# A LYAPUNOV FUNCTION FOR SYSTEMS WHOSE LINEAR PART IS ALMOST CLASSICALLY DAMPED

Steven J. Cox† and Julio Moro‡

†Department of Computational and Applied Mathematics Rice University, PO Box 1892, Houston, TX 77251

‡Departamento de Matemáticas, Universidad Carlos III, Butarque 15, 28911 Leganes (Madrid) Spain

**Abstract:** We show that one may construct a Lyapunov function for any classically damped linear system. The explicit nature of the construction permits us to show that it remains a Lyapunov function under both perturbation of the linear part and introduction of a nonlinear term. We apply our findings to a stability analysis of the discrete, as well as continuous, damped mechanical transmission line.

## 1. **Introduction**

In this work we address the global stability of the zero solution of

$$u''(t) + Du'(t) + Ku(t) = f(u(t), u'(t)), \quad t > 0, \tag{1.1}$$

where $' \equiv \frac{d}{dt}$ and $u(t) \in \mathbf{R}^n$. One, now classical, approach to such a problem is through the methods of A. M. Liapunov (1966). In particular, one constructs a positive definite functional whose time derivative along solutions of (1.1) is negative definite. Though the import of these ideas has been widely appreciated since their export to the west in the late fifties, the Achilles' heel has remained the lack of a systematic procedure by which one may arrive at a Lyapunov function. Indeed, as evidenced by the text of Gajić and Qureshi (1995), contemporary work has rather focused on the numerical approximation of such functions.

In this paper we shall show that one may construct, by hand, a Lyapunov function for (1.1) when $f$ is small and $D$ and $K$ almost commute. In §2 we shall specify our exact hypotheses and show that under these (1.1) indeed possesses global solutions. In §3 we proceed to construct the Lyapunov function. There we present inequalities which facilitate two distinct modes of application. In the first we imagine that one begins with a classically damped linear system and wishes to determine its robustness. That is, the extent to which the stiffness and damping may be perturbed and nonlinearity introduced without causing instability of the trivial solution. The second scenario deals with the situation in which one wishes to determine the amount of damping necessary to stabilize the trivial solution of a given nonlinear system. As an instance of this second scenario we consider, in §4, the stabilization of the mechanical transmission line. This line was first introduced as a discrete analog of a nonlinear Klein–Gordon equation. We close with an application of our method to this nonlinear partial differential equation.

## 2. **Preliminaries**

We ask that $f$ vanish at the origin in order that zero be a steady solution of (1.1) and we ask the gradient of $f$ at zero to vanish in order that it not interfere with the linear terms involving $D$ and $K$. To be precise, we require

$$f(0,0)=0 \quad \text{and} \quad \nabla f(0,0)=0. \tag{2.1}$$

We shall also require $f$ to be globally Lipschitz. More precisely, we assume the existence of a finite $C$ for which

$$\|f(p_1,q_1)-f(p_2,q_2)\| \le C(\|p_1-p_2\|+\|q_1-q_2\|), \tag{2.2}$$

for all $p_1, q_1, p_2$, and $q_2$ in $\mathbf{R}^n$. This shall be enough to establish the global existence of a unique solution to (1.1). Such a result is most easily obtained with reference to the equivalent first order system. To wit, we assign

$$V(t) \equiv \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$$

and note that $u$ satisfies (1.1) if and only if $V$ satisfies

$$V'(t) = AV(t) + F(V(t)), \tag{2.3}$$

where

$$A = \begin{pmatrix} 0 & I \\ -K & -D \end{pmatrix} \quad \text{and} \quad F(V(t)) = \begin{pmatrix} 0 \\ f(V(t)) \end{pmatrix}.$$

**Proposition 2.1.** *If $f$ satisfies (2.2) and (2.1) and $V_0 \in \mathbf{R}^{2n}$ then there exists a unique $V \in C^1(\mathbf{R}, \mathbf{R}^{2n})$ satisfying (2.3) for all $t$ and $V(0) = V_0$.*

