Another Inverse Inequality

For future analysis it will be useful to bound the norm of the derivative of a $p$th order polynomial.

We define the $L_2$ norm of $\rho$ on $I_j = [x_j, x_{j+1}]$ by:

$$\|\rho\|_{L_2(I_j)} := \sqrt{\int_{x_j}^{x_{j+1}} \rho^2 \, dx}$$

We can find the norm of the derivative of $\rho$ by

$$\left\| \frac{d\rho}{dx} \right\|_{L_2(I_j)} = \sqrt{\int_{x_j}^{x_{j+1}} \left| \frac{d\rho}{dx} \right|^2 \, dx}$$

We can bound the norm of the derivative by

$$\left\| \frac{d\rho}{dx} \right\|_{L_2(I_j)} \leq C \rho^2 \frac{1}{x_{j+1} - x_j}$$

$\forall \rho \in P^p(I_j)$
Theorem

• Let $I=(a,b)$ and $h=b-a$. Then for every polynomial $\rho \in P^p(I)$ it holds that

$$\left\| \frac{d \rho}{dx} \right\|_{L^2(I)} \leq 2\sqrt{3} \frac{p^2}{h} \|\rho\|_{L^2(I)}$$
Proof Part 1

• Assume for the moment that $a=-1, b=1, h=2$

• We have for every $n \geq 1$ that

\[
\int_{-1}^{1} \left( \frac{dL_n}{dx} \right)^2 \, dx = n(n+1)
\]

• To see this we apply integration by parts

\[
\int_{-1}^{1} \left( \frac{dL_n}{dx} \right)^2 \, dx = \left[ L_n \left( \frac{dL_n}{dx} \right) \right]_{-1}^{1} - \int_{-1}^{1} L_n \frac{d^2L_n}{dx^2} \, dx
\]

\[
= \left[ L_n \left( \frac{dL_n}{dx} \right) \right]_{-1}^{1}
\]

\[
= L_n(1) \frac{dL_n}{dx}(1) - L_n(-1) \frac{dL_n}{dx}(-1)
\]

\[
= n(n+1)
\]

1) Integrate by parts

2) Use the orthogonality of the Legendre polynomials

3) $L_m(\pm 1) = (\pm 1)^m$

\[
dL_m \left( \pm 1 \right) = (\pm 1)^m \frac{m(m+1)}{2}
\]
Proof Part 2

• Every $\rho \in P^p (I)$ may be expanded into a Legendre series:

$$\rho(x) = \sum_{n=0}^{n=p} \rho_n L_n(x)$$

where

$$\|\rho\|^2_{L^2(I)} = \sum_{n=0}^{n=p} \frac{2}{2n+1} |\rho_n|^2$$

by the triangle inequality and Step 1 we obtain:

$$\left\| \frac{d \rho}{dx} \right\|_{L^2(I)} \leq \sum_{n=0}^{n=p} |\rho_n| \left\| \frac{dL_n}{dx} \right\|_{L^2(I)} \leq \sum_{n=0}^{n=p} |\rho_n| \sqrt{n(n+1)}$$
Proof Part 2 cont

\[
\left\| \frac{d \rho}{dx} \right\|_{L^2(I)} \leq \sum_{n=0}^{n=p} |\rho_n| \sqrt{n(n+1)}
\]

\[\downarrow\]

\[
\left\| \frac{d \rho}{dx} \right\|_{L^2(I)}^2 \leq \left\{ \sum_{n=0}^{n=p} \frac{2}{2n+1} |\rho_n|^2 \right\} \left\{ \sum_{n=1}^{n=p} n(n+1) \left( \frac{2n+1}{2} \right) \right\}
\]

\[
\leq \left\| \rho \right\|_{L^2(I)}^2 \ p \max_{1 \leq n \leq p} \left( n(n+1) \left( \frac{2n+1}{2} \right) \right)
\]

\[
\leq \left\| \rho \right\|_{L^2(I)}^2 \ p^2 \ (p+1) \left( \frac{2p+1}{2} \right)
\]

\[
\leq \left\| \rho \right\|_{L^2(I)}^2 \ p^2 \ (p+p) \left( \frac{2p+p}{2} \right) \ \forall p > 0
\]

\[
\leq 3p^4 \left\| \rho \right\|_{L^2(I)}^2
\]
Proof Step 2 finished

• From the result for $I=[-1, 1]$

$$\left\| \frac{d \rho}{dx} \right\|_{L^2(I)}^2 \leq 3 p^4 \| \rho \|_{L^2(I)}^2$$

• a standard scaling argument (change of variables) gives us the final result..

$$\left\| \frac{d \rho}{dx} \right\|_{L^2(I)} \leq \sqrt{3} p^2 \frac{2}{h} \| \rho \|_{L^2(I)}$$
Lax-Friedrichs Fluxes

• Suppose we know the maximum wave speed of the hyperbolic system:

\[
\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = 0
\]

