Sources on the Internet

• http://www.psc.edu/~burkardt/math2070/lab_09.html#LEGENDRE_POLYNOMIALS

• http://www.efunda.com/math/legendre/index.cfm
This Lecture (7)

• Ok – so far we have used a piecewise constant approximation to the solution of:

\[
\frac{d}{dt}(dx_q(t)) = -\overline{u}q(x_{i+1}, t) + \overline{u}q(x_i, t) \\
\cong \{-\overline{u}q_i + \overline{u}q_{i-1}\}
\]

• We are now going to generalize this so that we may use approach to allow the use of higher order piecewise polynomial approximation
First: Review of Polynomial Approximation

- First we define the n-th order, univariate, polynomial space as the span of the monomials of order less than or equal to n:

\[ P^n = \langle 1, x, x^2, ..., x^n \rangle \]

- We call the set of functions the monomial basis for \( P^n \) \[ \{1, x, x^2, ..., x^n\} \]

- It is certainly not the only choice!

- Any set of \( p \), non-degenerate, linear combinations of these monomials will provide an equivalent basis.
Discontinuous Galerkin Scheme

• Recall that under sufficient conditions (i.e. smoothness) on an interval we obtained the density advection PDE
\[
\frac{\partial q}{\partial t} + \bar{u} \frac{\partial q}{\partial x} = 0
\]

• We again divide the pipe into sections divided by \(x_1, x_2, \ldots, x_N\).

• On each section we are going to solve a weak version of the advection equation.

Find \(\rho_i \in P^p(x_i, x_{i+1})\) such that \(\forall v \in P^p(x_i, x_{i+1})\)
\[
\int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = v(x_i) \bar{u} \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
\]
In the i’th cell we will calculate an approximation of q, denoted by $\rho_i$. Typically, this will be a p’th order polynomial on the i’th interval. We will solve the following weak equation for $\rho_i$ \( i=1,..,N-1 \)

Find $\rho_i \in P^p \left( x_i, x_{i+1} \right)$ such that $\forall v \in P^p \left( x_i, x_{i+1} \right)$

\[
\int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = v(x_i) \bar{u} \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
\]
Reducing DG to Finite Volume

- By choosing $p=0$ (i.e. the space of piecewise constant functions) then DG reduces to FV by:

$$
\int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = v(x_i) \bar{u} \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
$$

Find $\rho_i \in P^0(x_i, x_{i+1})$ such that $\forall v \in P^0(x_i, x_{i+1})$

$$
\int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} \right) dx = \bar{v} \bar{u} \left( \rho_{i-1} - \rho_i \right)
$$

$$
dx \bar{v} \left( \frac{d \rho_i}{dt} \right) = \bar{v} \bar{u} \left( \rho_{i-1} - \rho_i \right)
$$

$$
\frac{d \rho_i}{dt} = \bar{u} \frac{dx}{dx} \left( \rho_{i-1} - \rho_i \right)
$$
Stability of DG

• Recall: 

Find $\rho_i \in P^p(x_i, x_{i+1})$ such that $\forall v \in P^p(x_i, x_{i+1})$

$$\int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = v(x_i) (\rho_{i-1}(x_i) - \rho_i(x_i))$$

• Since $\rho_i \in P^p(x_i, x_{i+1})$ we can make the substitution $v = \rho_i \in P^p(x_i, x_{i+1})$ and find:

$$\int_{x_i}^{x_{i+1}} \rho_i \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = \rho_i(x_i) (\rho_{i-1}(x_i) - \rho_i(x_i))$$
DG Stability cont

- Simplifying

\[
\int_{x_i}^{x_{i+1}} \rho_i \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = \rho_i(x_i) \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
\]