Proof: With our Lipschitz hypothesis on $f$ we may invoke Miller and Michel (1982, Theorem 2.4.2) and conclude that (2.3), subject to $V(0) = V_0$, has a unique solution near $t = 0$. Also, it follows from (2.2) and (2.1) that

$$\|F(U)\| \le C\|U\| \qquad \forall\, U \in \mathbf{R}^{2n}. \tag{2.4}$$

Now suppose $[0,\tau)$ to be the interval of existence. As such,

$$V(t) = V_0 + \int_0^t \{AV(s) + F(V(s))\}\, ds, \quad 0 \le t < \tau,$$

and so, recalling (2.4),

$$\|V(t)\| \le \|V_0\| + (\|A\| + C) \int_0^t \|V(s)\|\, ds, \quad 0 \le t < \tau.$$

It now follows from Gronwall's inequality, see Miller and Michel (1982, Theorem 2.1.6), that

$$\|V(t)\| \le \|V_0\| \mathrm{e}^{(\|A\|+C)t}, \quad 0 \le t < \tau,$$

and hence $V(t)$ is bounded on the closed interval $[0,\tau]$. From this we may conclude, see, e.g., Miller and Michel (1982, Corollary 2.3.2), that $V$ is in fact defined on the whole real line. ∎

This result carried no restrictions on $K$ and $D$. We shall see that in order to guarantee stability both $K$ and $D$ will have to outweigh $f$. To state our precise conditions we follow Caughey and O'Kelly (1965) and recall that a system (whose mass matrix has been normalized to the identity) with symmetric and positive definite stiffness, $K_0$, and damping, $D_0$, matrices is said to be classically damped when

$$K_0 D_0 = D_0 K_0. \tag{2.5}$$

Such systems are also referred to as "normal mode" or "Caughey damped." With respect to (1.1) we suppose that $K$ and $D$ are both symmetric and positive definite and that there exists a classically damped pair, $K_0$ and $D_0$, for which

$$\|K - K_0\| \le \delta \quad \text{and} \quad \|D - D_0\| \le \delta. \tag{2.6}$$

Our main result states that if $\delta$ and $C$ are sufficiently small, relative to $K_0$ and $D_0$, then the zero solution of (1.1) is globally asymptotically stable. The precise statement requires

$$\lambda \equiv \min_{v \in \mathbf{R}^n} \frac{\langle K_0 v, v\rangle}{\langle v, v\rangle} \quad \text{and} \quad \gamma \equiv \min_{v \in \mathbf{R}^n} \frac{\langle D_0 v, v\rangle}{\langle v, v\rangle}, \tag{2.7}$$

the least eigenvalues of $K_0$ and $D_0$, respectively.

## 3. The Lyapunov Function

Presence of a Lyapunov function shall lead to a proof of

**Theorem 3.1.** *If $f$ satisfies* (2.2) *and $K$ and $D$ satisfy* (2.6) *and $K_0$ and $D_0$ commute and*

$$C+\delta<\frac{2\gamma\sqrt{(\gamma+2\lambda)^2+\lambda(2-\gamma)^2}-2\gamma(\gamma+2\lambda)}{(2-\gamma)^2}, \tag{3.1}$$

*then the zero solution of* (1.1) *is globally asymptotically stable.*

This inequality bounds the degree of the nonlinearity, $C$, and deviation from classically damped, $\delta$, in terms of the known stiffness, $\lambda$, and damping, $\gamma$. For reference let us refer to the right side of this inequality as $\xi_+(\gamma,\lambda)$. This function clearly approaches zero as $\lambda$ and/or $\gamma$ tend to zero. Three other limits of interest are

$$\lim_{\gamma\to 2}\xi_+(\gamma,\lambda)=\frac{\lambda}{\lambda+1},\quad \lim_{\gamma\to\infty}\xi_+(\gamma,\lambda)=2\sqrt{1+\lambda}-2\quad\text{and}\quad \lim_{\lambda\to\infty}\xi_+(\gamma,\lambda)=\gamma/2.$$

