THE GENERALIZED GRADIENT AT A MULTIPLE EIGENVALUE

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Abstract: When a symmetric, positive, isomorphism between a reflexive Banach space (that is densely and compactly embedded in a Hilbert space) and its dual varies smoothly over a Banach space, its eigenvalues vary in a Lipschitz manner. We calculate the generalized gradient of the extreme eigenvalues at an arbitrary crossing. We apply this to the generalized gradient, with respect to a coefficient in an elliptic operator, of (i) the gap between the operator's first two eigenvalues, and (ii) the distance from a prescribed value to the spectrum of the operator.

1. **The Result**

Intent on studying the variation in the eigenvalues of an elliptic operator under variations in its measurable coefficients we work in the context of a Gelfand triple. That is, a reflexive Banach space V, a Hilbert space H, and a continuous, injective imbedding $i : V \to H$ with dense image. The adjoint of i is consequently a continuous, injective imbedding $i' : H \to V'$ with image dense in V', the dual of V. We recall, see, e.g., Wloka [W], that this setting permits a representation of V' in terms of the continuous extension of $(\cdot,\cdot)_H$, the inner product on H, to $V \times V'$.

Within $L(V, V')$, the space of bounded linear transformations from V to V' we shall consider, after Spagnolo [S], the class $E(V)$ of symmetric, positive, isomorphisms. That is, those $A \in L(V, V')$ satisfying

$$(Au, v)_H = (u, Av)_H, \quad u, v \in V,$$
$$\alpha \|v\|_V^2 \le (Av, v)_H \le \beta \|v\|_V^2, \quad v \in V,$$

for some fixed positive α and β.

We shall assume that A depends smoothly on its "coefficients" by requiring that $A \in C^1(B_\delta(f_0), E(V))$ where $B_\delta(f_0) \equiv \{g \in X : \|f_0 - g\|_X < \delta\}$ and $E(V)$ inherits the uniform operator topology from $L(V, V')$.

An adequate spectral theory follows on assuming the imbedding $i : V \to H$ to be compact. In this case the Green's operator $iA^{-1}(f)i'$ is symmetric, positive, and compact on the Hilbert space H. As a result, $A(f)$ possesses a sequence of positive eigenvalues each of finite multiplicity, $0 < \lambda_1(f) \le \lambda_2(f) \le \cdots \to \infty$, and an orthonormal base of eigenfunctions. We denote by $\mathcal{E}_k(f)$ the span of the eigenfunctions of $A(f)$ associated with $\lambda_k(f)$ and by $\mathcal{E}^1_k(f)$ the intersection of $\mathcal{E}_k(f)$ with S^1_H, the unit sphere in H.

Under these hypotheses we find $f \mapsto \lambda_k(f)$ to be Lipschitz and so amenable to the calculus of Clarke [Cl]. Recall that the generalized directional derivative of the real valued Lipschitz function F on the Banach space X at f in the direction g is

$$F^o(f;g) \equiv \limsup_{\substack{h \to f \\ t \downarrow 0}} \frac{F(h+tg) - F(h)}{t}.$$

The generalized gradient of F at f is the nonempty, convex, weak* compact set

$$\partial F(f) \equiv \{\xi \in X';\ F^o(f;g) \ge \langle \xi, g\rangle_X\ \forall g \in X\}.$$

We state our main result. We denote convex hull by co and the derivative of A at f_0 by $DA(f_0) : X \to L(V, V')$.

Theorem 1. *If $A \in C^1(B_\delta(f_0), E(V))$ and*

$$\lambda_{j-1}(f_0) < \lambda_j(f_0) = \cdots = \lambda_{j+m_j-1}(f_0) < \lambda_{j+m_j}(f_0) \tag{1.1}$$

then

$$\partial\lambda_j(f_0) = \partial\lambda_{j+m_j-1}(f_0) = \mathrm{co}\,\{(\langle DA(f_0), \cdot\rangle u, u)_H : u \in \mathcal{E}^1_j(f_0)\}. \tag{1.2}$$

2. **Its Proof**

We find it more convenient to work with the eigenvalues of A^{-1}. To begin, we show them to be Lipschitz.

