Discontinuous Galerkin Scheme

- Recall that under sufficient conditions (i.e. smoothness) on an interval we obtained the density advection PDE
  \[ \frac{\partial q}{\partial t} + \bar{u} \frac{\partial q}{\partial x} = 0 \]

- We again divide the pipe into sections divided by \( x_1, x_2, \ldots, x_N \).

- On each section we are going to solve a weak version of the advection equation with the following scheme:

Find \( \rho_{i,m} \) such that

\[
\sum_{m=0}^{m=p} \left\{ M_{nm} \frac{d\rho_{i,m}}{dt} \right\} + \sum_{m=0}^{m=p} \bar{u} \{ D_{nm} \rho_{i,m} \} = \bar{u} L_n (-1) \sum_{m=0}^{m=p} \rho_{i-1,m} L_n (1) - \bar{u} L_n (-1) \sum_{m=0}^{m=p} \rho_{i,m} L_n (-1)
\]
Recall we derived the following matrices last lecture

\[
M_{nm} = \begin{cases} 
\frac{x_{i+1} - x_i}{2} & \text{if } j > m \text{ and } j + m \text{ odd} \\
0 & \text{otherwise}
\end{cases} \quad (n = m)
\]

\[
\hat{D}_{mj} = \begin{cases} 
(2m + 1) & \text{if } j > m \text{ and } j + m \text{ odd} \\
0 & \text{otherwise}
\end{cases}
\]

\[
D_{nm} = \left(\frac{2}{2n + 1}\right) \hat{D}_{nm}
\]

\[
V_{nm} = L_m(x_n)
\]
Recall the following Legendre Polynomial properties:

\[ P_n(x) = (1 + x^2)^{-\frac{1}{2}} \frac{d^n}{dx^n} \left( (1 + x^2)^{-\frac{1}{2}} \right) \]

\[ (x)_{I+} = (x)_{I}^{\pm} \]

\[ (x)_{I-} = (x)_{I}^{\pm} \]

\[ (x)^{-w} \frac{I + w}{w} \frac{I + w}{1 + w} = (x)^{1+w} \]

\[ (x)_{I-} \frac{I + w}{w} \frac{I + w}{1 + w} = (x)_{I+} \]
Simplifying, 

\[
\begin{align*}
\sum_{m=0}^{p} \rho_{i,m} L_{m} (1) - \bar{u} L_{n} (-1) \sum_{m=0}^{p} \rho_{i,m} L_{m} (-1) \\
\end{align*}
\]

Find \( \{ \rho_{i,m} \} \) such that

\[
\begin{align*}
d \rho_{i,m} = \left( \frac{2n+1}{x_{i+1} - x_{i}} \right) (D_{i,m} \rho_{i,m}) + \sum_{n=0}^{m} (\bar{u} - \bar{u}) \rho_{i,m} + (-1)^{n+1} \bar{u} \sum_{m=0}^{p} \rho_{i,m} \\
\end{align*}
\]

We have figured out all the coefficients of the right hand side. So we just have to discretize the time derivative.
A Note on the Taylor Expansion

\[ \exists t^* \in [t, t + \delta] \text{ such that } \]
\[ q_i(t + \delta) = q_i(t) + \delta \frac{dq_i}{dt}(t) + \frac{\delta^2}{2!} \frac{d^2 q_i}{dt^2}(t) + \frac{\delta^3}{3!} \frac{d^3 q_i}{dt^3}(t) + \frac{\delta^4}{4!} \frac{d^4 q_i}{dt^4}(t^*) \]

- We are going to assume that our approximation of \( q \) is a 3\textsuperscript{rd} order polynomial in the time variable, so we are going to drop that last term and obtain the following:

\[ \rho_i(t + \delta) = \rho_i(t) + \delta \frac{d\rho_i}{dt}(t) + \frac{\delta^2}{2!} \frac{d^2 \rho_i}{dt^2}(t) + \frac{\delta^3}{3!} \frac{d^3 \rho_i}{dt^3}(t) \]
A Time Marching Scheme

