MA557/MA578/CS557
Lecture 6

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Next Lecture (6)

- Alternative flux formulations
- Alternative time stepping schemes
• In detail:

\[
\frac{d}{dt} \int_{a}^{b} \rho(x,t) \, dx = -u(b,t) \rho(b,t) + u(a,t) \rho(a,t)
\]

The time rate of change of total mass in the section of pipe \([a,b]\)

The flux \textit{out} of the section at the \textbf{right} end of the section of pipe per unit time

The flux \textit{into} the section at the \textbf{left} end of the section of pipe per unit time
Recall We Wish to Discretize:

\[
\frac{d}{dt}(d\bar{q}_i(t)) = -\bar{u}q(x_{i+1},t) + \bar{u}q(x_i,t)
\]

1) We will consider alternatives for discretizing the time derivative on the left hand side

2) We will consider alternatives for discretizing the point values of the density on the right hand side.
Lax-Friedrichs Fluxes

• Consider the alternative:

\[
\frac{d}{dt}(dxq_i(t)) = -\bar{u}q(x_{i+1},t) + \bar{u}q(x_i,t)
\]

\[
\cong -\frac{dt}{2dx}\{\bar{u}q_{i+1} - \bar{u}q_{i-1}\}
\]

Note left biased jump term replaced with centered jump term
Numerical Scheme (LF fluxes)

\[
dx \left( \frac{\bar{\rho}_i^{n+1} - \bar{\rho}_{i-1}^{n} + \bar{\rho}_{i+1}^{n}}{2} \right) = -\frac{1}{2} \left\{ -u \bar{\rho}_{i+1} + u \bar{\rho}_{i-1} \right\}
\]

\[
\bar{\rho}_i^{n+1} = \frac{\bar{\rho}_{i-1}^{n} + \bar{\rho}_{i+1}^{n}}{2} + \frac{dt}{2dx} \left\{ -u \bar{\rho}_{i+1} + u \bar{\rho}_{i-1} \right\}
\]
Numerical Scheme Stability (LF fluxes)

\[ \bar{\rho}_i^{n+1} = \frac{\bar{\rho}_{i-1}^n + \bar{\rho}_{i+1}^n}{2} + \frac{dt}{2dx} (-\bar{u}\rho_{i+1}^n + \bar{u}\rho_{i-1}^n) \]

\[ = + \frac{1}{2} \left( 1 - \frac{\bar{u}dt}{dx} \right) \bar{\rho}_{i+1}^n + \frac{1}{2} \left( 1 + \frac{\bar{u}dt}{dx} \right) \bar{\rho}_{i-1}^n \]

\[ \leq \left( \sum_{i=1}^{N-1} \bar{\rho}_i^{n+1} \right) \leq \frac{1}{2} \left( 1 + \frac{\bar{u}dt}{dx} \right) \sum_{i=1}^{N-1} \left| \bar{\rho}_{i-1}^n \right| + \frac{1}{2} \left( 1 - \frac{\bar{u}dt}{dx} \right) \sum_{i=1}^{N} \left| \bar{\rho}_{i+1}^n \right| \]

triangle inequality, and assuming \( \left| \frac{\bar{u}dt}{2dx} \right| < 1 \)

\[ \leq \frac{1}{2} \left( 1 + \frac{\bar{u}dt}{dx} \right) \left\{ \left( \sum_{i=1}^{N-1} \left| \bar{\rho}_i^n \right| \right) - \left| \bar{\rho}_{N-1}^n \right| + \left| \bar{\rho}_0^n \right| \right\} \]

\[ + \frac{1}{2} \left( 1 - \frac{\bar{u}dt}{dx} \right) \left\{ \left( \sum_{i=1}^{N-1} \left| \bar{\rho}_i^n \right| \right) - \left| \bar{\rho}_1^n \right| + \left| \bar{\rho}_N^n \right| \right\} \]

\[ \leq \left( \sum_{i=1}^{N-1} \left| \bar{\rho}_i^n \right| \right) + \frac{1}{2} \left( 1 + \frac{\bar{u}dt}{dx} \right) \left| \bar{\rho}_0^n \right| + \frac{1}{2} \left( 1 - \frac{\bar{u}dt}{dx} \right) \left| \bar{\rho}_N^n \right| \]

