CAAM 436/CAAM 535, Fall 2021 : Homework 4
Posted on November 8, Due November 29 noon: drop it off at my office
If the homework is hand-written, it has to be legible.

Problems 1-4 are for students enrolled in CAAM 436. Problems 1-5 are for students enrolled in CAAM 535. Bonus problems A and B are optional and are for everyone.

You should be familiar with the notation used below as it is the same used in class.

Problem 1. (20 points)

Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function with compact support, i.e. $\phi \in \mathcal{D}$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index, $\alpha_i \in \mathbb{N}$. Let $u : \mathcal{D} \to \mathbb{R}$ be a distribution. The cross-correlation of u and ϕ is denoted by $u \star \phi$. Show that

$$u \star D^\alpha \phi = (D^\alpha)' u \star \phi$$

where $(D^\alpha)' u$ is the transpose of the distribution $D^\alpha u$.

Problem 2. (20 points)

Let $\mathbf{v}$ be a vector field, $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ that is divergent-free, i.e. $\nabla \cdot \mathbf{v} = 0$. Define the matrix:

$$\epsilon(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$$

Show that

$$2\nabla \cdot \epsilon(\mathbf{v}) = \Delta \mathbf{v}$$

Problem 3. (20 points)

Assume that $\mathbf{v}_0 \in \mathbb{R}^3, p_0 \in \mathbb{R}$ are constant vector and scalar respectively. We observe that they represent a slow, steady motion of a Newtonian fluid, i.e. the pair $(\mathbf{v}_0, p_0)$ solves the Stokes equations with vanishing external force ($\boldsymbol{f} = \mathbf{0}$). Let $\delta\mathbf{v}$ and δp be small perturbations of $\mathbf{v}_0$ and p_0. We write

$$\mathbf{v}(\boldsymbol{x}, t) = \mathbf{v}_0 + \delta\mathbf{v}(\boldsymbol{x}, t), \quad p(\boldsymbol{x}, t) = p_0 + \delta p(\boldsymbol{x}, t),$$

and let $\delta \boldsymbol{f}$ represents a small perturbation of the constant solution. Assume the perturbations are small. Derive the linearized Navier-Stokes equations:

$$\rho_0 \frac{\partial \delta\mathbf{v}}{\partial t} + \rho_0 (\nabla \delta\mathbf{v})\mathbf{v}_0 + \nabla \delta p - \mu \Delta(\delta\mathbf{v}) = \delta \boldsymbol{f}$$
$$\nabla \cdot \delta\mathbf{v} = 0$$

Problem 4. (40 points)

This problem is about the solution to the steady-state Navier-Stokes equations for a Newtonian fluid of viscosity μ, in a (horizontal) cylindrical pipe with x-axis for axis of symmetry. Denote by L the length of the pipe and by R its constant radius. We neglect external forces. We use the cylindrical coordinate system (r, θ, x) and we denote by $p(r, \theta, x)$ the fluid pressure and by (v_r, v_θ, v_x) the components of the velocity $\mathbf{v}$ in the cylindrical coordinate system.

(a) [20 points] Derive the system of equations satisfied by v_r, v_θ, v_x and p. Hint: use the fact that the flow is axisymmetric and laminar.

(b) [20 points] Assume no-slip boundary conditions at the pipe wall and assume that pressure takes constant value P_0 at $x = 0$ and P_L at $x = L$. Solve the system of equations obtained in (a) and give expressions for v_r, v_θ, v_x and p.

Problem 5. (30 points)

We consider in this problem the conservation of momentum law for materials in three dimensions. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be two coordinate systems related by the transformation:

$$\boldsymbol{x}'(\boldsymbol{x}, t) = \boldsymbol{x}'_0(t) + \boldsymbol{R}(t)(\boldsymbol{x} - \boldsymbol{x}_0)$$

where $\boldsymbol{x}'_0(t)$ and $\boldsymbol{R}(t)$ are continuously differentiable functions of t. The matrix $\boldsymbol{R}(t)$ is assumed to be orthogonal for each t and $\det(\boldsymbol{R}(t)) = 1$. Show that the conservation of momentum in the $\boldsymbol{x}'$ coordinate system takes the same form:

$$\rho' \frac{D\mathbf{v}'}{Dt} = \nabla_{\boldsymbol{x}'} \cdot \sigma' + \boldsymbol{f}'$$

as in the $\boldsymbol{x}$ coordinate system

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla_{\boldsymbol{x}} \cdot \sigma + \boldsymbol{f}$$

and state an explicit relation between the external force vector fields $\boldsymbol{f}'$ and $\boldsymbol{f}$.

Bonus problems are optional. Points from bonus problems will be added to the overall homeworks points, and will be counted toward the class grade. **Codes need to be emailed to me by the homework deadline**.

Bonus Problem A. (40 points)

(a) [20 points]
Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or $d = 3$. Define the function

$$u(\boldsymbol{x}, t) = \left(\frac{1}{4\pi t}\right)^{d/2} e^{-\frac{\|\boldsymbol{x}\|^2}{4t}}$$

Show that u satisfies the heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = 0, \quad \text{in} \quad \Omega, \quad \forall t > 0$$

(b) [20 points]
Modify the code `ft03_heat.py` (that was emailed to you) to solve for the problem described in (a) on the time interval $[0, 1]$ in the unit square. Choose for time step $\tau = 1/50$. The mesh is made of $N \times N$ squares. Give a table of L^2 errors and max-norm errors (at the vertices) for $N = 4, 8, 16, 32$. Give the plots of the numerical solution at final time for each mesh.

Bonus Problem B. (40 points)

Modify the code `NSEchannel.py` so that it solves the lid driven cavity problem in the unit square. Plot streamlines for viscosity values $\mu = 1, \mu = 0.01$ on a mesh made of 100×100 squares.