CAAM 570

Homework 2

Due at the beginning of class on Feb 13. No late submissions accepted.

$\star$**2.1.11** A *topological sort* of a digraph D is an linear ordering of its vertices such that, for every arc a of D, the tail of a precedes its head in the ordering.

a) Show that every acyclic digraph has at least one source and at least one sink.
b) Deduce that a digraph admits a topological sort if and only if it is acyclic.

2.1.17 TRIANGLE-FREE GRAPH

A *triangle-free* graph is one which contains no triangles. Let G be a simple triangle-free graph.

a) Show that $d(x) + d(y) \leq n$ for all $xy \in E$.
b) Deduce that $\sum_{v \in V} d(v)^2 \leq mn$.
c) Applying the Cauchy–Schwarz Inequality[1], deduce that $m \leq n^2/4$.
(W. MANTEL)
d) For each positive integer n, find a simple triangle-free graph G with $m = \lfloor n^2/4 \rfloor$.

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[1] $\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$ for real numbers a_i, b_i, $1 \leq i \leq n$.

2.2.8 Give an example to show that the following simple procedure, known as the *Greedy Heuristic*, is not guaranteed to solve the Travelling Salesman Problem.

- ▷ Select an arbitrary vertex v.
- ▷ Starting with the trivial path v, grow a Hamilton path one edge at a time, choosing at each iteration an edge of minimum weight between the terminal vertex of the current path and a vertex not on this path.
- ▷ Form a Hamilton cycle by adding the edge joining the two ends of the Hamilton path.

2.2.15

a) Show that an induced subgraph of a line graph is itself a line graph.
b) Deduce that no line graph can contain either of the graphs in Figure 1.19 as an induced subgraph.
c) Show that these two graphs are minimal with respect to the above property. Can you find other such graphs? (There are nine in all.)

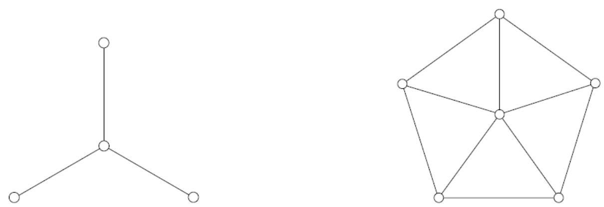

Fig. 1.19. Two graphs that are not line graphs

2.2.19 Read the 'Theorem' and 'Proof' given below, and then answer the questions which follow.

'Theorem'. Let G be a simple graph with $\delta \geq n/2$, where $n \geq 3$. Then G has a Hamilton cycle.

'Proof'. By induction on n. The 'Theorem' is true for $n = 3$, because $G = K_3$ in this case. Suppose that it holds for $n = k$, where $k \geq 3$. Let G' be a simple graph on k vertices in which $\delta \geq k/2$, and let C' be a Hamilton cycle of G'. Form a graph G on $k+1$ vertices in which $\delta \geq (k+1)/2$ by adding a new vertex v and joining v to at least $(k+1)/2$ vertices of G'. Note that v must be adjacent to two consecutive vertices, u and w, of C'. Replacing the edge uw of C' by the path uvw, we obtain a Hamilton cycle C of G. Thus the 'Theorem' is true for $n = k+1$. By the Principle of Mathematical Induction, it is true for all $n \geq 3$. □

a) Is the 'Proof' correct?
b) If you claim that the 'Proof' is incorrect, give reasons to support your claim.
c) Can you find any graphs for which the 'Theorem' fails? Does the existence or nonexistence of such graphs have any relationship to the correctness or incorrectness of the 'Proof'? (D.R. Woodall)