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Summary of LF Stability

\[ \bar{\rho}_{i}^{n+1} = \frac{1}{2} \left( 1 - \frac{\bar{u}dt}{dx} \right) \bar{\rho}_{i+1}^{n} + \frac{1}{2} \left( 1 + \frac{\bar{u}dt}{dx} \right) \bar{\rho}_{i-1}^{n} \]

\[ \downarrow \]

\[ \sum_{i=1}^{N-1} |\bar{\rho}_{i}^{n+1}| \leq \left( \sum_{i=1}^{N-1} |\bar{\rho}_{i}^{n}| \right) + \frac{1}{2} \left( 1 + \frac{\bar{u}dt}{dx} \right) |\bar{\rho}_{0}^{n}| + \frac{1}{2} \left( 1 - \frac{\bar{u}dt}{dx} \right) |\bar{\rho}_{N}^{n}| \]

- This basically shows that the LF fluxes formulation leads to a contraction map – although now we have the slightly trickier issue of dealing with the outflow boundary at \( x_N \).
- To get around this we would most likely drop to the upwind scheme at \( x_N \).
- In that case the scheme satisfies the contraction condition as before (assuming the error for the 0-cell is zero)
- CFL stability condition is
  \[ \left| \frac{\bar{u}dt}{2dx} \right| < 1 \quad \text{i.e.} \quad dt < \frac{2dx}{\bar{u}} \]
Consistency For LF
Local Truncation Error

- Suppose at the beginning of a time step we actually have the exact solution -- one question we can ask is how large is the error committed in the evaluation of the approximate solution at the end of the time step.

\[ \bar{\rho}_i^n = \bar{q}_i^n \]

- i.e. choose

\[ \bar{\rho}_i^{n+1} = \frac{1}{2} \left( 1 - \frac{\bar{u} dt}{dx} \right) \bar{q}_{i+1}^n + \frac{1}{2} \left( 1 + \frac{\bar{u} dt}{dx} \right) \bar{q}_{i-1}^n \]

- Then

\[ R_i^n := \frac{1}{dt} \left( \bar{\rho}_i^{n+1} - \bar{q}_i^{n+1} \right) \]
Taylor Series

- We expand $\bar{q}_{i-1}^n, \bar{q}_{i+1}^n, \bar{q}_i^{n+1}$ about $x_i, t_n$ with Taylor series:

\[
\bar{q}_{i+1}^n = \bar{q}_i^n + dx \left( \frac{\partial q}{\partial x} \right) + \frac{dx^2}{2} \left( \frac{\partial^2 q}{\partial x^2} \right) + O(dx^3)
\]

\[
\bar{q}_{i-1}^n = \bar{q}_i^n - dx \left( \frac{\partial q}{\partial x} \right) + \frac{dx^2}{2} \left( \frac{\partial^2 q}{\partial x^2} \right) + O(dx^3)
\]

\[
\bar{q}_i^{n+1} = \bar{q}_i^n + dt \left( \frac{\partial q}{\partial t} \right) + \frac{dt^2}{2} \left( \frac{\partial^2 q}{\partial t^2} \right) + O(dt^3)
\]
Estimating Truncation Error

• Inserting the formulas for the expanded q’s:

\[
R^n_i := \frac{1}{dt} \left( \frac{1}{2} \left( 1 - \frac{dt}{dx} \bar{u} \right) q^n_{i+1} + \frac{1}{2} \left( 1 + \frac{dt}{dx} \bar{u} \right) q^n_{i-1} - q^{n+1}_i \right) \\
= \frac{1}{dt} \left( \frac{1}{2} \left( 1 - \frac{dt}{dx} \bar{u} \right) \bar{q}^n_i + dx \left( \frac{\partial \bar{q}}{\partial x} \right) + \frac{dx^2}{2} \left( \frac{\partial^2 \bar{q}}{\partial x^2} \right) + O(dx^3) \right) \\
- \left( \bar{q}^n_i + dt \left( \frac{\partial \bar{q}}{\partial t} \right) + \frac{dt^2}{2} \left( \frac{\partial^2 \bar{q}}{\partial t^2} \right) + O(dt^3) \right)
\]
Estimating Truncation Error