- we obtain:

\[
\frac{d}{dt} \int_{x_i}^{x_{i+1}} \frac{\rho_i^2}{2} dx = -\bar{u} \left( \int_{x_i}^{x_{i+1}} \rho_i \frac{\partial \rho_i}{\partial x} dx \right) + \bar{u} \rho_i(x_i) \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
\]

\[
= -\bar{u} \left[ \frac{\rho_i^2}{2} \right]_{x_i}^{x_{i+1}} + \bar{u} \rho_i(x_i) \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
\]

\[
= \bar{u} \left( -\rho_i(x_i)^2 - \rho_i(x_{i+1})^2 \right) + \bar{u} \rho_i(x_i) \rho_{i-1}(x_i)
\]
DG Stability cont

- Summing over all the cells:

\[
\frac{d}{dt} \sum_{i=1}^{i=N-1} \left( \int_{x_i}^{x_{i+1}} \frac{\rho_i^2}{2} \, dx \right) = \sum_{i=1}^{i=N-1} \left\{ \bar{u} \left( \frac{-\rho_i(x_i)^2 - \rho_i(x_{i+1})^2}{2} \right) + \bar{u} \rho_i(x_i) \rho_{i-1}(x_i) \right\}
\]

\[\leq \sum_{i=1}^{i=N-1} \left\{ \bar{u} \left( \frac{-\rho_i(x_i)^2 - \rho_i(x_{i+1})^2}{2} \right) + \bar{u} \left( \frac{\rho_i(x_i)^2 + \rho_{i-1}(x_i)^2}{2} \right) \right\} \]

\[\leq \frac{\bar{u}}{2} \sum_{i=1}^{i=N-1} \left\{ -\rho_i(x_{i+1})^2 + \rho_{i-1}(x_i)^2 \right\} \]

\[\leq \frac{\bar{u}}{2} \left\{ \rho_0(x_1)^2 - \rho_{N-1}(x_N)^2 \right\} \]

- This **semi-discrete** analysis shows that if the time derivative is discretized exactly then the “energy” is non-increasing in time – except for the flux of density in at \(x_1\).
Defining a Norm

• We define a 2-norm on the interval \([x_1,x_N]\) by:

\[
\|\rho\|_2 = \sqrt{\sum_{i=1}^{i=N-1} \left( \int_{x_i}^{x_{i+1}} \frac{\rho_i^2}{2} \, dx \right)}
\]

• We have just calculated that:

\[
\frac{d}{dt} \left( \frac{\|\rho\|_2^2}{2} \right) = \sum_{i=1}^{i=N-1} \left( \int_{x_i}^{x_{i+1}} \frac{\rho_i^2}{2} \, dx \right) \leq \frac{\bar{u}}{2} \left\{ \rho_0 (x_1)^2 - \rho_{N-1} (x_N)^2 \right\}
\]

• So trivially:

\[
\frac{d}{dt} \|\rho\|_2 \leq \frac{\bar{u}}{2\|\rho\|_2} \left\{ \rho_0 (x_1)^2 - \rho_{N-1} (x_N)^2 \right\}
\]
Comparing Two Solutions

• Suppose we compute two solutions which are initially close, with the same boundary condition imposed weakly at $x_1$ then:

$$\frac{d}{dt} \| \rho - \sigma \|_2 \leq -\bar{u} \left( \rho_{N-1}(x_N) - \sigma_{N-1}(x_N) \right)^2 \frac{\rho_{N-1}(x_N)}{2 \| \rho - \sigma \|_2} \leq 0$$

• i.e. supposing the time derivative is computed exactly then the distance between the two solutions (measured in the $L_2$ norm) will decrease in time


**Consistency**

- We suppose that there is an exact solution $q$ to the advection equation.

