CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Problem Set 7

[Late papers are due Friday 21 November]

Young, 8.1 [10 points], 8.14 [15 points], 8.16 [35 points], either (both Ex. 8.6 and 8.x) or 8.y [40 points].

8.1 Let $E$ be a Banach space and let $A, B, C \in \mathcal{L}(E)$. Show that if $B$ is compact, then so is $ABC$.

8.14 Show that a compact Hermitian operator is the limit with respect to the operator norm (‘uniform limit’) of a sequence of finite rank operators.

8.16 Let $H$ be a separable Hilbert space and let $K$ be a compact operator on $H$. Let $(\varphi_j)_{j=1}^{\infty}$ be a complete orthonormal sequence of eigenvectors of $K^*K$ and let $(\lambda_j)_{j=1}^{\infty}$ be the sequence of corresponding eigenvalues. (Explain why this sequence always exists.) Show that the formula

$$U\left(\sum_{j=1}^{\infty} x_j \varphi_j\right) = \sum_{j=1}^{\infty} x_j \mu_j K \varphi_j,$$

where

$$\mu_j = \begin{cases} \lambda_j^{-1/2} & \text{if } \lambda_j > 0 \\ 0 & \text{if } \lambda_j = 0 \end{cases},$$

defines $U$ as a bounded linear operator on $H$. What is $\|U\|$? Show further that $K = U(K^*K)^{1/2}$. Deduce, using Problems 8.1 and 8.14 that $K$ is the limit with respect to the operator norm of a sequence of finite rank operators.

Ex. 8.6 Prove that the Volterra operator $V$ on $L^2(0, 1)$ defined by

$$(Vx)(t) = \int_0^t x(s) \, ds, \quad 0 < s < 1$$

is a Hilbert–Schmidt operator, and thus compact.

8.x The Volterra operator in Exercise 8.6 is non-self-adjoint.

(a) Determine the spectrum of $V$.

(b) What does your answer to part (a) suggest about the possibility of generalizing the Spectral Theorem to non-self-adjoint compact operators?

[You may use the following result: any nonzero $\lambda \in \sigma(K)$ must be an eigenvalue for any compact operator $K$, and $\sup_{\lambda \in \sigma(A)} |\lambda| = \lim_{n \to \infty} \|A^n\|^{1/n}$ for any bounded linear operator $A$.]
We are interested in approximating the spectrum of the Fredholm integral operator \( K : L^2[a, b] \to L^2[a, b] \), defined pointwise for \( t \in [a, b] \) by

\[
(Ku)(t) = \int_a^b k(t, s)u(s) \, ds.
\]

For purposes of this problem, assume that the kernel \( k(t, s) \) is sufficiently well-behaved so as not to complicate the calculations that are to follow. Since \( K \) is a compact operator, all nonzero points in its spectrum must be eigenvalues. Hence, we can approximate the spectrum by looking for nonzero solutions to the equation

\[
Ku = \lambda u. \tag{1}
\]

We can approximate an integral over \([a, b]\) via a quadrature rule of the form

\[
\int_a^b f(s) \, ds \approx \sum_{\ell=1}^n \omega_\ell f(s_\ell),
\]

where \( \omega_1, \ldots, \omega_n \) are the quadrature weights and \( s_1, \ldots, s_n \in [a, b] \) are the quadrature nodes.