- Removing canceling terms:

\[
R^n_i := \frac{1}{dt} \left[ \frac{1}{2} \left( 1 - \frac{dt}{dx} \bar{u} \right) \left\{ \partial \bar{q}_i \partial \bar{q}_i + dx \left( \frac{\partial \bar{q}}{\partial x} \right) + \frac{dx^2}{2} \left( \frac{\partial^2 \bar{q}}{\partial x^2} \right) + O(dx^3) \right\} \right. \\
\left. - \left\{ \partial \bar{q}_i \partial \bar{q}_i + dt \left( \frac{\partial \bar{q}}{\partial t} \right) + \frac{dt^2}{2} \left( \frac{\partial^2 \bar{q}}{\partial t^2} \right) + O(dt^3) \right\} \right]
\]
Simplifying

\[ R^n_i := \frac{1}{dt} \left\{ \frac{1}{2} \left( 1 - \frac{dt}{dx} \bar{u} \right) \left\{ dx \left( \frac{\partial q}{\partial x} \right) + \frac{dx^2}{2} \left( \frac{\partial^2 q}{\partial x^2} \right) + O(dx^3) \right\} \right. \]

\[ \left. - \left\{ dt \left( \frac{\partial q}{\partial t} \right) + \frac{dt^2}{2} \left( \frac{\partial^2 q}{\partial t^2} \right) + O(dt^3) \right\} \right\} \]
Rearranging

\[
R_i^n := \frac{1}{dt} \left\{ \begin{array}{l}
- \frac{dt}{dx} \frac{\partial}{\partial x} \left( \frac{\partial q}{\partial x} \right) + \frac{dx^2}{2} \left( \frac{\partial^2 q}{\partial x^2} \right) + O(dx^3) \\
- \left[ dt \left( \frac{\partial q}{\partial t} \right) + \frac{dt}{2} \left( \frac{\partial^2 q}{\partial t^2} \right) + O(dt^3) \right]
\end{array} \right\}
\]

\[
R_i^n := \frac{1}{dt} \left\{ \begin{array}{l}
\frac{dx^2}{2} \left( \frac{\partial^2 q}{\partial x^2} \right) + O(dx^3) \\
- \left[ \frac{dt}{2} \left( \frac{\partial^2 q}{\partial t^2} \right) + O(dt^3) \right]
\end{array} \right\}
\]
\[ R_i^n := \frac{1}{dt} \left\{ \frac{dx^2}{2} \left( \frac{\partial^2 q}{\partial x^2} \right) + O(dx^3) \right\} \]

\[ R_i^n := \frac{1}{dt} \left\{ \frac{dx^2}{2} \left( \frac{\partial^2 q}{\partial x^2} \right) + O(dx^3) \right\} - \left\{ \frac{dt^2}{2} \left( \frac{\partial^2 q}{\partial t^2} \right) + O(dt^3) \right\} \]

\[ = \frac{1}{dt} \left( \frac{dx^2}{2} - \bar{u} \frac{dt^2}{2} \right) \left( \frac{\partial^2 q}{\partial x^2} \right) + O(dx^3) \]

\[ = \bar{u}dx \left( \frac{dx}{dt} - \frac{dt}{dx} \right) + O(dx^2) \]
Local Truncation Error: LF

\[ R_i^n = \frac{\overline{u dx}}{2} \left( \frac{dx}{dt} - \frac{dt}{dx} \right) + O(dx^2) \]

Leading order behavior of local truncation error is again O(dx)
Consistency + Stability

• By showing that the LF formulation is consistent (R->0 as dt->0) and contraction mapping we have proved convergence…
Class Group Exercise

• Prove the following is consistent and stable:

• Lax-Wendroff formula:

\[
\bar{\rho}_i^{n+1} = \bar{\rho}_i^n - \frac{1}{2}\left( \frac{\bar{u}dt}{dx} \right) \left( \bar{\rho}_{i+1}^n - \bar{\rho}_{i-1}^n \right) + \frac{1}{2}\left( \frac{\bar{u}dt}{dx} \right)^2 \left( \bar{\rho}_{i-1}^n - 2\bar{\rho}_i^n + \bar{\rho}_{i+1}^n \right)
\]

• Describe any difficulties with boundary terms…

• Estimate the order of accuracy

• Volunteer to write results on the board 😊
Lecture 7

• Introducing the Discontinuous Galerkin Method for the advection equation.

• Review of some special univariate polynomials.