Discontinuous Galerkin Scheme

• Recall that under sufficient conditions (i.e. smoothness) on an interval we obtained the density advection PDE
\[ \frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} = 0 \]

• We again divide the pipe into sections divided by \( x_1, x_2, \ldots, x_N \).

• On each section we are going to solve a weak version of the advection equation.

Find \( \rho_i \in P^p \left( x_i, x_{i+1} \right) \) such that \( \forall v \in P^p \left( x_i, x_{i+1} \right) \)
\[ \int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + u \frac{\partial \rho_i}{\partial x} \right) dx = v \left( x_i \right) \bar{u} \left( \rho_{i-1} \left( x_i \right) - \rho_i \left( x_i \right) \right) \]
In the i’th cell we will calculate an approximation of \( q \), denoted by \( \rho_i \). Typically, this will be a p’th order polynomial on the i’th interval. We will solve the following weak equation for \( \rho_i \) \( \quad i=1,..,N-1 \)

\[
\int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = v(x_i) \bar{u} \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
\]
Discretization of $P^p$

• In order to work with the DG scheme we need to choose a basis for $P^p$ and formulate the scheme as a finite dimensional problem

\[
\text{Find } \rho_i \in P^p \left(x_i, x_{i+1}\right) \text{ such that } \forall v \in P^p \left(x_i, x_{i+1}\right) \quad \int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = v(x_i) \bar{u} \left( \rho_{i-1}(x_i) - \rho_i(x_i) \right)
\]

• We expand the density in terms of the Legendre polynomials

\[
\rho_i = \sum_{m=0}^{m=p} \rho_{i,m} L_m \left( \tilde{x} \right)
\]

where:

$\rho_{i,m}$ is the $m$'th coefficient of the Legendre expansion of the density in the $i$'th cell.

\[
\tilde{x} = \frac{2x - x_i - x_{i+1}}{x_{i+1} - x_i}
\]
Discretization of \( P^p \)

- The scheme:
  
  Find \( \rho_i \in P^p \left( x_i, x_{i+1} \right) \) such that \( \forall v \in P^p \left( x_i, x_{i+1} \right) \)

  \[
  \int_{x_i}^{x_{i+1}} v \left( \frac{\partial \rho_i}{\partial t} + \overline{u} \frac{\partial \rho_i}{\partial x} \right) dx = v \left( x_i \right) \overline{u} \left( \rho_{i-1} \left( x_i \right) - \rho_i \left( x_i \right) \right)
  \]

- Becomes:

Find \( \rho_{i,m} \left\{ m = 0, \ldots, p \right\} \) such that

\[
\sum_{m=0}^{m=p} \left\{ \left( \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n L_m d\tilde{x} \right) \frac{d \rho_{i,m}}{dt} \right\} + \overline{u} \sum_{m=0}^{m=p} \left\{ \left( \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n \frac{dL_m}{dx} d\tilde{x} \right) \rho_{i,m} \right\} = \overline{u} L_n \left( -1 \right) \sum_{m=0}^{m=p} \rho_{i-1,m} L_m \left( 1 \right) - \overline{u} L_n \left( -1 \right) \sum_{m=0}^{m=p} \rho_{i,m} L_m \left( -1 \right)
\]
**Simplify, simplify**

- Starting with:

$$\text{Find } \rho_{i,m} \{m = 0, \ldots, p\} \text{ such that}$$

$$\sum_{m=0}^{m=p} \left\{ \left( \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n L_m \, d\tilde{x} \right) \frac{d\rho_{i,m}}{dt} \right\} + \sum_{m=0}^{m=p} \left\{ \left( \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n \frac{dL_m}{dx} \, d\tilde{x} \right) \rho_{i,m} \right\} = \bar{u} L_n \left( -1 \right)^{m=p} \sum_{m=0}^{m=p} \rho_{i-1,m} L_m \left( 1 \right) - \bar{u} L_n \left( -1 \right)^{m=p} \sum_{m=0}^{m=p} \rho_{i,m} L_m \left( -1 \right)$$