Lemma 1. *If $A \in C^1(B_\delta(f_0), E(V))$ and $r < \delta$ then, for each k,*

$$\left| \frac{1}{\lambda_k(f)} - \frac{1}{\lambda_k(g)} \right| \le K\|f - g\|_X, \quad \forall f, g \in B_r(f_0),$$

where

$$K = \alpha^{-2} \sup_{f \in B_r(f_0)} \|DA(f)\|_{L(X, L(V,V'))}.$$

Proof: We exploit the characterization

$$\frac{1}{\lambda_k(f)} = \max_{H_k \subset H} \min_{u \in H_k \cap S_H^1} (A^{-1}(f)u, u)_H, \tag{2.1}$$

where H_k varies over the k–dimensional subspaces of H. If $f, g \in B_r(f_0)$ and $u \in S_H^1$ then

$$\begin{aligned} |((A^{-1}(f) - A^{-1}(g))u, u)_H| &= |(A^{-1}(g)(A(f) - A(g))A^{-1}(f)u, u)_H| \\ &\le \alpha^{-2} \|A(f) - A(g)\|_{L(V,V')} \\ &= \alpha^{-2} \|\langle DA(tf + (1-t)g), f - g\rangle\|_{L(V,V')} \\ &\le K\|f - g\|_X. \end{aligned}$$

That is,

$$(A^{-1}(g)u, u)_H - K\|f - g\|_X \le (A^{-1}(f)u, u)_H \le (A^{-1}(g)u, u)_H + K\|f - g\|_X.$$

Applying the maximin of (2.1) throughout now gives the desired result. ∎

Of course the ease with which one is able to calculate the generalized gradient will depend greatly on one's choice of representation for $1/\lambda_k$. When $k = 1$ there is general agreement that (2.1) does the job. For here

$$\frac{1}{\lambda_1(f)} = \max_{u \in S_H^1} (A^{-1}(f)u, u)_H,$$

with the maximum being attained at $\mathcal{E}_1^1(f)$. Formally, the generalized gradient of a maximum is the convex hull of the gradient of the function being maximized, evaluated at it's maximizers. That is

$$\begin{aligned} \partial\lambda_1^{-1}(f) &= \mathrm{co}\,\{(\langle DA^{-1}(f), \cdot\rangle u, u)_H ; u \in \mathcal{E}_1^1(f)\} \\ &= \mathrm{co}\,\{-(A^{-1}(f)\langle DA(f), \cdot\rangle A^{-1}(f)u, u)_H ; u \in \mathcal{E}_1^1(f)\} \\ &= -\lambda_1^{-2}(f)\,\mathrm{co}\,\{(\langle DA(f), \cdot\rangle u, u)_H ; u \in \mathcal{E}_1^1(f)\}. \end{aligned} \tag{2.2}$$

Special cases of this formula have appeared in Polak and Wardi [PW] (symmetric matrices) and Cox and Overton [CO] (a Sturm–Liouville problem).

When $k > 1$ (2.1) is no longer a simple maximum principle and moreover the class of maximizing subspaces remains poorly understood. Three alternate representations have appeared in existing work on the generalized gradient at a multiple eigenvalue. All were confined to the treatment of symmetric matrices and the two earlier attempts required considerable digressions. We take up these works in the order of their appearance.

Gollan [G], with a representation

$$\lambda_k^{-1}(f) = \max_{\substack{u \in S_H^1 \\ u \perp \oplus_{i=1}^{k-1} \mathcal{E}_i(f)}} (A^{-1}(f)u, u)_H,$$

that places f in the constraint set, was compelled to draw on difficult results regarding the sensitivity of the Value function. In the finite dimensional context he found

$$\partial \lambda_k^{-1}(f) \subset -\lambda_k^{-2} \operatorname{co}\{(\langle DA(f), \cdot \rangle u, u)_H \,;\, u \in \mathcal{E}_k^1(f)\}. \tag{2.3}$$