With \( t_m = s_m \) for \( m = 1, \ldots, n \), we can approximate the action of the Fredholm integral operator as

\[
(Ku)(t_m) \approx \sum_{\ell=1}^n \omega_\ell k(t_m, s_\ell)u(s_\ell).
\]

Each value of \( m = 1, \ldots, n \) thus provides a row in a finite-dimensional approximation of equation (1):

\[
\begin{bmatrix}
\omega_1 k(t_1, s_1) & \ldots & \omega_n k(t_1, s_n) \\
\vdots & \ddots & \vdots \\
\omega_1 k(t_n, s_1) & \ldots & \omega_n k(t_n, s_n)
\end{bmatrix}
\begin{bmatrix}
u_1 \\
\vdots \\
u_n
\end{bmatrix} = \lambda
\begin{bmatrix}
u_1 \\
\vdots \\
u_n
\end{bmatrix},
\]

which we write as \( Ku = \lambda u \). The goal of this problem is to (computationally) study how well the \( n \times n \) matrix \( K \) approximate (some subset of) the spectrum of \( K \), as a function of the kernel \( k \) and quadrature rule.

For all these examples, we shall use the \( n \)-point Gauss–Legendre quadrature rules; a MATLAB code to produce the nodes and weights (adapted from a code in Trefethen’s Spectral Methods in MATLAB) can be found on the class website.

(a) If \([a, b] = [-\pi, \pi]\) and the kernel \( k(t, s) \) has the special form \( k(t, s) = \kappa(t - s) \), where \( \kappa \) is a continuous, \( 2\pi \)-periodic function, we saw in class (see Young, Example 7.24) the eigenvectors \( \phi_m \) and associated eigenvalues \( \lambda_m \) of \( K \) satisfy, for \( m \in \mathbb{Z} \),

\[
\phi_m(x) = \frac{e^{imx}}{\sqrt{2\pi}}, \quad \lambda_m = \int_{-\pi}^\pi \kappa(\tau)e^{-im\tau} \, d\tau = (\kappa, e^{im\tau}).
\]

Compute the eigenvalues of the matrix \( K \) for modest values of \( n \) for the kernels

\[
\kappa(t) = 1, \quad \kappa(t) = e^{it}, \quad \kappa(t) = \sin(t).
\]

Explain your results in terms of the exact eigenvalues for this problem.

(b) Now consider the kernel

\[
\kappa(t) = |\sin(t)|,
\]

which gives exact eigenvalues

\[
\lambda_m = \begin{cases} 
4 & m \text{ even;} \\
1 - m^2 & m \text{ odd.}
\end{cases}
\]

Produce a plot (e.g., \texttt{loglog} in MATLAB) showing how the error in eigenvalues \( \lambda_0, \lambda_2, \lambda_4, \) and \( \lambda_6 \) decreases with growing value of \( n \) (say, \( n = 8, 16, \ldots, 512 \)).
(c) Repeat the experiment in part (b) with the functions

\[ \kappa(t) = |\sin(t)|^3, \quad \lambda_m = \begin{cases} 
24 & m \text{ even;} \\
\frac{24}{m^2 - 10n^2 + 9} & m \text{ odd;}
\end{cases} \]

and

\[ \kappa(t) = e^{\sin(t)} \]

with

\[
\begin{align*}
\lambda_0 &= 7.9549265210128452745132196 \ldots \\
\lambda_2 &= -0.852977641641214869989135 \ldots \\
\lambda_4 &= 0.0171978335568658124299194 \ldots \\
\lambda_6 &= -0.0001413004273713492084865 \ldots 
\end{align*}
\]

Speculate about the reason for the different convergence behavior you observe for the three kernels in parts (b) and (c).

(d) Now consider the non-self-adjoint integral operator

\[ (K_Fu)(t) = \sqrt{\frac{iF}{\pi}} \int_{-1}^{1} e^{-iF(t-s)^2} u(s) \, ds \]

on \( L^2[-1, 1] \), which arises in a model for light propagating in a laser cavity (see Trefethen and E., *Spectra and Pseudospectra*, §60, and references therein). The parameter \( F \) is called the Fresnel number.

Produce plots of the spectrum of \( K_F \) in the complex plane for (i) \( F = 16\pi \) and (ii) \( F = 64\pi \). In each case, select the discretization parameter \( n \) sufficiently large that the eigenvalues have converged to plotting accuracy, i.e., they do not appear to move when you increase \( n \).