- In addition we suppose that there exists a projection function to the polynomial space

\[ \Pi^p : H^1(x_1, x_N) \to \bigcup_{i=1}^{i=N-1} P^p(x_i, x_{i+1}) \]

- Then we consider three equations

\[
\begin{align*}
\int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx &= v(x_i) \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
\end{align*}
\]
Three Key Equations

1) \[ \int_{x_i}^{x_{i+1}} \nu \Pi^p \left( \frac{\partial q_i}{\partial t} + \bar{u} \frac{\partial q_i}{\partial x} \right) = 0 \]

2) \[ \int_{x_i}^{x_{i+1}} \nu \left( \frac{\partial \Pi^p q_i}{\partial t} + \bar{u} \frac{\partial \Pi^p q_i}{\partial x} \right) dx = \nu(x_i) \left( \Pi^p q_{i-1}(x_i) - \Pi^p q_i(x_i) \right) + \int_{x_i}^{x_{i+1}} \nu R_i \]

3) \[ \int_{x_i}^{x_{i+1}} \nu \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = \nu(x_i) \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right) \]

1) We project the exact equation
2) We plug the projection of the exact solution into the numerical operator
3) The basic numerical scheme
Estimating the Truncation Error

1) \[ \int_{x_i}^{x_{i+1}} v \left( \frac{\partial \Pi^p q_i}{\partial t} + \bar{u} \Pi^p \frac{\partial q_i}{\partial x} \right) dx = 0 \]

2) \[ \int_{x_i}^{x_{i+1}} v \left( \frac{\partial \Pi^p q_i}{\partial t} + \bar{u} \frac{\partial \Pi^p q_i}{\partial x} \right) dx = v(x_i) \left( \Pi^p q_{i-1}(x_i) - \Pi^p q_i(x_i) \right) + \int_{x_i}^{x_{i+1}} vR_i \]

1) + 2) \[ \Rightarrow \int_{x_i}^{x_{i+1}} v \left( \bar{u} \frac{\partial \Pi^p q_i}{\partial x} - \bar{u} \Pi^p \frac{\partial q_i}{\partial x} \right) dx = \]

\[ v(x_i) \left( \Pi^p q_{i-1}(x_i) - q_{i-1}(x_i) + q_i(x_i) - \Pi^p q_i(x_i) \right) + \int_{x_i}^{x_{i+1}} vR_i \]

Here we assumed the projection and the time derivative commute and in the jump term we assumed that the exact solution is continuous at \( x_i \)
Estimating the Truncation Error

\[
\int_{x_i}^{x_{i+1}} v R_i = \int_{x_i}^{x_{i+1}} v \left( \frac{\partial \Pi^p q_i}{\partial x} - \bar{u} \frac{\partial q_i}{\partial x} \right) dx - v(x_i) \left( \Pi^p q_{i-1}(x_i) - q_{i-1}(x_i) + q_i(x_i) - \Pi^p q_i(x_i) \right)
\]

Using the fact that the truncation term is in $P^p$, setting $v=R_i$ and using Cauchy-Schwarz, and the triangle inequality:

\[
\| R \|_2 \leq \sum_{i=1}^{i=N-1} \left\| \frac{\partial \Pi^p q_i}{\partial x} - \bar{u} \frac{\partial q_i}{\partial x} \right\|_2 + \\
\sum_{i=1}^{i=N-1} \left\{ \left| \Pi^p q_{i-1}(x_i) - q_{i-1}(x_i) \right| + \left| q_i(x_i) - \Pi^p q_i(x_i) \right| \right\}
\]

Later on we will cover a detailed description of the projection operator – but for now trust me that the difference between the projection of derivative of $q$ and the derivative of the projection of $q$ will be approximately $Ch^s \left\| \frac{d^p q}{dx^p} \right\|_2$ for some order $s$. 
Semi-discrete Convergence Analysis

- Subtracting the third equation from the second equation

\[
2) \int_{x_i}^{x_{i+1}} v \left( \frac{\partial \Pi^p q_i}{\partial t} + \bar{u} \frac{\partial \Pi^p q_i}{\partial x} \right) dx = v(x_i) \left( \Pi^p q_{i-1}(x_i) - \Pi^p q_i(x_i) \right) + \int_{x_i}^{x_{i+1}} vR_i
\]

\[
3) \int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = v(x_i) \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
\]

\[
2) + 3) \int_{x_i}^{x_{i+1}} v \left( \frac{\partial \Pi^p q_i - \rho_i}{\partial t} + \bar{u} \frac{\partial \Pi^p q_i - \rho_i}{\partial x} \right) dx =
\]

\[
v(x_i) \left( \Pi^p q_{i-1}(x_i) - \rho_{i-1}(x_i) - \Pi^p q_i(x_i) + \rho_i(x_i) \right) + \int_{x_i}^{x_{i+1}} vR_i
\]
Semi-discrete Convergence Analysis