The author [C], on choosing

$$\lambda_k^{-1}(f) = \max_{u \in H} \; 2((I - P_{k-1}(f))u, u)_H^{1/2} - (A(f)u, u)_H,$$

where $P_{k-1}(f)$ is the orthogonal projection of H onto $\oplus_{i=1}^{k-1} \mathcal{E}_i(f)$, simply shifted the difficulty from the constraint to the objective function. When $\lambda_k(f)$ is distinct from $\lambda_{k-1}(f)$ (e.g., if k and f are the j and f_0 of (1.1)) the projection is differentiable and $\langle DP_{k-1}(f), g \rangle : \mathcal{E}_k(f) \to \mathcal{E}_k^{\perp}(f)$. In this case one achieves equality in (2.3). The apparent drawback to the approaches of [G] and [C] is that while the generalized gradient of a higher eigenvalue was found to be free of the derivatives of any previous eigenfunctions, the proof of the formula nonetheless required one to differentiate eigenfunctions. This is simply an artifact of poor representation of $1/\lambda_k(f)$. Hiriart–Urruty and Ye [HY] have chosen the more clever characterization

$$\lambda_k^{-1}(f) = \sum_{i=1}^{k} \lambda_i^{-1}(f) - \sum_{i=1}^{k-1} \lambda_i^{-1}(f).$$

Clever in light of Ky Fan's representation

$$\sum_{i=1}^{m} \lambda_i^{-1}(f) = \max_{Q \in \mathcal{Q}_m} \operatorname{tr}(A^{-1}(f)Q), \tag{2.4}$$

where $\mathcal{Q}_m$ is the class of rank m projections on H. Overton and Womersley [OW] and Hiriart–Urruty and Ye [HY] have independently computed the generalized gradient of (2.4). In this way one arrives at the same formula as [C].

We now extend these results to the infinite dimensional setting of the previous section via a representation of $1/\lambda_k(f)$ that is no more costly to differentiate than that for $1/\lambda_1(f)$. This will permit us to essentially lift the corresponding result from [CO, Theorem 4.3], which, admittedly, was lifted from [Cl, Theorem 2.8].

In particular, with respect to (1.1), we choose a value $\omega \in (\lambda_{j-1}(f_0), \lambda_j(f_0))$ and note that under the hypotheses of Lemma 1 there exist positive ϵ and δ for which

$$\lambda_{j-1}(f) + \epsilon < \omega < \lambda_j(f) - \epsilon, \quad \forall f \in B_\delta(f_0). \tag{2.5}$$

As a result, the resolvent $R(f, \omega) \equiv (iA(f)i' - \omega)^{-1}$ is symmetric, compact, and uniformly bounded on H over $f \in B_\delta(f_0)$. This affords us the very simple representation

$$\frac{1}{\lambda_j(f) - \omega} = \sup_{u \in S_H^1} (R(f, \omega)u, u)_H, \tag{2.6}$$

the sup being attained on $\mathcal{E}_j^1(f)$. Out of convenience we define $\mu(f) \equiv 1/(\lambda_j(f) - \omega)$ and label our candidate for its generalized gradient,

$$\Xi \equiv -\mu^2(f_0)\,\mathrm{co}\,\{(\langle DA(f_0), \cdot\rangle u, u)_H : u \in \mathcal{E}_j(f_0)\}.$$

Lemma 2. *$f \mapsto \mu(f)$ is Lipschitz on $B_\delta(f_0)$ and $\partial\mu(f_0) = \Xi$.*

Proof: For $f, g \in B_\delta(f_0)$ and $u \in S_H^1$ we find

$$\begin{aligned}
|((R(f,\omega) - R(g,\omega))u, u)_H| &\le \|R(f,\omega) - R(g,\omega)\|_{L(V',V)} \\
&= \|R(f,\omega)(A(f) - A(g))R(g,\omega)\|_{L(V'V)} \\
&\le K(\omega)\|f_1 - f_2\|_X,
\end{aligned}$$

where, as in the proof of Lemma 1, $K(\omega)$ is the maximum of $\|R(f,\omega)\|^2\|DA(f)\|$ over $B_\delta(f_0)$. As a supremum of uniformly Lipschitz functions, $f \mapsto \mu(f)$ is Lipschitz.