• As long as the matrix \( A \in \mathbb{R}^n \times \mathbb{R}^n \) has all real eigenvalues (i.e. the system is hyperbolic) and

\[
\max \left\{ |\lambda| \mid Ax = \lambda x \text{ for some } x \in \mathbb{R}^n \right\} \leq \tilde{\lambda}
\]

• then we can use the following DG scheme:
DG Scheme with Lax-Friedrichs Fluxes

Not that we are going to use this in the end, but for ease let’s change basis with the diagonalizing matrix. Then consider a single component $q$ (associated with eigenvalue lambda) :

$$
\int_{x_j}^{x_{j+1}} \phi \frac{\partial q_j}{\partial t} \, dx + \int_{x_j}^{x_{j+1}} \phi \lambda \frac{\partial q_j}{\partial x} \, dx =
$$

$$
- \phi(x_{j+1}) \left( \frac{\lambda - \tilde{\lambda}}{2} \right) \left( q_{j+1}(x_{j+1}) - q_j(x_{j+1}) \right)
$$

$$
- \phi(x_j) \left( \frac{-\lambda - \tilde{\lambda}}{2} \right) \left( q_{j-1}(x_j) - q_j(x_j) \right)
$$

Notice that if we set $\tilde{\lambda} = \lambda$ and $\lambda \geq 0$ then the red terms disappear and the scheme reduces to the upwind DG scheme.
DG Scheme with Lax-Friedrichs Fluxes

Set \( \phi = q \)

\[
\int_{x_j}^{x_{j+1}} q_j \frac{\partial q_j}{\partial t} dx + \lambda \int_{x_j}^{x_{j+1}} q_j \frac{\partial q_j}{\partial x} dx = \\
- q_j (x_{j+1}) \left( \frac{\lambda - \tilde{\lambda}}{2} (q_{j+1}(x_{j+1}) - q_j(x_{j+1})) \right) \\
- q_j (x_j) \left( \frac{-\lambda - \tilde{\lambda}}{2} (q_{j-1}(x_j) - q_j(x_j)) \right)
\]
Quick Reminder – Integration by Parts

\[ \int_{x_j}^{x_{j+1}} q_j^t \frac{\partial q_j}{\partial x} dx = \frac{1}{2} \left[ q_j^2 \right]_{x_j}^{x_{j+1}} \]
DG Scheme with Lax-Friedrichs Fluxes

Rephrasing the time derivative and space derivative

\[
\frac{1}{2} \frac{d}{dt} \int_{x_j}^{x_{j+1}} q_j^2 dx = -\frac{\lambda}{2} \left[ q_j^2 \right]_{x_j}^{x_{j+1}}
\]

\[
-q_j(x_{j+1}) \left( \frac{\lambda - \tilde{\lambda}}{2} \right) \left( q_{j+1}(x_{j+1}) - q_j(x_{j+1}) \right)
\]

\[
-q_j(x_j) \left( \frac{-\lambda - \tilde{\lambda}}{2} \right) \left( q_{j-1}(x_j) - q_j(x_j) \right)
\]
DG Scheme with Lax-Friedrichs Fluxes

Summing over all the cells:

\[
\frac{d}{dt} \sum_{j=1}^{j=N-1} \int_{x_j}^{x_{j+1}} q_j^2 \, dx = - \sum_{j=1}^{j=N-1} \left\{ \lambda \left[ q_j^2 \right]_{x_j}^{x_{j+1}} \right. \\
\left. \quad + (\lambda - \tilde{\lambda}) q_j \left( x_{j+1} \right) \left( q_{j+1} \left( x_{j+1} \right) - q_j \left( x_{j+1} \right) \right) \right. \\
\left. \quad + (-\lambda - \tilde{\lambda}) q_j \left( x_j \right) \left( q_{j-1} \left( x_j \right) - q_j \left( x_j \right) \right) \right\}
\]

\[
\frac{d}{dt} \sum_{j=1}^{j=N-1} \int_{x_j}^{x_{j+1}} q_j^2 \, dx = - \sum_{j=1}^{j=N-1} \left\{ \lambda \left( q_j^2 \left( x_{j+1} \right) - q_j^2 \left( x_j \right) \right) \right. \\
\left. \quad + (\lambda - \tilde{\lambda}) q_j \left( x_{j+1} \right) \left( q_{j+1} \left( x_{j+1} \right) - q_j \left( x_{j+1} \right) \right) \right. \\
\left. \quad + (-\lambda - \tilde{\lambda}) q_j \left( x_j \right) \left( q_{j-1} \left( x_j \right) - q_j \left( x_j \right) \right) \right\}
\]
DG Scheme with Lax-Friedrichs Fluxes

