CAAM 436/CAAM 535, Fall 2021 : Homework 2
Posted on September 21 , Due October 5 in class.
If the homework is hand-written, it has to be legible.

Problems 1-3 are for students enrolled in CAAM 436. Problems 1-5 are for students enrolled in CAAM 535.

Problem 1. (40 points)

Let $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ be a symmetric real matrix. It is diagonalizable, meaning that there exists an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the diagonal entries of $\boldsymbol{D}$.
(a) [10 points] Recall that the trace of a matrix $\boldsymbol{B}$ is the sum of the diagonal entries of $\boldsymbol{B}$. Show for any square matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ of same size that

$$\mathrm{tr}(\boldsymbol{BC}) = \mathrm{tr}(\boldsymbol{CB})$$

(b) [10 points] Show that

$$\mathrm{tr}(\boldsymbol{A}) = \lambda_1 + \lambda_2 + \lambda_3$$

(c) [10 points] Show that

$$\frac{1}{2}\mathrm{tr}(\boldsymbol{A}^2) - \frac{1}{2}(\mathrm{tr}(\boldsymbol{A})^2 = -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3$$

(d) [10 points] Show that

$$\det(\boldsymbol{A}) = \lambda_1\,\lambda_2\,\lambda_3$$

Problem 2. (20 points)

(Notation follows what is used in class). Let $\boldsymbol{v}(\boldsymbol{x}, t)$ be the velocity and let $\boldsymbol{u}(\boldsymbol{x}, t)$ be the displacement. Show that

$$\boldsymbol{v} = \frac{D\boldsymbol{u}}{Dt}$$

In other words, the velocity is the material derivative of the displacement.

Problem 3. (30 points)

Let $\boldsymbol{e}$ be a non-zero vector in $\mathbb{R}^3$. A surface extension in the direction $\boldsymbol{e}$ is a homogeneous deformation of the form

$$\boldsymbol{x} = \boldsymbol{x}_a + \boldsymbol{A}(\boldsymbol{X} - \boldsymbol{X}_b), \quad \boldsymbol{A} = \boldsymbol{I} + (\lambda - 1)\boldsymbol{e}\boldsymbol{e}^T,$$

where λ is a positive scalar, and $\boldsymbol{x}_a, \boldsymbol{X}_b$ are fixed vectors.

Suppose for this problem, that $\boldsymbol{e} = \boldsymbol{e}_1 = (1, 0, 0)^T$.
(a) [15 points] Compute the strain tensors $\boldsymbol{L}$ and $\boldsymbol{E}$ explicitely (i.e. give the matrix entries in terms of the scalar λ). Both $\boldsymbol{L}$ and $\boldsymbol{E}$ have been defined in class, and the definition of $\boldsymbol{L}$ is recalled in Problem 4 below.

(b) [15 points] Compute the eigenvalues and eigenvectors of $\boldsymbol{L}$ and $\boldsymbol{E}$. Eigenvalues are called principal strains and eigenvectors are called principal axes.

Problem 4. (20 points)

(This problem follows closely what was done in class).
Let $\boldsymbol{L}$ be the Lagrangian strain tensor:

$$\boldsymbol{L}(\boldsymbol{X}) = \frac{1}{2}((\nabla_{\boldsymbol{X}}\boldsymbol{x})^T \nabla_{\boldsymbol{X}}\boldsymbol{x} - \boldsymbol{I}).$$

Show that if $\boldsymbol{L}(\boldsymbol{X}) = \boldsymbol{0}$ for all $\boldsymbol{X}$, then the deformation map $\boldsymbol{x}(\boldsymbol{X})$ is a rigid deformation.

Problem 5 (20 points)

Recall that the Euclidean norm of a vector $\boldsymbol{x} = (x_1, x_2, x_3)^T$ is

$$\|\boldsymbol{x}\|_2 = (\sum_{i=1}^{3} x_i^2)^{1/2}$$

(a) [10 points] Show the identity

$$\nabla\|\boldsymbol{x}\|_2 = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}$$

(b) [10 points] Show for $\boldsymbol{x} \neq \boldsymbol{0}$ that

$$\nabla \cdot \nabla \frac{1}{\|\boldsymbol{x}\|_2} = 0$$