), \(\sigma(C) = \{z \in \mathbb{C} : |z| = 1\}\), and that all \(z \in \sigma(C)\) are eigenvalues. Are these same \(z\) eigenvalues if \(C\) acts on \(\ell^p(\mathbb{Z})\) for \(1 \leq p < \infty\)?
(b) Now consider general banded Laurent operators on $\ell^\infty(\mathbb{Z})$ determined by the coefficients $a_{-m}, \ldots, a_n$:

$$A = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & a_{-m} & a_{-1} & a_0 & a_1 & a_n & \ldots & \ldots \\ \ldots & a_{-m} & a_{-1} & a_0 & a_1 & a_n & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$ 

Use the result of Problem 7.7 (whether you solved that problem or not) to show that

$$\sigma(A) = \{a(z) : |z| = 1\},$$

where $a$ is the symbol of the operator, defined by the Laurent polynomial

$$a(z) = \sum_{k=-m}^{n} a_k z^k.$$

(c) Plot the spectrum $\sigma(A) = \{a(z) : |z| = 1\}$ for a few interesting symbols of your choosing (e.g., using MATLAB). Below are several examples to whet your appetite.

8.13 Let $K$ be a compact Hermitian operator on a Hilbert space $H$. Show that if $K$ has infinite rank then the range of $K$ is not closed in $H$.

— Optional Supplemental Problems —

7.27 Let $V$ be the Volterra operator $V$ from Problem 7.8:

$$(Vx)(t) = \int_0^t x(s) \, ds, \quad 0 < t < 1.$$ 

Show that $V + V^*$ has rank 1.

7.x Suppose $A$ is a bounded linear operator on the Hilbert space $H$. Show that

$$\frac{1}{2} \|A\| \leq \sup_{\|x\|=1} |(Ax, x)|.$$ 

(You may find it useful to write $A$ as the sum of Hermitian and skew-Hermitian operators.)

Use this result to show that if $(Ax, x) = (Bx, x)$ for all $x \in H$, then $A = B$.

An operator $A \in \mathcal{L}(H)$ is called normal provided $A$ commutes with its adjoint, i.e., $A^* A = AA^*$. Show that $A$ is normal if and only if $\|A^* x\| = \|Ax\|$ for all $x$. 