- We define two matrices:

$$M_{nm} = \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n \left( \tilde{x} \right) L_m \left( \tilde{x} \right) \, d\tilde{x}$$

$$D_{nm} = \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n \left( \tilde{x} \right) \frac{d}{dx} L_m \left( \tilde{x} \right) \, d\tilde{x}$$

$$= \frac{dx}{d\tilde{x}} \frac{d\tilde{x}}{dx} \int_{-1}^{1} L_n \left( \tilde{x} \right) \frac{d}{d\tilde{x}} L_m \left( \tilde{x} \right) \, d\tilde{x}$$

$$= \int_{-1}^{1} L_n \left( \tilde{x} \right) \frac{d}{d\tilde{x}} L_m \left( \tilde{x} \right) \, d\tilde{x}$$
Then:

Find $\{\rho_{i,m} \mid m = 0, \ldots, p\}$ such that

\[
\sum_{m=0}^{p} \left\{ \frac{dL_{m}}{dx} \frac{1}{\bar{u}} \right\} \rho_{i,m} + \sum_{m=0}^{p} \left\{ \frac{1}{\bar{u}} \frac{dL_{m}}{dx} \frac{1}{\bar{u}} \right\} \rho_{i,m} = \bar{u} L_{n} (-1) \sum_{m=0}^{p} \rho L_{m} (-1)
\]

Becomes:

Find $\{\rho_{i,m} \mid m = 0, \ldots, p\}$ such that

\[
\sum_{m=0}^{p} \left\{ M_{nm} \frac{d\rho_{i,m}}{dt} \right\} + \bar{u} \sum_{m=0}^{p} \left\{ \rho_{i,m} \frac{dL_{m}}{dx} \frac{1}{\bar{u}} \right\} = \bar{u} L_{n} (-1) \sum_{m=0}^{p} \rho L_{m} (-1)
\]
Questions

- This raises the following questions how can we compute $M_{nm}$ and $D_{nm}$

- We will need some of the following results.
The Legendre Polynomials
(used to discretize $P^p$)

- Let’s introduce the Legendre polynomials.

- The Legendre $L_m(x), m = 0, 1, \ldots$ polynomials are defined as eigenfunctions of the singular Sturm-Liouville problem:

$$\frac{d}{dx}\left( (1-x)(1+x) \frac{dL_m}{dx}(x) \right) + m(m+1)L_m(x) = 0$$
**Closed Form Formula for $L_m$**

- If we choose $L_m(1) = 1$ then:

  $$L_m(x) = \frac{1}{2^m} \sum_{l=0}^{l=\text{floor}(m/2)} (-1)^l \binom{m}{l} \binom{2m-2l}{m} x^{m-2l}$$

- Examples:

  $$L_0(x) = 1$$

  $$L_1(x) = \frac{1}{2} \begin{pmatrix} 1 \end{pmatrix} = x$$

  $$L_2(x) = \frac{1}{2^2} \begin{pmatrix} 2 \end{pmatrix} \begin{pmatrix} 4 \end{pmatrix} x^2 + (-1)^1 \begin{pmatrix} 2 \end{pmatrix} \begin{pmatrix} 2 \end{pmatrix} x^0 = \frac{3x^2 - 1}{2}$$

- Warning: this is not a stable way to compute the value of a Legendre polynomial (unstable as $m$ increases)
Stable Formula for Computing $L_m$

- We can also use the following recurrence relation to compute $L_m$

$$L_{m+1}(x) = \frac{2m+1}{m+1} xL_m(x) - \frac{m}{m+1} L_{m-1}(x)$$

- Starting with $L_0=1$ $L_1=x$ we obtain:

$$L_2(x) = \frac{3}{2} xL_1(x) - \frac{1}{2} L_0(x) = \frac{1}{2} \left(3x^2 - 1\right)$$

- This is a more stable way to compute the Legendre polynomials at some $x$. 
The Rodriguez Formula

• We can also obtain the m’th order Legendre polynomial using the Rodriguez formula:

\[ L_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} \left(x^2 - 1\right)^m \]
Orthogonality of the Legendre Polynomials