As mentioned in the introduction we may also wish to know how large $\gamma$ must be in order to stabilize a given nonlinear system. This is addressed by

**Corollary 3.2.** *Suppose that $f$ satisfies* (2.2) *and $K$ and $D$ satisfy* (2.6) *and $K_0$ and $D_0$ commute. Let us call $\xi=C+\delta$. If*

$$\xi+\xi^2/4<\lambda\quad\textit{and}\quad\gamma>\frac{2\xi}{4\lambda-\xi^2-4\xi}\left(2\sqrt{\lambda^2+\lambda(1-\xi)-\xi}+2\lambda-\xi\right).$$

*then the zero solution of* (1.1) *is globally asymptotically stable.*

We build a Lyapunov function for the classically damped problem and then show that it is robust enough to serve as a Lyapunov function for the almost classically damped problem. To this end, let us first consider the linear classically damped system

$$U_0'(t)=A_0U_0(t),\quad\text{where}\quad A_0=\begin{pmatrix}0 & I\\ -K_0 & -D_0\end{pmatrix}. \tag{3.2}$$

As usual, we derive the Lyapunov function from energy considerations. We denote the instantaneous energy in $U_0$ by

$$E_0(t)=\langle S_0U_0(t),U_0(t)\rangle\quad\text{where}\quad S_0=\begin{pmatrix}K_0 & 0\\ 0 & I\end{pmatrix},$$

and the total energy by

$$L_0=\int_0^\infty E_0(t)\,dt.$$

As $U_0(t)$ is simply $\mathrm{e}^{A_0t}U_0(0)$ we recognize in $L_0$ the quadratic form

$$L_0(U_0(0))=\langle\mathcal{E}_0U_0(0),U_0(0)\rangle\quad\text{where}\quad\mathcal{E}_0=\int_0^\infty \mathrm{e}^{A_0^Tt}S_0\mathrm{e}^{A_0t}\,dt.$$

It is not difficult to show that $L_0$ serves as a Lyapunov function for the full nonlinear system under the assumption that $F$ is sufficiently small. However, as $\mathcal{E}_0$, and therefore $L_0$, is in general difficult to compute and/or estimate, one does not find sharp conditions in terms of bounds on $f$ and $K_0$ and $D_0$ sufficient to guarantee the global stability of the zero solution of (1.1). Our contribution amounts to exploiting the fact, first noted by Cox (1997), that $\mathcal{E}_0$ may be computed in closed form when $K_0$ and $D_0$ commute.

This is most easily seen by recalling, see, e.g., Bellman, (1970, §13.5), that $\mathcal{E}_0$ is the solution of the Lyapunov equation

$$A_0^T\mathcal{E}_0 + \mathcal{E}_0 A_0 = -S_0. \tag{3.3}$$

Given this form, one easily checks that

$$\mathcal{E}_0 = \begin{pmatrix} \frac{1}{2}D_0 + K_0 D_0^{-1} & \frac{1}{2}I \\ \frac{1}{2}I & D_0^{-1} \end{pmatrix}$$

is the symmetric, positive definite solution. Now, in order to establish Theorem 3.1, we must show that

$$\frac{d}{dt}L_0(V(t)) < 0$$

for all $t$ and for every nontrivial solution $V$ of (2.3). Using only (2.3) and (3.3) we find

$$\begin{aligned}\frac{d}{dt}L_0(V(t)) &= \langle \mathcal{E}_0 V(t), V'(t)\rangle + \langle \mathcal{E}_0 V'(t), V(t)\rangle \\ &= \langle \mathcal{E}_0 V(t), AV(t) + F(V(t))\rangle + \langle \mathcal{E}_0\{AV(t) + F(V(t))\}, V(t)\rangle \\ &= 2\langle \mathcal{E}_0 F(V(t)), V(t)\rangle + \langle\{A^T\mathcal{E}_0 + \mathcal{E}_0 A\}V(t), V(t)\rangle.\end{aligned}$$

