MA557/MA578/CS557
Lecture 30

Spring 2003
Prof. Tim Warburton
timwar@math.unm.edu
Special Edition of ANUM on Absorbing Boundary Conditions

Applied Numerical Mathematics

Volume 27, Issue 4, Pages 327-560 (August 1998)
Special Issue on Absorbing Boundary Conditions
Edited by Eli Turkel
How Thick Does the PML Region Need To Be

• Suppose we consider the region in blue [a,a+\(\delta\)]
• And we set:

\[
\sigma_x = \sigma_x^* = C \left( \frac{x-a}{\delta} \right)^n , \sigma_y = \sigma_y^* = 0
\]
Good Papers on PML Stability


Today

- Last class we examined Berenger’s split field, PML, TE Maxwell’s equations.

- Berenger introduced anisotropic dissipative terms which allow plane waves to pass into an absorbing region without reflection.

- However – Abarbanel, Gottlieb, and Hesthaven later showed that the split PML may suffer explosive instability due to the fact that it is only weakly well posed:


- Yet – even later, Becache and Joly showed that the split PML has at worst a linearly growing solution in the late-time.


Catalogue of Some PMLs

• There are three well known PML formulations.

1) Berenger’s split PML

2) Ziolkowski’s PML based on a Lorentz material.

3) Abarbanel & Gottlieb’s mathematically derived PML.
Recall Berenger’s Split PML

$H_{zx}, H_{zy}$ are defined so that $H_z = \left( H_{zx} + H_{zy} \right)$

and:

$$\frac{\partial E_x}{\partial t} - \frac{\partial}{\partial y} \left( H_{zx} + H_{zy} \right) = -\sigma_y E_x$$

$$\frac{\partial E_y}{\partial t} + \frac{\partial}{\partial x} \left( H_{zx} + H_{zy} \right) = -\sigma_x E_y$$

$$\frac{\partial H_{zx}}{\partial t} + \frac{\partial E_y}{\partial x} = -\sigma_x^* H_{zx}$$

$$\frac{\partial H_{zy}}{\partial t} - \frac{\partial E_x}{\partial y} = -\sigma_y^* H_{zy}$$
Recall: Wave Speeds

- In the previous notation we looked at eigenvalues of linear combination of the flux matrices:

\[
A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\Rightarrow C = \begin{pmatrix} 0 & 0 & -\beta & -\beta \\ 0 & 0 & \alpha & \alpha \\ 0 & \alpha & 0 & 0 \\ -\beta & 0 & 0 & 0 \end{pmatrix}
\]

- The eigenvalues computed by Matlab:

i.e. 0,0,1,-1 under constraint on (alpha,beta)

- So for Lax-Friedrichs we take \( \tilde{\lambda} = 1 \)
Eigenvectors of $\mathbf{C}$

- Using Matlab we can determine the eigenvectors of $\mathbf{C}$

```matlab
>> a = sym('a');
>> b = sym('b');
>> C = [[0,0,-b,-b],[0,0,a,a],[0,a,0,0],[-b,0,0,0]];
>> [v,d]=eig(C);
```

$v = \begin{bmatrix}
0, & -b(a^2+b^2)^{(1/2)/a^2}, & b(a^2+b^2)^{(1/2)/a^2} \\
0, & 1/a(a^2+b^2)^{(1/2)/a^2}, & -1/a(a^2+b^2)^{(1/2)/a^2} \\
-1, & 1, & 1 \\
1, & 1/a^2*b^2, & 1/a^2*b^2 \\
\end{bmatrix}$

- So $\mathbf{C}$ does not have a full space of eigenvectors, which in turn means that $\mathbf{C}$ cannot be diagonalized.
- So the split PML equations are hyperbolic but only weakly well posed.

$$\begin{pmatrix}
-b & a \\
a & -a \\
a^2 & a^2 \\
b^2 & b^2
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
-1 \\
1
\end{pmatrix}$$

where $|\mathbf{a}| = \sqrt{a^2 + b^2}$
Ziolkowski’s PML

• Ziolkowski proposed a method based on a physical polarized absorbing Lorenz material.

