CAAM 440 · APPLIED MATRIX ANALYSIS

Problem Set 1

Posted Friday 13 January 2012. Due Friday 20 January 2012.

Complete any four problems, 25 points each.

You are welcome to complete more problems if you like. The latter problems are generally more challenging than the early problems. If you are already familiar with this material, please tackle the latter problems. If you submit more than four solutions, specify those four you would like to be graded.

1. Compute the resolvent $\mathbf{R}(z) = (z\mathbf{I} - \mathbf{A})^{-1}$ for the following three matrices.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (These are simple enough to compute by hand.)

2. The structure of the resolvents in Problem 1 reveal a considerable amount about the sensitivity of the eigenvalues to small perturbations. We will develop the full theory later in the semester; for now, build your intuition by conducting the following computational experiment.

   For each matrix in Problem 1, compute the eigenvalues of $\mathbf{A} + \mathbf{E}$ for random matrices $\mathbf{E}$ having norm $10^{-15}, 10^{-12}, 10^{-9}, 10^{-6}, 10^{-3}, 10^{0}$. (For example, use `E = randn(3); E = 1e-15*E/norm(E)`.) Produce a `loglog` plot in MATLAB comparing the size of the perturbation, $\|\mathbf{E}\|$ (horizontal axis) versus the maximum amount the eigenvalues of $\mathbf{A}+\mathbf{E}$ differ from those of $\mathbf{A}$ (vertical axis). Your plot should have three lines, one for each of the three $\mathbf{A}$ matrices.

   Can you spot a relationship between the entries in your resolvents and the slopes your lines?

3. Compute, by hand, *two* Schur factorizations *with different values of* $\mathbf{T}$ for each of the following two matrices.

$$\mathbf{A} = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}; \qquad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

   (You should have a total of four Schur factorizations for this problem.)

4. (a) Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$ is a *normal* matrix (meaning $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$) and has the Schur factorization $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^*$. Show that the upper triangular matrix $\mathbf{T}$ is actually *diagonal*, i.e., $t_{j,k} = 0$ if $j \neq k$.

   (b) What does this imply about the eigenvectors of $\mathbf{A}$?

   (c) Suppose $\mathbf{A}$ is a normal matrix and all of its eigenvalues are real. Show that $\mathbf{A}$ must be Hermitian, i.e., $\mathbf{A}^* = \mathbf{A}$.

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5. You were probably not surprised by our result that every square matrix must have at least one eigenvalue. However, slightly more general settings can give rather more exotic behavior.

   The differential equation $\mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ (which comes from a variety of finite element problems, for example) gives rise to the *generalized eigenvalue problem* $\mathbf{A}\mathbf{u} = \lambda\mathbf{B}\mathbf{u}$.

   (a) Show that $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{u}$ is a solution to the differential equation $\mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ if and only if $\mathbf{A}\mathbf{u} = \lambda\mathbf{B}\mathbf{u}$.

   (b) For each of the following three problems, list all values of λ for which $\mathbf{A}\mathbf{u} = \lambda\mathbf{B}\mathbf{u}$ for some nonzero $\mathbf{u} \in \mathbb{C}^n$. The spectrum of this generalized eigenvalue problem is denoted $\sigma(\mathbf{A}, \mathbf{B})$.

   (i) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (ii) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

   (iii) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (iv) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

   (c) To appreciate the difference between cases (i) and (ii) in part (b), compute $\sigma(\mathbf{A}, \mathbf{B})$ for the pair

   $$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$$

   with $\varepsilon \neq 0$. How does $\sigma(\mathbf{A}, \mathbf{B})$ evolve as $\varepsilon \to 0$?

6. More sophisticated dynamical systems give rise to still more subtle eigenvalue problems than those encountered in the last problem.

   Damped vibrating systems gives second order differential equations of the form

   $$\mathbf{C}\mathbf{x}''(t) + \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t).$$

   (a) Show that $\mathrm{e}^{\lambda t}\mathbf{u}$ is a solution to $\mathbf{C}\mathbf{x}''(t) + \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ if and only if $\lambda^2\mathbf{C}\mathbf{u} + \lambda\mathbf{B}\mathbf{u} = \mathbf{A}\mathbf{u}$. (This is called a *quadratic eigenvalue problem*.)

   (b) Find *all* solutions $\lambda \in \mathbb{C}$ for which $\lambda^2\mathbf{C}\mathbf{u} + \lambda\mathbf{B}\mathbf{u} = \mathbf{A}\mathbf{u}$ for some nonzero $\mathbf{u} \in \mathbb{C}^n$, where

   $$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Are the eigenvectors $\mathbf{u}$ associated with distinct eigenvalues always linearly independent?

   In recent years there has been increasing interest in *delay differential equations*, the simplest example of which is $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t - s)$ for some $s > 0$.

   (c) Show that $\mathrm{e}^{\lambda t}\mathbf{u}$ is a solution to $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t - s)$ if and only if $\mathbf{A}\mathbf{u} = \lambda\mathrm{e}^{s\lambda}\mathbf{u}$.

   (d) This *nonlinear eigenvalue problem* is even interesting in the scalar case, $n = 1$.
   Try to find/estimate eigenvalues $\lambda \in \mathbb{C}$ for which $\mathbf{A}\mathbf{u} = \lambda\mathrm{e}^{s\lambda}\mathbf{u}$ for some nonzero $\mathbf{u} \in \mathbb{C}^n$, where

   $$\mathbf{A} = [\,1\,], \quad s = 1.$$

   Find all real eigenvalues λ, or justify why there are none (e.g., with a plot).
   Estimate some complex eigenvalues, e.g., by using a Taylor series for e^{λ}, and/or Newton's method.
   How many eigenvalues do you think this problem has?

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7. (a) Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and suppose z is not an eigenvalue of $\mathbf{A}$. Prove that $\mathbf{A}$ commutes with its resolvent, i.e., $\mathbf{A}(z\mathbf{I}-\mathbf{A})^{-1} = (z\mathbf{I}-\mathbf{A})^{-1}\mathbf{A}$.

   (b) Show that if $\mathbf{S}$ is skew-Hermitian (i.e., $\mathbf{S}^* = -\mathbf{S}$), then

$$\mathbf{Q} = (\mathbf{I}-\mathbf{S})(\mathbf{I}+\mathbf{S})^{-1}$$

   is a unitary matrix. (Note that $(\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1}$ for any invertible matrix $\mathbf{A}$.)

   (c) Show that all unitary matrices $\mathbf{Q} \in \mathbb{C}^{n\times n}$ are normal. What can be said of the eigenvalues of unitary matrices?

   (d) [not required]
   Show that, if $\mathbf{Q}$ is a unitary matrix and -1 is *not* an eigenvalue of $\mathbf{Q}$, then there exists a skew-Hermitian matrix $\mathbf{S}$ such that

$$\mathbf{Q} = (\mathbf{I}-\mathbf{S})(\mathbf{I}+\mathbf{S})^{-1}.$$

   [Stewart and Sun; Loewy (1898)]

8. A matrix $\mathbf{P} \in \mathbb{C}^{n\times n}$ is a *projector* provided $\mathbf{P}^2 = \mathbf{P}$. We say that such a $\mathbf{P}$ is an *orthogonal projector* provided that $\mathbf{P}$ is also Hermitian, $\mathbf{P}^* = \mathbf{P}$.

   Prove that a nonzero projector $\mathbf{P}$ is an orthogonal projector if and only if $\|\mathbf{P}\| = 1$.
   [Trefethen and Bau]