For $\rho_i \in P^3([t, t + \delta])$

$$\rho_i(t + \delta) = \rho_i(t) + \delta \frac{d \rho_i}{dt}(t) + \frac{\delta^2}{2!} \frac{d^2 \rho_i}{dt^2}(t) + \frac{\delta^3}{3!} \frac{d^3 \rho_i}{dt^3}(t)$$

From which we create the scheme

$$\sigma = \rho_i(t)$$

$$\sigma = \rho_i(t) + \frac{\delta}{3} \frac{d}{dt} \sigma$$

$$\sigma = \rho_i(t) + \frac{\delta}{2} \frac{d}{dt} \sigma$$

$$\rho_i(t + \delta) = \rho_i(t) + \frac{\delta}{1} \frac{d}{dt} \sigma$$
Verifying the Scheme

\[ \sigma = \rho_i(t) \]

\[ \sigma = \rho_i(t) + \frac{\delta d}{3 \sigma} \]

\[ \sigma = \rho_i(t) + \frac{\delta d}{2 \sigma} \]

\[ \sigma = \rho_i(t) + \frac{\delta d}{1 \sigma} \]

\[ \rho_i(t + \delta) = \rho_i(t) + \frac{\delta d}{1 \sigma} \]

\[ \sigma = \rho_i(t) + \delta \left( \frac{d \rho_i(t)}{dt} + \frac{\delta d}{2 \sigma} \left( \frac{d \rho_i(t)}{dt} + \frac{\delta d}{3 \sigma} \left( \frac{d \rho_i(t)}{dt} + \frac{\delta d}{1 \sigma} \left( \rho_i(t) \right) \right) \right) \]

\[ = \rho_i(t) + \frac{d \rho_i(t)}{dt} + \frac{\delta d}{2 \sigma} \left( \frac{d \rho_i(t)}{dt} + \frac{\delta d}{3 \sigma} \left( \frac{d \rho_i(t)}{dt} + \frac{\delta d}{1 \sigma} \left( \rho_i(t) \right) \right) \right) \]

Correct!
General Time Integrator

• The scheme we just covered is the 3rd order version of the general s’th order Runge-Kutta scheme by Jameson, Schmidt and Turkel:

\[
\begin{align*}
\text{Set} & \quad U = U^n \\
\text{for } k = s : -1 : 1 & \quad U = U^n + \frac{d}{dt}U \\
\text{end} & \quad U^{n+1} = U
\end{align*}
\]
Putting it all Together

\[ \frac{d\rho_{i,n}}{dt} = \left( \frac{n+1}{x_{i+1} - x_i} \right) \left\{ -\bar{u} \sum_{m=0}^{m=p} \left( D_{nm} + (-1)^{n+m} \right) \rho_{i,m} \right\} + \bar{u} (-1)^n \sum_{m=0}^{m=p} \rho_{i-1,m} \] 

Becomes with \( \delta t = dt \)

\[ \sigma_{i,n} = \rho_{i,n}(t) \]

\[ \sigma_{i,n} = \rho_{i,n}(t) + \frac{dt}{3} \left( \frac{2n+1}{x_{i+1} - x_i} \right) \left\{ -\bar{u} \sum_{m=0}^{m=p} \left( D_{nm} + (-1)^{n+m} \right) \sigma_{i,m} \right\} + \bar{u} (-1)^n \sum_{m=0}^{m=p} \sigma_{i-1,m} \] 

\[ \sigma_{i,n} = \rho_{i,n}(t) + \frac{dt}{2} \left( \frac{2n+1}{x_{i+1} - x_i} \right) \left\{ -\bar{u} \sum_{m=0}^{m=p} \left( D_{nm} + (-1)^{n+m} \right) \sigma_{i,m} \right\} + \bar{u} (-1)^n \sum_{m=0}^{m=p} \sigma_{i-1,m} \] 

\[ \sigma_{i,n} = \rho_{i,n}(t) + \frac{dt}{1} \left( \frac{2n+1}{x_{i+1} - x_i} \right) \left\{ -\bar{u} \sum_{m=0}^{m=p} \left( D_{nm} + (-1)^{n+m} \right) \sigma_{i,m} \right\} + \bar{u} (-1)^n \sum_{m=0}^{m=p} \sigma_{i-1,m} \] 

\[ \rho_{i,n}(t + dt) = \sigma_{i,n} \]
Initial Conditions

- Ok – we know how the Legendre coefficients will involve in discrete time...