Rephrase in terms of $x_j$ (ignoring bc nodes)

\[
\frac{d}{dt} \sum_{j=1}^{j=N-1} \int_{x_j}^{x_{j+1}} q_j^2 \, dx = - \sum_{j=1}^{j=N-1} \left\{ \lambda \left( -q_j^2 (x_j) \right) \right. \\
\left. + \left( -\lambda - \tilde{\lambda} \right) q_j (x_j) \left( q_{j-1} (x_j) - q_j (x_j) \right) \right\} \\
- \sum_{j=1}^{j=N-1} \left\{ \lambda \left( q_j^2 (x_{j+1}) \right) \\
+ \left( \lambda - \tilde{\lambda} \right) q_j (x_{j+1}) \left( q_{j+1} (x_{j+1}) - q_j (x_{j+1}) \right) \right\}
\]

\[
= - \sum_{j=1}^{j=N-1} \left\{ \tilde{\lambda} q_j^2 (x_j) \\
+ \left( -\lambda - \tilde{\lambda} \right) q_j (x_j) q_{j-1} (x_j) \right\} \\
- \sum_{j=2}^{j=N} \left\{ \tilde{\lambda} q_{j-1}^2 (x_j) \\
+ \left( \lambda - \tilde{\lambda} \right) q_{j-1} (x_j) q_j (x_j) \right\}
\]
DG Scheme with Lax-Friedrichs Fluxes

Rephrase in terms of $x_j$

\[
\frac{d}{dt} \sum_{j=1}^{j=N-1} \int_{x_j}^{x_{j+1}} q_j^2 dx = - \sum_{j=1}^{j=N-1} \left\{ \tilde{\lambda} q_j^2 (x_j, t) + \left( -\lambda - \tilde{\lambda} \right) q_j(x_j, t) q_{j-1}(x_j, t) \right\} \\
- \sum_{j=2}^{j=N} \left\{ \tilde{\lambda} q_{j-1}^2 (x_j, t) + \left( \lambda - \tilde{\lambda} \right) q_{j-1}(x_j, t) q_j(x_j, t) \right\} \\
= - \sum_{j=1}^{j=N-1} \left\{ \tilde{\lambda} q_j^2 (x_j, t) + \tilde{\lambda} q_{j-1}^2 (x_j, t) \right\} \\
- 2 \tilde{\lambda} q_j(x_j, t) q_{j-1}(x_j, t) \\
+ \text{some boundary terms} \\
= - \tilde{\lambda} \sum_{j=1}^{j=N-1} \left( q_j(x_j, t) - q_{j-1}(x_j, t) \right)^2 + \text{some boundary terms}
\]
Summary Stability

• Ignoring the boundary terms:

\[
\frac{d}{dt} \left\{ \sum_{j=1}^{j=N-1} x_{j+1} \int_{x_j} q_j^2 dx + \tilde{\lambda} \int_0^t \left( q_j(x_j, \tilde{t}) - q_{j-1}(x_j, \tilde{t}) \right)^2 d\tilde{t} \right\} = 0 \text{ plus some boundary terms}
\]
Back To The System

• Recall our goal was to use the Lax-Friedrichs fluxes for a system.

• We just proved stability for a characteristic component of the state vector.

• We now revert back to the full system and the scheme is:

\[
\begin{align*}
\int_{x_j}^{x_{j+1}} \phi \frac{\partial q_j}{\partial t} \, dx + \int_{x_j}^{x_{j+1}} \phi A \frac{\partial q_j}{\partial x} \, dx &= \\
- \phi(x_{j+1}) \left( \frac{(A - \tilde{\lambda} I)}{2} \left( q_{j+1}(x_{j+1}) - q_j(x_{j+1}) \right) \right) \\
- \phi(x_j) \left( \frac{(-A - \tilde{\lambda} I)}{2} \left( q_{j-1}(x_j) - q_j(x_j) \right) \right)
\end{align*}
\]
Discrete Version

- In each cell we keep a vector of Legendre coefficients for each component of $\mathbf{q}$

\[
\frac{x_{j+1} - x_j}{2} \int_{-1}^{1} L_n L_m d\tilde{x} \frac{d\mathbf{q}_{j,m}}{dt} + \int_{-1}^{1} L_n \frac{dL_m}{dx} d\tilde{x} \mathbf{A} \mathbf{q}_{j,m} =
\]

\[
- \left( \frac{(\mathbf{A} - \tilde{\lambda} \mathbf{I})}{2} \left( (-1)^m \mathbf{q}_{j+1,m} - \mathbf{q}_j \left( x_{j+1} \right) \right) \right)
\]

\[
- (-1)^n \left( \frac{(-\mathbf{A} - \tilde{\lambda} \mathbf{I})}{2} \left( \mathbf{q}_{j-1,m} - (-1)^m \mathbf{q}_{j,m} \right) \right)
\]
Homework 6 (due 02/21)

• Form into pairs

• Each pair will code a Lax-Friedrichs solver for an arbitrary size system – using an arbitrary order Runge-Kutta time integration and arbitrary order Legendre approximations.

• Create a non-trivial hyperbolic system

• Concoct a set of test cases – at least two test case.

• Verify convergence with decreasing cell size and with increasing polynomial order.
Presentation

• On 02/21 you will be required to make a presentation.

• Be ready to explain your approaches, difficulties……

• You will be given time in the 02/19 class to work on your code.