A: For all students

The purpose of this problem is to solve numerically the following two-points boundary value problem by three finite difference schemes. The problem is: Find $u$ such that

$$-u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \quad \text{in } (0, 1),$$  
with the boundary conditions

$$u(0) = u(1) = 0,$$

where

$$b(x) = x^2, \quad c(x) = 1 + x, \quad f(x) = -2 + 13x^2 + 3x^3 - x^4 - 5x^5.$$ 

The exact solution is

$$u(x) = x^2(1 - x^2).$$

Let $N \geq 1$ be an integer, define the mesh points

$$x_i = ih, \quad h = \frac{1}{N+1}, \quad i = 0, \ldots, N + 1,$$

set

$$c_i = c(x_i), \quad b_i = b(x_i), \quad f_i = f(x_i),$$

and approximate $u(x_i)$ respectively by $U_i$, $V_i$, and $W_i$, where $U_i$ is the solution of the finite-difference scheme:

$$\frac{1}{h^2}[-U_{i-1} + 2U_i - U_{i+1}] + \frac{b_i}{h}[U_i - U_{i-1}] + c_iU_i = f_i,$$

$V_i$ is the solution of

$$\frac{1}{h^2}[-V_{i-1} + 2V_i - V_{i+1}] + \frac{b_i}{h}[V_{i+1} - V_i] + c_iV_i = f_i,$$

and $W_i$ is the solution of

$$\frac{1}{h^2}[-W_{i-1} + 2W_i - W_{i+1}] + \frac{b_i}{2h}[W_{i+1} - W_{i-1}] + c_iW_i = f_i,$$

with $U_0 = V_0 = W_0 = 0 = U_{N+1} = V_{N+1} = W_{N+1} = 0$.

**Problem 1.** (10 points)

Let $v \in C^2(0, 1)$. Using Taylor’s formula, show that for any $h > 0$

$$u''(x) - \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} = O(h^2), \quad 0 < x < 1.$$

**Problem 2.** (30 points)

Using Taylor’s formula, derive an upper bound for the local truncation error of the schemes (3), (4), and (5) in terms of $h$ and derivatives of $u$. Which scheme has the highest order?

**Problem 3.** (15 points)

Write the first line, the last line and a general line of the matrix of each scheme. Are any of these matrices symmetric?
Problem 4. (45 points) Write a Matlab code implementing the schemes (3), (4), and (5). Run this code with $h = \frac{1}{20}$, $h = \frac{1}{40}$, $h = \frac{1}{80}$. Make a table of the relative max-norm errors for each value of $h$ and each solution:

$$\frac{\text{Max}_{1\leq i \leq N} |U_i - u(x_i)|}{\text{Max}_{1\leq i \leq N} |u(x_i)|}, \frac{\text{Max}_{1\leq i \leq N} |V_i - u(x_i)|}{\text{Max}_{1\leq i \leq N} |u(x_i)|}, \frac{\text{Max}_{1\leq i \leq N} |W_i - u(x_i)|}{\text{Max}_{1\leq i \leq N} |u(x_i)|}.$$

Obtain the numerical convergence rate of each scheme. Compare the accuracy of the three schemes in this example. Is it consistent with the order found in Problem 2?

B: Additional problems for students enrolled in CAAM 536/CEVE 555.

The purpose of this problem is to solve numerically the following two-points boundary value problem. The problem is: Find $u$ such that

$$-u''(x) + (2 + x^2)u(x) = f(x) \quad \text{in} \ (0,1),$$

with the boundary conditions

$$u'(0) = 1, \quad u(1) = 0.$$  \hspace{1cm} (7)

Problem 5. (10 points)
Compute $f$ such that the exact solution is

$$u(x) = \cos(\pi x) + x.$$  \hspace{1cm} (6)

Problem 6. (40 points)
In class, we saw two second-order finite difference methods for solving this problem. Let $U = (U_0, U_1, \ldots, U_{N+1})$ be the solution vector. The value $U_i$ is the numerical solution at the node $x_i$. We use a grid similar to part A. Describe the system of equations satisfied by the $U_i$'s for the two methods. Implement the two methods. Solve the problem on different grids ($h = 1/10, 1/20, 1/40$) and compute the numerical errors and convergence rate. The errors are

$$\|e\|_{2,h} = \left(h \sum_{j=0}^{N+1} (U_j - u(x_j))^2 \right)^{1/2}.$$