Problem 1

The following questions concern the proof of the sensitivity theorem given on the last three pages of Dense_LA_Notes: Set 7 (or Theorem 5.3.1 in Golub and Van Loan 3rd ed.).

a) Where in the proof is right hand inequality in the assumption

\[ \varepsilon \equiv \max \left\{ \frac{\| \hat{A} - A \|}{\| A \|}, \frac{\| \hat{b} - b \|}{\| b \|} \right\} < \frac{\sigma_n}{\sigma_1} \]

required?

b) Explain why the vector valued function \( x(t) \) is well defined (exists and is unique) at each \( t \in [0, \varepsilon] \). Why is this function continuously differentiable with respect to \( t \)? Which assumption assures that this holds?

c) How is the smoothness (cont. differentiability) of \( x(t) \) used in the proof?

d) Verify that \( \| f \| \leq \| b \| \) and that \( \| E \| \leq \| A \| \).

Problem 2

Let \( AV = VH + f e_k^T \) be a \( k \)-step Arnoldi factorization of \( A \) with \( V e_1 = b \) (assume \( \| b \| = 1 \)) where \( \| \cdot \| \) is the 2-norm. Assume \( A \) is nonsingular.

Let \( r_k = p_k(A)b \) be the GMRES residual (i.e. \( r_k = b - Ax_k \), where \( x_k \) is computed with the GMRES method).

Prove the following results:

1. The \( j + 1 \)-st column \( v_{j+1} \) of \( V \) is of the form \( \phi_j(A)b \) where \( \phi_j(\tau) = \gamma_j \text{det}(\tau I - H_j) \) with \( H_j \) the leading \( j \times j \) submatrix of \( H \) (i.e. the value of \( H \) at step \( j \) of the Arnoldi process).

2. The roots \( \theta_j \) of the GMRES polynomial \( p_k(\tau) \) are the eigenvalues of the generalized eigenvalue problem

\[ \overline{H}^T H z = H^T z \theta \]

where \( \overline{H}^T = [H^T, e_k \beta] \) with \( \beta = \| f \| \).

These are called Harmonic Ritz values.
3. When \( A \) is symmetric, demonstrate that these Harmonic Ritz values are reciprocals of the critical points of the generalized Rayleigh quotients

\[
\frac{w^T A^{-1} w}{w^T w}, \text{ where } w \in \text{AK}_k(A, b).
\]

4. Let \( A \) be symmetric and indefinite with \( \lambda_- \) the (algebraically) largest negative eigenvalue and \( \lambda_+ \) the smallest positive eigenvalue of \( A \). Prove there are no Harmonic Ritz values in the open interval \((\lambda_-, \lambda_+)\).

5. If \( x_k \in \text{K}_k(A, b) \) is any approximate solution drawn from the Krylov space (not necessarily the GMRES approximation) then \( b - Ax_k = p(A)b \) for some polynomial \( p \) of degree \( k \) such that \( p(0) = 1 \) and where \( x_k = \phi(A)b \) with

\[
p(\tau) = 1 - \tau \phi(\tau).
\]

Show that \( \phi \) is the unique polynomial of degree \( k-1 \) that interpolates the function \( \eta(\tau) \equiv \frac{1}{\tau} \) at the \( k \) roots of \( p \).

6. Graph the polynomial \( \phi \) from GMRES with \( k = 20 \) for \( A \) as the negative of the 1-D discrete Laplacian of order 100 (\( A = \text{trid}[-1, 2, -1] \)). Your graph should go over the interval \((\lambda_1, \lambda_n)\), the eigenvalue range of \( A \). On the same plot, graph the function \( \eta(\tau) = \frac{1}{\tau} \) over the same interval and show the interpolation points. Repeat this for \( A - 2I \) in place of \( A \).

**Problem 3**

Let

\[
A = \begin{bmatrix} 0 & 1 \\ I & 0 \end{bmatrix},
\]

where \( I \) is the \( n - 1 \times n - 1 \) identity. Note \( A \) is the \( n \times n \) left circular shift operator.

1. What is the \( k \)-step Arnoldi factorization \( AV = VH + f e_k^T \) for any \( k < n \)? What is it when \( k = n \)?

2. Let \( b = e_1 \). Give both the FOM and GMRES approximate solutions to \( Ax = b \) for a general iteration \( k \). What happens when \( k = n \)?

3. What are the Ritz values and the Harmonic Ritz values at each \( k \)?