CAAM 436/CAAM 535, Fall 2021 : Homework 1
Posted on September 8 , Due September 21 in class.
If the homework is hand-written, it has to be legible.

Problems 1-4 are for students enrolled in CAAM 436. Problems 1-5 are for students enrolled in CAAM 535.

Vocabulary: A motion of a material body is simply a time-parametrized family of deformations, $\boldsymbol{x}(\cdot, t)$, defined for an interval (t_0, t_1) of time. That is, for each $t \in (t_0, t_1)$, the map $\boldsymbol{x}(\cdot, t) : \Omega \mapsto \mathbb{R}^3$ is a deformation.

Problem 1. (30 points)

Suppose that Ω (the volume occupied by the undeformed body) is the unit cubit meter: $\{\boldsymbol{X} : 0 \leq X_i \leq 1,\ i = 1, 2, 3\}$, which deforms by stretching uniformly under the motion

$$\boldsymbol{x}(\boldsymbol{X}, t) = (1 + t)\,\boldsymbol{X}$$

The initial density is homogeneous, $\rho_{\text{ref}} = 1000$ kg/m^3.
(a) [10 points] Verify that this formula defines a motion for $t \geq 0$.
(b) [10 points] Compute the velocity $\boldsymbol{v}(\boldsymbol{x}, t)$
(c) [10 points] Compute the density $\rho(\boldsymbol{x}, t)$

Problem 2. (20 points)

Prove the following identities, where $\boldsymbol{u}, \boldsymbol{v}$ are vector-valued function with components $(u_i)_i, (v_i)_i$ for $1 \leq i \leq 3$ and where f is a scalar function.
(a) [10 points]

$$\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = (\nabla \boldsymbol{u})^T\, \boldsymbol{v} + (\nabla \boldsymbol{v})^T\, \boldsymbol{u}$$

(b) [10 points]

$$\nabla(f\boldsymbol{u}) = \boldsymbol{u}(\nabla f)^T + f\nabla \boldsymbol{u}$$

Problem 3. (10 points)

Derive the vector version of Reynold's transport theorem: for a vector valued function $\boldsymbol{w}(\boldsymbol{x}, t)$, and a moving part p_t within its domain of definition,

$$\frac{d}{dt}\int_{p_t} \boldsymbol{w}\,d\boldsymbol{x} = \int_{p_t} \left(\frac{\mathrm{D}\boldsymbol{w}}{\mathrm{D}t} + (\nabla \cdot \boldsymbol{v})\boldsymbol{w}\right) d\boldsymbol{x}$$

in which the material derivative is defined by

$$\frac{\mathrm{D}\boldsymbol{w}}{\mathrm{D}t} = \frac{\partial \boldsymbol{w}}{\partial t} + (\nabla \boldsymbol{w})\boldsymbol{v}$$

Problem 4 (40 points)

Suppose that Ω (the volume occupied by the undeformed body) is the cylinder

$$\Omega = \{\boldsymbol{X} : 0 \leq X_1 \leq 1,\ X_1^2 + X_2^2 \leq 1\}$$

and that the body undergoes the rotation about the X_3-axis with angular frequency ω:

$$\begin{aligned} x_1(\boldsymbol{X}, t) &= X_1 \cos(\omega t) + X_2 \sin(\omega t) \\ x_2(\boldsymbol{X}, t) &= -X_1 \sin(\omega t) + X_2 \cos(\omega t) \\ x_3(\boldsymbol{X}, t) &= X_3 \end{aligned}$$

The initial density is homogeneous: $\rho_{\text{ref}} = 1000\ \text{kg/m}^3$.
(a) [10 points] Verify that this is a motion for all t.
(b) [10 points] Compute $\boldsymbol{v}(\boldsymbol{x}, t)$ (units of m/s).4
(c) [10 points] Compute $\nabla \cdot \boldsymbol{v}(\boldsymbol{x}, t)$. What are the units?
(d) [10 points] Compute $\rho(\boldsymbol{x}, t)$

Problem 5 (30 points)
Suppose that $\Phi(\boldsymbol{x}, t)$ is a differentiable scalar-valued function, defined on a subset of $\mathbb{R}^4$ including all positions of the body.
(a) [20 points] Show that for any part p_t,

$$\frac{d}{dt} \int_{p_t} \rho(\boldsymbol{x}, t)\, \Phi(\boldsymbol{x}, t) d\boldsymbol{x} = \int_{p_t} \rho(\boldsymbol{x}, t) \frac{\mathrm{D}\Phi}{\mathrm{D}t}(\boldsymbol{x}, t) d\boldsymbol{x}$$

(b) [10 points] Deduce that

$$\frac{\mathrm{D}(\rho\Phi)}{\mathrm{D}t} + (\nabla \cdot \boldsymbol{v})\rho\Phi = \rho \frac{\mathrm{D}\Phi}{\mathrm{D}t}$$