Carrying this argument slightly further we find $f \mapsto (R(f,\omega)u, u)_H$ to be $C^1(B_\delta(f_0))$. For

$$\begin{aligned}
&\lim_{t\to 0} \frac{(R(f+tg,\omega)u,u)_H - (R(f,\omega)u,u)_H}{t} \\
&\qquad = -\lim_{t\to 0} (R(f,\omega)\frac{A(f+tg) - A(f)}{t} R(f+tg,\omega)u, u)_H \\
&\qquad = -(\langle DA(f), g\rangle R(f,\omega)u, R(f,\omega)u)_H\,.
\end{aligned}$$

Now, if $\xi \in \Xi$ then for some $\gamma_i \geq 0$, $\sum_{i=1}^n \gamma_i = 1$, and $u_i \in \mathcal{E}_j^1(f_0)$,

$$\begin{aligned}
\langle \xi, g \rangle &= -\mu^2(f_0) \sum_{i=1}^n \gamma_i(\langle DA(f_0), g\rangle u_i, u_i)_H \\
&= \sum_{i=1}^n \gamma_i \lim_{t\downarrow 0} \frac{(R(f_0+tg,\omega)u_i, u_i)_H - (R(f_0,\omega)u_i,u_i)_H}{t} \\
&\leq \sum_{i=1}^n \gamma_i \limsup_{t\downarrow 0} \frac{\mu(f_0+tg) - \mu(f_0)}{t} \\
&\leq \mu^o(f_0; g),
\end{aligned}$$

and so $\xi \in \partial\mu(f_0)$.

Regarding the reverse inclusion, we define

$$\phi(f_0; g) \equiv \max_{\xi\in\Xi} \langle \xi, g\rangle, \quad g \in X$$

and prove $\mu^o(f_0; g) \leq \phi(f_0; g)$.

Select $f_n \to f_0$ and $t_n \downarrow 0$ such that

$$q_n \equiv \frac{\mu(f_n + t_n g) - \mu(f_n)}{t_n} \to \mu^o(f_0; g).$$

Pick $u_n \in \mathcal{E}_j^1(f_n + t_n g)$ and note that

$$\begin{aligned}
q_n &\leq \frac{(R(f_n+t_ng,\omega)u_n,u_n)_H - (R(f_n,\omega)u_n,u_n)_H}{t_n} \\
&= -(\langle DA(f_n+\bar{t}_n g), g\rangle R(f_n+\bar{t}_n g,\omega)u_n, R(f_n+\bar{t}_n g,\omega)u_n)_H,
\end{aligned} \tag{2.7}$$

where $0 \leq \bar{t}_n \leq t_n$ by the mean value theorem. In passing to the limit in (2.7) we first establish that $u_n \to u_0 \in \mathcal{E}_j(f_0)$ in H. By definition,

$$(A(f_n + t_n g)u_n, u_n)_H = \lambda_j(f_n + t_n g).$$

As λ_j and A are respectively uniformly bounded and coercive over $B_\delta(f_0)$, we have $\|u_n\|_V \leq C$. Via the reflexivity of V we find that $u_n \rightharpoonup u_0$ in V and therefore strongly in H and V'. The H convergence yields $u_0 \in S_H^1$ so u_0 is nontrivial. That u_0 is indeed an eigenfunction follows from passing to the limit in

$$(A(f_n + t_n g)u_n, v)_H = \lambda_j(f_n + t_n g)(u_n, v)_H, \quad v \in V.$$

The right hand side poses no problem while the left is best written $(u_n, A(f_n + t_n g)v)_H$. As $A(f_n + t_n g) \to A(f_0)$ in $L(V, V')$ we find $A(f_n + t_n g)v \to A(f_0)v$ in V' for each $v \in V$. Hence,

$$(A(f_0)u_0, v)_H = \lambda(f_0)(u_0, v)_H, \quad v \in V.$$

That $\lambda(f_0)$ is in fact $\lambda_j(f_0)$ follows from the ordering of the eigenvalues at each $f_n + t_n g$ and the fact that the limiting eigenvalues completely exhaust the spectrum of $A(f_0)$, see Boccardo and Marcellini [BM].

As $R(f_n + \bar{t}_n g, \omega) \to R(f_0, \omega)$ in $L(V', V)$ we find

$$R(f_n + \bar{t}_n g, \omega)u_n \to R(f_0, \omega)u_0 = -\mu(f_0)u_0 \quad \text{in } V.$$

As $\langle DA(f_n + \bar{t}_n g), g\rangle \to \langle DA(f_0), g\rangle$ in $L(V, V')$ we may finally pass to the limit in (2.7), and so find

$$\mu^o(f_0; g) = \lim_{n\to\infty} q_n \leq -\mu^2(f_0)(\langle DA(f_0), g\rangle u_0, u_0)_H \leq \phi(f_0; g).$$