• Setting \( v = \Pi^p q_i - \rho_i \)

\[
\int_{x_i}^{x_{i+1}} \left( \Pi^p q_i - \rho_i \right) \left( \frac{\partial \Pi^p q_i - \rho_i}{\partial t} + \bar{u} \frac{\partial \Pi^p q_i - \rho_i}{\partial x} \right) dx = \\
\left( \Pi^p q_i (x_i) - \rho_i (x_i) \right) \left( \Pi^p q_{i-1} (x_i) - \rho_{i-1} (x_i) - \Pi^p q_i (x_i) + \rho_i (x_i) \right) \\
+ \int_{x_i}^{x_{i+1}} \left( \Pi^p q_i - \rho_i \right) R_i
\]

• We can now bound the time growth of

\( \Pi^p q_i - \rho_i \)
\[ \int_{x_i}^{x_{i+1}} \left( \Pi^p q_i - \rho_i \right) \left( \frac{\partial \Pi^p q_i - \rho_i}{\partial t} + \bar{u} \frac{\partial \Pi^p q_i - \rho_i}{\partial x} \right) dx = \left( \Pi^p q_i(x_i) - \rho_i(x_i) \right) \left( \Pi^p q_{i-1}(x_i) - \rho_{i-1}(x_i) - \Pi^p q_i(x_i) + \rho_i(x_i) \right) + \int_{x_i}^{x_{i+1}} \left( \Pi^p q_i - \rho_i \right) R_i \]
Partial Summary

\[
\frac{d}{dt} \left\| \Pi^p q - \rho \right\|_2 \leq \left\| R \right\|_2 \leq Ch^s \left\| \frac{d^p q}{dx^p} \right\|_2
\]

- The consistency analysis basically shows us that the numerical solution \( \rho \), and some projection of the exact solution \( q \) to the same space of polynomials which we use to represent \( \rho \) – grow apart slowly in time.
Full Summary

\[
\frac{d}{dt} \left\| \Pi^p q - \rho \right\|_2 \leq \left\| R \right\|_2 \leq Ch^s \left\| \frac{d^p q}{dx^p} \right\|_2
\]

Integrate to time T:

\[
\left\| \Pi^p q - \rho \right\|_2 \leq \left\| \Pi^p q(t = 0) - \rho(t = 0) \right\|_2 + Ch^s T \max_{0 \leq s \leq T} \left\| \frac{d^p q}{dx^p}(t = s) \right\|_2
\]

Full case:

\[
\left\| q - \rho \right\|_2 = \left\| q - \Pi^p q + \Pi^p q - \rho \right\|_2
\]

\[
\leq \left\| q - \Pi^p q \right\|_2 + \left\| \Pi^p q(t = 0) - \rho(t = 0) \right\|_2 + Ch^s T \max_{0 \leq s \leq T} \left\| \frac{d^p q}{dx^p}(t = s) \right\|_2
\]

a) Part of q which only exists outside of the fixed polynomial space
b) Error in initial guess
c) Accumulation of truncation errors in time
Final Summary

\[ \| q - \rho \|_2 \leq \| q - \Pi^p q \|_2 \]
\[ + \| \Pi^p q(t = 0) - \rho(t = 0) \|_2 \]
\[ + Ch^s T \max_{0 \leq s \leq T} \| \frac{d^p q}{dx^p}(t = s) \|_2 \]

The first term we have no way to deal with, it is the result of our finite polynomial space.

The second term tells us to compute the initial conditions accurately.

The third term tells us that there is a possibility of linear growth of the error in time.
Next: Lecture 8

• Details on polynomial approximation and estimates on errors.