Lorentz Material Model Based PML

• Introduce 3 new auxiliary variables $K, J_x, J_y$:

\[
\frac{\partial E_x}{\partial t} - \frac{\partial H_z}{\partial y} = -\sigma_y E_x
\]
\[
\frac{\partial E_y}{\partial t} + \frac{\partial H_z}{\partial x} = -\sigma_x E_y
\]
\[
\frac{\partial H_z}{\partial t} - \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} = -\left(\sigma_x + \sigma_y\right)H_z - K
\]
\[
\frac{\partial J_x}{\partial t} = -\sigma_x \frac{\partial H_z}{\partial y}
\]
\[
\frac{\partial J_y}{\partial t} = +\sigma_y \frac{\partial H_z}{\partial x}
\]
\[
\frac{\partial K}{\partial t} = +\sigma_x \sigma_y H_z
\]
Ziolkowski’s Lorentz Material Model Based PML

• Modification proposed by Abarbanel and Gottlieb:

\[
\begin{align*}
\frac{\partial E_x}{\partial t} - \frac{\partial H_z}{\partial y} &= -\sigma_y E_x \\
\frac{\partial E_y}{\partial t} + \frac{\partial H_z}{\partial x} &= -\sigma_x E_y \\
\frac{\partial H_z}{\partial t} - \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} &= -\left(\sigma_x + \sigma_y\right) H_z - K \\
\frac{\partial P_x}{\partial t} &= -\sigma_x \sigma_y E_x \\
\frac{\partial P_y}{\partial t} &= -\sigma_x \sigma_y E_x \\
\frac{\partial K}{\partial t} &= +\sigma_x \sigma_y H_z
\end{align*}
\]

Set:

\[
P_x = J_x + \sigma_x E_x \\
P_y = J_y + \sigma_y E_y
\]
Reconfigured Lorentz Material Model Based PML

• Note that now the corrections are all lower order terms:

\[
\begin{align*}
\frac{\partial E_x}{\partial t} - \frac{\partial H_z}{\partial y} &= -\sigma_y E_x \\
\frac{\partial E_y}{\partial t} + \frac{\partial H_z}{\partial x} &= -\sigma_x E_y \\
\frac{\partial H_z}{\partial t} - \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} &= -\left(\sigma_x + \sigma_y\right) H_z - K \\
\frac{\partial P_x}{\partial t} &= -\sigma_x \sigma_y E_x \\
\frac{\partial P_y}{\partial t} &= -\sigma_x \sigma_y E_x \\
\frac{\partial K}{\partial t} &= +\sigma_x \sigma_y H_z
\end{align*}
\]
Abarbanel and Gottlieb’s PML

- Abarbanel and Gottlieb proposed a mathematically derived PML.
- Like the Ziolkowski’s PML it is constructed by adding lower order terms and auxiliary variables.
Abarbanel & Gottlieb’s PML

\[
\frac{\partial E_x}{\partial t} - \frac{\partial H_z}{\partial y} = -2\sigma_y E_x - \sigma_y P_y
\]

\[
\frac{\partial E_y}{\partial t} + \frac{\partial H_z}{\partial x} = -2\sigma_x E_y - \sigma_x P_x
\]

\[
\frac{\partial H_z}{\partial t} - \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} = + \left( \frac{d\sigma_x}{dx} \right) Q_x + \left( \frac{d\sigma_y}{dy} \right) Q_y
\]

\[
\frac{\partial P_x}{\partial t} = \sigma_x E_y
\]

\[
\frac{\partial P_y}{\partial t} = \sigma_y E_x
\]

\[
\frac{\partial Q_x}{\partial t} = -\sigma_x Q_x - E_y
\]

\[
\frac{\partial Q_y}{\partial t} = -\sigma_y Q_y - E_x
\]
Converting Berenger To An Unsplit PML

• It is possible to start from the Berenger split PML and return to the Maxwell’s TE equations with additional, lower order terms.

• See:

Berenger To Robust PML

- We will start with the Berenger PML equations:

\[
H_z = (H_{zx} + H_{zy})
\]

\[
\frac{\partial E_x}{\partial t} - \frac{\partial H_z}{\partial y} = -\sigma_y E_x
\]

\[
\frac{\partial E_y}{\partial t} + \frac{\partial H_z}{\partial x} = -\sigma_x E_y
\]

\[
\frac{\partial H_{zx}}{\partial t} + \frac{\partial E_y}{\partial x} = -\sigma_x^* H_{zx}
\]

\[
\frac{\partial H_{zy}}{\partial t} - \frac{\partial E_x}{\partial y} = -\sigma_y^* H_{zy}
\]
Next we Fourier transform in time: $f(\omega) = \int e^{i\omega t} f(t) dt$

