CAAM 452 - 536, CEVE 555, Spring 2020 : Homework 1
Posted on January 23 , Due February 6 in class
Printout of codes are to be included, and codes are to be uploaded in Canvas

A: For all students

The purpose of this problem is to solve numerically the following two-points boundary value problem by three finite difference schemes. The problem is: Find u such that

$$-u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \quad \text{in } (0,1)\,, \tag{1}$$

with the boundary conditions

$$u(0) = u(1) = 0\,, \tag{2}$$

where

$$b(x) = x^2\ ,\ c(x) = 1 + x\ ,\ f(x) = -2 + 13x^2 + 3x^3 - x^4 - 5x^5\,.$$

The exact solution is

$$u(x) = x^2(1 - x^2)\,.$$

Let $N \geq 1$ be an integer, define the mesh points

$$x_i = ih\ ,\ h = \frac{1}{N+1}\,,\ i = 0, \ldots, N+1\,,$$

set

$$c_i = c(x_i)\ ,\ b_i = b(x_i)\ ,\ f_i = f(x_i)\,,$$

and approximate $u(x_i)$ respectively by U_i, V_i, and W_i, where U_i is the solution of the finite-difference scheme:

$$\frac{1}{h^2}[-U_{i-1} + 2U_i - U_{i+1}] + \frac{b_i}{h}[U_i - U_{i-1}] + c_iU_i = f_i\,, \tag{3}$$

V_i is the solution of

$$\frac{1}{h^2}[-V_{i-1} + 2V_i - V_{i+1}] + \frac{b_i}{h}[V_{i+1} - V_i] + c_iV_i = f_i\,, \tag{4}$$

and W_i is the solution of

$$\frac{1}{h^2}[-W_{i-1} + 2W_i - W_{i+1}] + \frac{b_i}{2h}[W_{i+1} - W_{i-1}] + c_iW_i = f_i\,, \tag{5}$$

with $U_0 = V_0 = W_0 = U_{N+1} = V_{N+1} = W_{N+1} = 0$.

Problem 1. (10 points)

Let $v \in \mathcal{C}^2(0,1)$. Using Taylor's formula, show that for any $h > 0$

$$u''(x) - \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = \mathcal{O}(h^2), \quad 0 < x < 1.$$

Problem 2. (30 points)
Using Taylor's formula, derive an upper bound for the local truncation error of the schemes (3), (4), and (5) in terms of h and derivatives of u. Which scheme has the highest order?

Problem 3. (15 points)
Write the first line, the last line and a general line of the matrix of each scheme. Are any of these matrices symmetric?

Problem 4. (45 points) Write a Matlab code implementing the schemes (3), (4), and (5). Run this code with $h = \frac{1}{20}$, $h = \frac{1}{40}$, $h = \frac{1}{80}$. Make a table of the relative max-norm errors for each value of h and each solution:

$$\frac{\mathrm{Max}_{1\le i\le N}|U_i - u(x_i)|}{\mathrm{Max}_{1\le i\le N}|u(x_i)|}\,,\quad \frac{\mathrm{Max}_{1\le i\le N}|V_i - u(x_i)|}{\mathrm{Max}_{1\le i\le N}|u(x_i)|}\,,\quad \frac{\mathrm{Max}_{1\le i\le N}|W_i - u(x_i)|}{\mathrm{Max}_{1\le i\le N}|u(x_i)|}\,.$$

Obtain the numerical convergence rate of each scheme. Compare the accuracy of the three schemes in this example. Is it consistent with the order found in Problem 2?

B: Additional problems for students enrolled in CAAM 536/CEVE 555.

The purpose of this problem is to solve numerically the following two-points boundary value problem. The problem is: Find u such that

$$-u''(x) + (2 + x^2)u(x) = f(x) \ \text{ in } (0,1)\,, \tag{6}$$

with the boundary conditions

$$u'(0) = 1, \quad u(1) = 0\,. \tag{7}$$

Problem 5. (10 points)
Compute f such that the exact solution is

$$u(x) = cos(\pi\, x) + x.$$

Problem 6. (40 points)
In class, we saw two second-order finite difference methods for solving this problem. Let $U = (U_0, U_1, \ldots, U_{N+1})$ be the solution vector. The value U_i is the numerical solution at the node x_i. We use a grid similar to part **A**. Describe the system of equations satisfied by the U_i's for the two methods. Implement the two methods. Solve the problem on different grids ($h = 1/10, 1/20, 1/40$) and compute the numerical errors and convergence rate. The errors are

$$\|e\|_{2,h} = \left(h \sum_{j=0}^{N+1} (U_j - u(x_j))^2 \right)^{1/2}.$$