• The Legendre polynomials are orthogonal to each other:

\[
\int_{-1}^{1} L_p(x) L_m(x) \, dx = \int_{-1}^{1} \frac{1}{2^p \, p!} \frac{d^p}{dx^p} (x^2 - 1)^p \frac{1}{2^m \, m!} \frac{d^m}{dx^m} (x^2 - 1)^m \, dx
\]

\[
= \frac{1}{2^p \, p!} \frac{1}{2^m \, m!} \int_{-1}^{1} \frac{d^p}{dx^p} (x^2 - 1)^p \frac{d^m}{dx^m} (x^2 - 1)^m \, dx
\]

\[
= -\frac{1}{2^p \, p!} \frac{1}{2^m \, m!} \int_{-1}^{1} \frac{d^{p-1}}{dx^{p-1}} (x^2 - 1)^p \frac{d^{m+1}}{dx^{m+1}} (x^2 - 1)^m \, dx
\]

\[
= (-1)^p \frac{1}{2^p \, p!} \frac{1}{2^m \, m!} \int_{-1}^{1} (x^2 - 1)^p \frac{d^{m+p}}{dx^{m+p}} (x^2 - 1)^m \, dx
\]

\[
= 0
\]

• Since \( 2m < m+p \)
Miscellaneous Relations

\[ |L_m(x)| \leq 1, \quad -1 \leq x \leq 1 \]

\[ L_m(\pm 1) = (\pm 1)^m \]

\[ \left| \frac{dL_m(x)}{dx} \right| \leq \frac{m(m+1)}{2}, \quad -1 \leq x \leq 1 \]

\[ \left| \frac{dL_m(x)}{dx} \right| = (\pm 1)^m \]

\[ dx \]

\[ \int_{-1}^{1} L_m(x)^2 \, dx = \frac{2}{2m+1} \]
Differentiation

• Suppose we have a Legendre expansion for the density in a cell:

\[ \rho_i = \sum_{m=0}^{m=p} \rho_{i,m} L_m (\tilde{x}) \]

• We can find the coefficients of the Legendre expansion for the derivative of the function as:

\[ \frac{d\rho_i}{d\tilde{x}} (\tilde{x}) = \sum_{m=0}^{m=p} \rho_{i,m}^{(1)} L_m (\tilde{x}) \]

where:

\[ \rho_{i,m}^{(1)} = (2m+1) \sum_{\substack{j=m+1 \atop j+m \text{ odd}}}^{j=p} \rho_{i,j} \]
Coefficient Differentiation Matrix

\[ \rho_{i,m}^{(1)} = \frac{2}{x_{i+1} - x_i} (2m + 1) \sum_{j=m+1}^{j=p} \rho_{i,j} \]

\[ \hat{D}_{mj} = \begin{cases} (2m + 1) & \text{if } j > m \text{ and } j + m \text{ odd} \\ 0 & \text{otherwise} \end{cases} \]

\[ \rho_{i,m}^{(1)} = \frac{2}{x_{i+1} - x_i} \sum_{j=0}^{j=p} \hat{D}_{mj} \rho_{i,j} \]

or \[ \rho_{i,m}^{(1)} = \frac{2}{x_{i+1} - x_i} \hat{D}\rho_i \text{ in matrix-vector notation} \]

i.e. if we create a vector of the Legendre coefficients for rho in the I’th cell then we can compute the Legendre coefficients of the derivative of rho in the I’th cell by pre-multiplying the vector of coefficients with the matrix:

\[ \frac{2}{x_{i+1} - x_i} \hat{D} \]
The D Matrix

- We can use a simple trick to compute D

- Recall:

\[
M_{nm} = \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n(\tilde{x}) L_m(\tilde{x}) d\tilde{x}
\]

\[
D_{nm} = \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n(\tilde{x}) \frac{d}{d\tilde{x}} L_m(\tilde{x}) d\tilde{x}
\]