- But first we have to determine the Legendre coefficients at time $t=0$.

\[ \bar{x}_j = \sin \left( \frac{\pi j}{2p} \right) \quad j = 0, \ldots, p \]

\[ x_{i,j} = \left( \frac{x_{i+1} - x_i}{2} \right) \bar{x}_j + \left( \frac{x_i + x_{i+1}}{2} \right) \]
Projecting The Initial Conditions

• Suppose in the i’th cell we know $q(x_{i,j})$ then we can compute the coefficients in the Legendre expansion by:

$$\rho_{i,n}(t = 0) = \sum_{j=0}^{j=p} (V^{-1})_{nj} q(x_{i,j})$$
Evaluating the Final Solution

• After $T/dt$ time steps we end up with the solution we wished to compute. However – we are not really that interested in the Legendre coefficients so we evaluate the Legendre expansion at the same nodes we started at:

$$\rho_i \left( \tilde{x}_j , T \right) = \sum_{n=0}^{n=p} V_{jn} \rho_{i,n}$$
Finally The Inflow Boundary Condition

- For now let’s just set:

\[ \sum_{m=0}^{m=p} \sigma_{0,m} = 0 \]  

i.e. the density outside the tube is zero
Summary

• So you now have everything you need to write the scheme.

• Have at it…..

• Now’s the time to start coding and asking questions.

• Try an initial condition of a Gaussian pulse which decays before it hits the boundary…
\[ \rho_{i,m}(t=0) = \sum_{j=0}^{j=p} (V^{-1})_{mj} q(x_{i,j}) \]

\[ \sum_{m=0}^{m=p} \sigma_{0,m}(t) := 0 \]

For \( \text{tstep}=1:\text{Ntsteps} \)

For \( i=\text{N-1}:1 \)

\[ \sigma_{i,n}^1 = \rho_{i,n} \]

\[ \sigma_{i,n}^2 = \rho_{i,n} + \frac{dt}{3} \left( \frac{2n+1}{x_{i+1} - x_i} \right) \left\{ -\bar{u} \sum_{m=0}^{m=p} \left( D_{nm} + (-1)^{n+m} \right) \sigma_{i,m}^1 \right\} + \left( -1 \right)^n \bar{u} \sum_{m=0}^{m=p} \sigma_{i-1,m}^1 \]  

\[ \sigma_{i,n}^3 = \rho_{i,n} + \frac{dt}{2} \left( \frac{2n+1}{x_{i+1} - x_i} \right) \left\{ -\bar{u} \sum_{m=0}^{m=p} \left( D_{nm} + (-1)^{n+m} \right) \sigma_{i,m}^2 \right\} + \left( -1 \right)^n \bar{u} \sum_{m=0}^{m=p} \sigma_{i-1,m}^2 \]  

\[ \sigma_{i,n}^4 = \rho_{i,n} + \frac{dt}{1} \left( \frac{2n+1}{x_{i+1} - x_i} \right) \left\{ -\bar{u} \sum_{m=0}^{m=p} \left( D_{nm} + (-1)^{n+m} \right) \sigma_{i,m}^3 \right\} + \left( -1 \right)^n \bar{u} \sum_{m=0}^{m=p} \sigma_{i-1,m}^3 \]  

\[ \rho_{i,n} = \sigma_{i,n}^4 \]

end

end

\[ \rho(t=T,x_{i,j}) = \sum_{m=0}^{m=p} (V^{-1})_{jm} \rho_{i,m} \]
Deadlines

• HW3 is due on Monday at the start of class

• The DG solver is due on Wednesday 12\textsuperscript{th} Feb

• Time will be given in the Monday and Wednesday class to work on the code with assistance.

• When we have the codes running – we will run lots of tests on them so make sure you have an easy to use code.