Hence, if $\zeta \in \partial\mu(f_0)$ then $\langle \zeta, g\rangle \leq \phi(f_0; g)$ for each $g \in X$. That is

$$0 \leq \max_{\xi\in\Xi} \langle \xi - \zeta, g\rangle, \quad \forall g \in X.$$

Now minimizing in g over X clearly produces

$$0 = \min_{g\in X} \max_{\xi\in\Xi} \langle \xi - \zeta, g\rangle.$$

As $\mathcal{E}_j^1(f_0)$ is compact in V and $\|DA(f_0)\|$ is finite we find Ξ to be compact in X'. That a continuous bilinear functional over a product of convex sets (with at least one being compact), possesses a saddle point follows, e.g., from Barbu and Precupanu [BP, Theorem 3.7, Chapter 2]. If $(\hat{g}, \hat{\xi})$ is this saddle point then

$$\langle \hat{\xi} - \zeta, g\rangle = 0, \quad \forall g \in X,$$

that is $\zeta = \hat{\xi}$ and so $\partial\mu(f_0) \subset \Xi$. ∎

To recover our claim on $\partial\lambda_j(f_0)$ we observe that

$$\begin{aligned}\partial\lambda_j(f_0) &= \partial(\lambda_j - \omega)(f_0) \\ &= -\mu^{-2}(f_0)\partial\mu(f_0) \\ &= \mathrm{co}\,\{(\langle DA(f_0), \cdot\rangle u, u)_H : u \in \mathcal{E}_j^1(f_0)\}.\end{aligned}$$

The first equality follows from [Cl, Proposition 2.3.3, Corollary 1], the second from the chain rule [Cl, Theorem 2.3.9(ii)], and the third from our Lemma 2.

That this set also agrees with $\partial\lambda_{j+m_j-1}(f_0)$ follows from the above upon choosing ω in the gap to the right of $\lambda_{j+m_j-1}(f_0)$. That is, one picks

$$\lambda_{j+m_j-1}(f_0) < \omega < \lambda_{j+m_j}(f_0)),$$

notes the existence of $R(f,\omega)$ in a ball about f_0, and proceeds to differentiate

$$\frac{1}{\lambda_{j+m_j-1}(f)-\omega} = \inf_{u\in S_H^1} (R(f,\omega)u,u)_H.$$

3. Its Application

In practice one is more often faced with the so–called generalized eigenvalue problem

$$A(f)u = \lambda M(f)u \tag{3.1}$$

where $M \in C^1(B_\delta(f_0), E(W))$ and $W \subset H \subset W'$ comprise a Gelfand triple. Typically M is of lower order than A and so $V \subset W$. With respect to the least eigenvalue of (3.1), upon differentiating the characterization

$$\lambda_1(f) = \min_{v\in V} \frac{(A(f)v,v)_H}{(M(f)v,v)_H}$$

one finds

$$\partial\lambda_1(f_0) = \mathrm{co}\,\{(\langle DA(f_0) - \lambda_1(f_0)DM(f_0),\cdot\rangle u, u) : u \in \mathcal{E}_1^1(f_0)\}, \tag{3.2}$$

where $\mathcal{E}_1^1(f_0)$ is the intersection of the first eigenspace of (3.1) and the unit sphere in H. Gollan [G] established the above with $\subset$ rather than $=$ for *each* of the eigenvalues of a generalized matrix problem. Haeberly and Overton [HO] have shown that equality holds in the matrix case for the least eigenvalue.

To extend our result to the higher eigenvalues we should of course address the extent to which (2.6) obtains in the generalized context. We avoid a number of the strictly technical issues and yet still arrive at a useful result upon assuming that $H = L^2(\Omega)$ and that $M(f)$ corresponds to multiplication by a bounded nonnegative function. In particular, we suppose

$$M(f)u(x) = m(f(x))u(x), \tag{3.3}$$

where $m : \mathbf{R} \to [0, \overline{m} < \infty]$ is smooth. With this restriction $M(f)$ possesses a natural square–root.