\[
\begin{align*}
\hat{H}_z = \hat{H}_x + \hat{H}_y \\
\frac{\partial \hat{H}_x}{\partial y} - \frac{\partial \hat{H}_y}{\partial x} &= -\sigma_x \hat{E}_x \\
\frac{\partial \hat{H}_y}{\partial z} + \frac{\partial \hat{H}_z}{\partial x} &= -\sigma_y \hat{E}_y \\
\frac{\partial \hat{H}_z}{\partial t} + \frac{\partial \hat{E}_x}{\partial t} &= -\sigma_z \hat{E}_x \\
\frac{\partial \hat{H}_x}{\partial z} + \frac{\partial \hat{E}_y}{\partial t} &= -\sigma_z \hat{E}_y \\
\frac{\partial \hat{E}_x}{\partial y} - \frac{\partial \hat{E}_y}{\partial x} &= \omega \sigma_x \\
\frac{\partial \hat{E}_y}{\partial z} + \frac{\partial \hat{E}_z}{\partial x} &= \omega \sigma_y \\
\frac{\partial \hat{E}_z}{\partial t} + \frac{\partial \hat{H}_x}{\partial t} &= \omega \sigma_z
\end{align*}
\]
Berenger To Robust PML

• Gather like terms:

\[
i\omega \hat{E}_x - \frac{\partial \hat{H}_z}{\partial y} = -\sigma_y \hat{E}_x
\]
\[
i\omega \hat{E}_y + \frac{\partial \hat{H}_z}{\partial x} = -\sigma_x \hat{E}_y
\]
\[
i\omega \hat{H}_{zx} + \frac{\partial \hat{E}_y}{\partial x} = -\sigma_x \hat{H}_{zx}
\]
\[
i\omega \hat{H}_{zy} - \frac{\partial \hat{E}_x}{\partial y} = -\sigma_y \hat{H}_{zy}
\]

\[
(i\omega + \sigma_y) \hat{E}_x - \frac{\partial \hat{H}_z}{\partial y} = 0
\]
\[
(i\omega + \sigma_x) \hat{E}_y + \frac{\partial \hat{H}_z}{\partial x} = 0
\]
\[
(i\omega + \sigma_x) \hat{H}_{zx} + \frac{\partial \hat{E}_y}{\partial x} = 0
\]
\[
(i\omega + \sigma_y) \hat{H}_{zy} - \frac{\partial \hat{E}_x}{\partial y} = 0
\]
Berenger To Robust PML

- Multiply Hzx, Hzy terms with new factors:

\[
(i\omega + \sigma_x) \hat{E}_x - \frac{\partial \hat{H}_z}{\partial y} = 0
\]
\[
(i\omega + \sigma_x) \hat{E}_y + \frac{\partial \hat{H}_z}{\partial x} = 0
\]
\[
(i\omega + \sigma_x) \hat{H}_{zx} + \frac{\partial \hat{E}_y}{\partial x} = 0
\]
\[
(i\omega + \sigma_y) \frac{\partial \hat{E}_x}{\partial y} = 0
\]
Berenger To Robust PML

- Eliminate split variables:

\[
\begin{align*}
(i\omega + \sigma_y) \hat{E}_x - \frac{\partial \hat{H}_z}{\partial y} &= 0 \\
(i\omega + \sigma_x) \hat{E}_y + \frac{\partial \hat{H}_z}{\partial x} &= 0 \\
(i\omega + \sigma_y)(i\omega + \sigma_x) \hat{H}_{zx} + (i\omega + \sigma_y) \frac{\partial \hat{E}_y}{\partial x} &= 0 \\
(i\omega + \sigma_x)(i\omega + \sigma_y) \hat{H}_{zy} - (i\omega + \sigma_x) \frac{\partial \hat{E}_x}{\partial y} &= 0
\end{align*}
\]

\[
\begin{align*}
i\omega \left(1 + \frac{\sigma_y}{i\omega}\right) \hat{E}_x - \frac{\partial \hat{H}_z}{\partial y} &= 0 \\
i\omega \left(1 + \frac{\sigma_x}{i\omega}\right) \hat{E}_y + \frac{\partial \hat{H}_z}{\partial x} &= 0 \\
(i\omega + \sigma_y)(i\omega + \sigma_x) \hat{H}_z - i\omega \left(1 + \frac{\sigma_x}{i\omega}\right) \frac{\partial \hat{E}_x}{\partial y} + i\omega \left(1 + \frac{\sigma_y}{i\omega}\right) \frac{\partial \hat{E}_y}{\partial x} &= 0
\end{align*}
\]