- We use:

\[
D_{nm} = \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n(\tilde{x}) \frac{d}{d\tilde{x}} L_m(\tilde{x}) d\tilde{x}
\]

\[
= \frac{dx}{d\tilde{x}} \int_{-1}^{1} L_n(\tilde{x}) \frac{d}{d\tilde{x}} L_m(\tilde{x}) d\tilde{x}
\]

\[
= \int_{-1}^{1} L_n(\tilde{x}) \frac{d}{d\tilde{x}} L_m(\tilde{x}) d\tilde{x}
\]

\[
= \sum_{j=0}^{j=p} M_{nj} \hat{D}_{jm}
\]

\[
= \frac{2}{2n+1} \hat{D}_{nm}
\]
The Vandermonde Matrix

• It will be convenient for future applications to be able to transform between data at a set of nodes and the Legendre coefficients of the p’th order polynomial which interpolates the data at these nodes.

• Consider the I’th cell again, but imagine that there are p+1 nodes in that cell then we have:

\[ \rho_i(\tilde{x}_n) = \sum_{m=0}^{m=p} \rho_{i,m} L_m(\tilde{x}_n) \quad n=0,\ldots,p \]

• Notice that the Vandermonde matrix defined by \( V_{nm} = L_m(\tilde{x}_n) \) is square and we can obtain the coefficients by multiplying both sides by the inverse of \( V \)

\[ \rho_i(\tilde{x}_n) = \sum_{m=0}^{m=p} V_{nm} \rho_{i,m} \quad \Rightarrow \quad \rho_{i,m} = \sum_{n=0}^{n=p} (V^{-1})_{mn} \rho_i(\tilde{x}_n) \]
Vandermonde

• Reiterate:
  – that the inverse of the Vandermonde matrix is very useful for transforming nodal data to Legendre coefficients

  – that the Vandermonde matrix is very useful for transforming Legendre coefficients to nodal data
Results So Far

- $0 \leq n, j, m \leq p$

Mass matrix:

$$M_{nm} = \frac{x_{i+1} - x_i}{2} \int_{-1}^{1} L_n(\tilde{x})L_m(\tilde{x}) d\tilde{x} = 0 \quad (n \neq m)$$

$$= \begin{cases} 
\frac{x_{i+1} - x_i}{2} \frac{2}{2n+1} & (n=m) \\
0 & (n \neq m)
\end{cases}$$

Coeff. Differentiation

$$\hat{D}_{mj} = \begin{cases} 
(2m+1) & \text{if } (j>m) \text{ and } (j+m \text{ odd}) \\
0 & \text{otherwise}
\end{cases}$$

Stiffness matrix

$$D_{nm} = \int_{-1}^{1} L_n(\tilde{x}) \frac{dL_m}{d\tilde{x}}(\tilde{x}) d\tilde{x}$$

$$= \left( \frac{2}{2n+1} \right) \hat{D}_{nm}$$

Vandermonde matrix

$$V_{nm} = L_m(x_n)$$
Homework 3

Q1) Code up a function (LEGENDRE) which takes a vector $x$ and $m$ as arguments and computes the $m$’th order Legendre polynomial at $x$ using the stable recurrence relation.

Q2) Create a function (LEGdiff) which takes $p$ as argument and returns the coefficient differentiation matrix $D$.

Q3) Create a function (LEGmass) which takes $p$ as argument and returns the mass matrix $M$. 
Homework 3 cont

Q4) Create a function LEGvdm which returns the following matrix:

$$V_{ij} = L_j(x_i) \quad 0 \leq i, j \leq p$$

Q5) Test them with the following matlab routine:

```
3) Compute the Chebychev nodes
4) Build the Vandermonde matrix using the Chebychev nodes and Legendre polynomials
7) Build the coefficient differentiation matrix
9) Build the Legendre mass matrix
12) Evaluate x^m at the Chebychev nodes
14) Compute the Legendre coefficients
16) Multiply by Dhat (i.e. differentiate in coeff space)
18) Evaluate at the nodes
20) Compute the error in the derivative
```
Homework 3 cont

Q5 cont) Explain the output from this test case
Next Lecture (9)

• We will put everything together to build a 1D DG scheme for advection...