Suppose that ω lies between two eigenvalues of (3.1) at f_0, i.e., $\lambda_{j-1}(f_0) < \omega < \lambda_j(f_0)$. As above it follows that for δ sufficiently small there exists a positive ϵ for which

$$\lambda_{j-1}(f) + \epsilon < \omega < \lambda_j(f) - \epsilon$$

holds for each $f \in B_\delta(f_0)$. For such f it is now a simple matter to check that (2.6) takes the form

$$\frac{1}{\lambda_j(f) - \omega} = \sup_{u \in S_H^1} (M^{1/2}(f) R(f,\omega) M^{1/2}(f)u, u)_H, \tag{3.4}$$

where $R(f,\omega)$ is now the generalized resolvent

$$R(f,\omega) \equiv (iA(f)i' - \omega M(f))^{-1}.$$

The calculation of the generalized gradient of $f \mapsto \lambda(f)$ at f_0 now proceeds exactly as in §2.

Theorem 2. *Suppose that $A \in C^1(B_r(f_0), E(V))$ and $M \in C^1(L^2, L^2)$ and M is multiplication by a positive bounded function, (see (3.3)). If*

$$\lambda_{j-1}(f_0) < \lambda_j(f_0) = \cdots \lambda_{j+m_j-1}(f_0) < \lambda_{j+m_j}(f_0)$$

then

$$\partial\lambda_j(f_0) = \partial\lambda_{j+m_j-1}(f_0) = \mathrm{co}\,\{(\langle DA(f_0) - \lambda_j(f_0)DM(f_0), \cdot\rangle u, u) : u \in \mathcal{E}_j^1(f_0)\}.$$

We proceed to calculate the generalized gradient of two model functionals,

$$\mathcal{G}(f) = \lambda_2(f) - \lambda_1(f) \quad \text{and} \quad \mathcal{D}(f,\omega) = \min_k |\lambda_k(f) - \omega|$$

denoting respectively the gap and the distance to the spectrum, in the context of (3.1) where

$$V = W_0^{1,2}(\Omega),\ H = L^2(\Omega),\ X = L^\infty(\Omega),\ A(f) = -\nabla \cdot f\nabla,\ M(f) = f,$$

and Ω is some open bounded connected subset of $\mathbf{R}^n$. If $f_0 \in L^\infty(\Omega)$ is almost everywhere greater than some positive constant then $f \mapsto A(f)$ is smooth in some

$L^\infty(\Omega)$ ball about f_0. In this case it is well known that $\lambda_1(f)$ is simple. Hence $\partial\lambda_1(f_0)$ is a singleton, λ_1 is therefore, see [Cl, Proposition 2.2.4], strictly differentiable at f_0 and, see [Cl, Proposition 2.3.3, Corollary 2],

$$\begin{aligned}\partial\mathcal{G}(f_0) &= \partial\lambda_2(f_0) - \partial\lambda_1(f_0)\\ &= \mathrm{co}\,\{|\nabla u|^2 - \lambda_2(f_0)u^2 - |\nabla u_1|^2 + \lambda_1(f_0)u_1^2 : u \in \mathcal{E}_2^1(f_0)\},\end{aligned}$$

where u_1 is the positive element of $\mathcal{E}_1^1(f_0)$.

We characterize the reciprocal of $\mathcal{D}$ in terms of

$$\mathcal{D}^{-1}(f,\omega) = \sup_{u\in S_H^1} |(M^{1/2}(f)R(f,\omega)M^{1/2}(f)u,u)_H|. \tag{3.5}$$

With respect to (3.4), the effect of the absolute value here is simply to increase the size of the set on which the maximum is attained. At worst, and, in those applications where f is chosen to maximize $\mathcal{D}$, most likely, ω is the midpoint of the two multiple eigenvalues $\lambda_k(f)$ and $\lambda_{k+1}(f)$. In this case, the maximum in (3.5) is attained at $\mathcal{E}_k^1(f) \cup \mathcal{E}_{k+1}^1(f)$, and

$$\partial_f\mathcal{D}^{-1}(f,\omega) = -\mathcal{D}^{-2}(f,\omega)\,\mathrm{co}\,\{\partial\lambda_{k+1}(f) \cup -\partial\lambda_k(f)\}$$

The generalized gradient of $f \mapsto \mathcal{D}(f,\omega)$ follows, as above, from the chain rule.

Regarding less synthetic applications, the generalized gradient of an eigenvalue with respect to a drum's shape and a conductor's composition has been evaluated and applied in [CR] and [CL] respectively.

4. **References**

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