22
Berenger To Robust PML

- Expand out Hz terms in 3rd equation:
Berenger To Robust PML

- Expand out Hz terms in 3rd equation:

\[
\begin{align*}
  i\omega \left(1 + \frac{\sigma_y}{i\omega}\right) \hat{E}_x - \frac{\partial \hat{H}_z}{\partial y} &= 0 \\
  i\omega \left(1 + \frac{\sigma_x}{i\omega}\right) \hat{E}_y + \frac{\partial \hat{H}_z}{\partial x} &= 0 \\
  \left( (i\omega)^2 + (\sigma_x + \sigma_y)(i\omega) + \sigma_x \sigma_y \right) \hat{H}_z - i\omega \left(1 + \frac{\sigma_x}{i\omega}\right) \frac{\partial \hat{E}_x}{\partial y} + i\omega \left(1 + \frac{\sigma_y}{i\omega}\right) \frac{\partial \hat{E}_y}{\partial x} &= 0
\end{align*}
\]

- Also divide 3rd by \(i^*\omega\):
Berenger To Robust PML

- Create Auxiliary variables and substitute into PML

\[
\begin{align*}
\hat{P}_x & := \frac{1}{i\omega} \hat{E}_x \\
\hat{P}_y & := \frac{1}{i\omega} \hat{E}_y \\
\hat{Q}_z & := \frac{1}{i\omega} \hat{H}_z \\
\end{align*}
\]

\[
\begin{align*}
\left( i\omega + \left( \sigma_x + \sigma_y \right) + \frac{\sigma_x \sigma_y}{i\omega} \right) \hat{H}_z & = \left( 1 + \frac{\sigma_x}{i\omega} \right) \frac{\partial \hat{E}_x}{\partial y} + \left( 1 + \frac{\sigma_y}{i\omega} \right) \frac{\partial \hat{E}_y}{\partial x} \quad & (1) \\
\left( i\omega + \left( \sigma_x + \sigma_y \right) \right) \hat{H}_z & + \sigma_x \sigma_y \hat{Q}_z = \frac{\partial \left( \hat{E}_x + \sigma_x \hat{P}_x \right)}{\partial y} + \frac{\partial \left( \hat{E}_y + \sigma_y \hat{P}_y \right)}{\partial x} \quad & (2)
\end{align*}
\]

\[
\begin{align*}
i\omega \left( 1 + \frac{\sigma_y}{i\omega} \right) \hat{E}_x - \frac{\partial \hat{H}_z}{\partial y} & = 0 \\
i\omega \left( 1 + \frac{\sigma_x}{i\omega} \right) \hat{E}_y + \frac{\partial \hat{H}_z}{\partial x} & = 0
\end{align*}
\]

\[
\begin{align*}
i\omega \left( 1 + \frac{\sigma_y}{i\omega} \right) \hat{E}_x - \frac{\partial \hat{H}_z}{\partial y} & = 0 \\
i\omega \left( 1 + \frac{\sigma_x}{i\omega} \right) \hat{E}_y + \frac{\partial \hat{H}_z}{\partial x} & = 0
\end{align*}
\]
Berenger To Robust PML

- Inverse Fourier transform:

\[
\begin{align*}
&i\omega \left( 1 + \frac{\sigma_y}{i\omega} \right) \hat{E}_x - \frac{\partial \hat{H}_z}{\partial y} = 0 \\
&i\omega \left( 1 + \frac{\sigma_x}{i\omega} \right) \hat{E}_y + \frac{\partial \hat{H}_z}{\partial x} = 0 \\
&\left( (i\omega) + (\sigma_x + \sigma_y) \right) \hat{H}_z + \sigma_x \sigma_y \hat{Q}_z - \frac{\partial}{\partial y} \left( \hat{E}_x + \sigma_x \hat{P}_x \right) + \frac{\partial}{\partial x} \left( \hat{E}_y + \sigma_y \hat{P}_y \right) = 0 \\
&i\omega \hat{P}_x = \hat{E}_x \\
&i\omega \hat{P}_y = \hat{E}_y \\
&i\omega \hat{Q}_z = \hat{H}_z
\end{align*}
\]
Berenger To Robust PML

- Inverse Fourier transform:

\[
\frac{\partial E_x}{\partial t} + \sigma_y E_x - \frac{\partial H_z}{\partial y} = 0
\]

\[
\frac{\partial E_y}{\partial t} + \sigma_x E_y + \frac{\partial H_z}{\partial x} = 0
\]

\[
\frac{\partial H_z}{\partial t} + \left(\sigma_x + \sigma_y\right) H_z + \sigma_x \sigma_y Q_z - \frac{\partial \left( E_x + \sigma_x P_x \right)}{\partial y} + \frac{\partial \left( E_y + \sigma_y P_y \right)}{\partial x} = 0
\]

\[
\frac{\partial P_x}{\partial t} = E_x
\]

\[
\frac{\partial P_y}{\partial t} = E_y
\]

\[
\frac{\partial Q_z}{\partial t} = H_z
\]
Berenger To Robust PML

- Change of variables:
  \[ \tilde{E}_x = E_x + \sigma_x P_x \]
  \[ \tilde{E}_y = E_y + \sigma_y P_y \]
  \[ \tilde{E}_x = H_z \]

- Manipulate equations:
  \[ \frac{\partial \tilde{E}_x}{\partial t} - \frac{\partial \tilde{H}_z}{\partial y} = - (\sigma_y - \sigma_x) \tilde{E}_x - \sigma_x (\sigma_x - \sigma_y) P_x \]
  \[ \frac{\partial \tilde{E}_y}{\partial t} + \frac{\partial \tilde{H}_z}{\partial x} = - (\sigma_x - \sigma_y) \tilde{E}_y - \sigma_y (\sigma_y - \sigma_x) P_y \]
  \[ \frac{\partial \tilde{H}_z}{\partial t} + \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} = - (\sigma_x + \sigma_y) \tilde{H}_z - \sigma_x \sigma_y Q_z \]
  \[ \frac{\partial P_x}{\partial t} = \tilde{E}_x - \sigma_x P_x \]
  \[ \frac{\partial P_y}{\partial t} = \tilde{E}_y - \sigma_y P_y \]
  \[ \frac{\partial Q_z}{\partial t} = \tilde{H}_z \]
Comments

• Notice that the additional auxiliary variables $P_x, P_y, Q_z$ are defined as solutions of ODEs.

• Corrections to TE Maxwell’s are linear corrections in $E_x, E_y, H_z, P_x, P_y, Q_z \rightarrow$ strongly hyperbolic equations $\rightarrow$ well posed.

• Technically, one should verify that this is still a PML.

\[
\frac{\partial \tilde{E}_x}{\partial t} - \frac{\partial \tilde{H}_z}{\partial y} = -(\sigma_y - \sigma_x) \tilde{E}_x - \sigma_x (\sigma_x - \sigma_y) P_x \\
\frac{\partial \tilde{E}_y}{\partial t} + \frac{\partial \tilde{H}_z}{\partial x} = -(\sigma_x - \sigma_y) \tilde{E}_y - \sigma_y (\sigma_y - \sigma_x) P_y \\
\frac{\partial \tilde{H}_z}{\partial t} + \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} = -(\sigma_x + \sigma_y) \tilde{H}_z - \sigma_x \sigma_y Q_z \\
\frac{\partial P_x}{\partial t} = \tilde{E}_x - \sigma_x P_x \\
\frac{\partial P_y}{\partial t} = \tilde{E}_y - \sigma_y P_y \\
\frac{\partial Q_z}{\partial t} = \tilde{H}_z
\]
Surce Term Stability

- We can verify that the source matrix is a non-positive matrix:

$$
\begin{bmatrix}
-b+a & 0 & 0 & -a^*(-b+a) & 0 & 0 \\
0 & b-a & 0 & 0 & -b^*(b-a) & 0 \\
0 & 0 & -a-b & 0 & 0 & -a^*b \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

$$
\text{eig}(c)
$$

Eigenvalues are:

$$
0,0,-\sigma_x,-\sigma_x,-\sigma_y,-\sigma_y
$$
Eigenvectors

Full set of vectors (at least in the corners):

\[
\begin{pmatrix}
0 & 0 & -\sigma_y + \sigma_x & 0 & 0 & \sigma_x & 0 \\
\sigma_y - \sigma_x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma_x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Message on Derivation of a General PML

• After reviewing the literature it appears that there is a certain art to constructing a PML for a given set of PDEs.

• For a possible generic approach see:

• Hagstrom et al:

  • http://www.math.unm.edu/~hagstrom/papers/aero.ps
  • http://www.math.unm.edu/~hagstrom/papers/newpml.ps