The strategy is now to show that as $A$ is close to $A_0$ the second term is sufficiently negative to counteract the indefiniteness of the first term. In particular, given (2.6), we may write

$$K = K_0 + \delta\widetilde{K} \quad \text{and} \quad D = D_0 + \delta\widetilde{D} \quad \text{where} \quad \|\widetilde{K}\| \le 1 \quad \text{and} \quad \|\widetilde{D}\| \le 1.$$

This produces

$$A = A_0 + \delta\widetilde{A}, \quad \text{where} \quad \widetilde{A} = \begin{pmatrix} 0 & 0 \\ -\widetilde{K} & -\widetilde{D} \end{pmatrix},$$

and hence

$$\begin{aligned}\langle\{A^T\mathcal{E}_0 + \mathcal{E}_0 A\}V(t), V(t)\rangle &= \langle\{A_0^T\mathcal{E}_0 + \mathcal{E}_0 A_0\}V(t), V(t)\rangle + \langle\{\widetilde{A}^T\mathcal{E}_0 + \mathcal{E}_0\widetilde{A}\}V(t), V(t)\rangle \\ &= -\langle S_0 V(t), V(t)\rangle - \langle \widetilde{S}_0 V(t), V(t)\rangle,\end{aligned}$$

where

$$\widetilde{S}_0 \equiv \widetilde{A}^T\mathcal{E}_0 + \mathcal{E}_0\widetilde{A} = \begin{pmatrix} \widetilde{K} & \frac{1}{2}\widetilde{D} + \widetilde{K}D_0^{-1} \\ \frac{1}{2}\widetilde{D} + D_0^{-1}\widetilde{K} & D_0^{-1}\widetilde{D} + \widetilde{D}D_0^{-1} \end{pmatrix}.$$

We shall find it more convenient to now return to the original coordinates $V = [u\ u']$. In particular,

$$L_0(u,u') = L_0(V) = \langle \mathcal{E}_0 V, V\rangle = \langle (K_0 D_0^{-1} + D_0/2)u, u\rangle + \langle D_0^{-1}u', u'\rangle + \langle u, u'\rangle. \quad (3.4)$$

and

$$\begin{aligned}
\frac{d}{dt}L_0(u,u') &= \langle u, f\rangle + 2\langle u', D_0^{-1} f\rangle - \langle u, K_0 u\rangle - \langle u', u'\rangle \\
&\quad - \delta\{\langle u, \widetilde{K}u\rangle + \langle u, (\widetilde{D} + 2\widetilde{K}D_0^{-1})u'\rangle + 2\langle u', D_0^{-1}\widetilde{D}u'\rangle\}. \\
&\le C\|u\|(\|u\| + \|u'\|) + (2C/\gamma)\|u'\|(\|u\| + \|u'\|) - \lambda\|u\|^2 - \|u'\|^2 \\
&\quad + \delta\{\|u\|^2 + (1 + (2/\gamma))\|u\|\|u'\| + (2/\gamma)\|u'\|^2\} \\
&= (C + \delta - \lambda)\|u\|^2 + (C+\delta)(1 + (2/\gamma))\|u\|\|u'\| + ((C+\delta)(2/\gamma) - 1)\|u'\|^2
\end{aligned}$$

Assigning $\xi = C + \delta$ the above is negative so long as

$$\xi < \lambda \quad \text{and} \quad \xi^2(1 + (2/\gamma))^2 < 4(\xi - \lambda)((2\xi/\gamma) - 1). \quad (3.5)$$

Collecting terms in the latter we find that

$$(2-\gamma)^2\xi^2 + 4\gamma(2\lambda + \gamma)\xi - 4\lambda\gamma^2 < 0. \quad (3.6)$$

So long as $\gamma \neq 2$ this polynomial has the roots

$$\xi_\pm = \left(-2\lambda - \gamma \pm \sqrt{(2\lambda+\gamma)^2 + \lambda(2-\gamma)^2}\right)\frac{2\gamma}{(2-\gamma)^2}.$$

The polynomial is clearly positive for $\xi$ of large magnitude and so it can only be negative when $\xi$ lies between $\xi_-$ and $\xi_+$. We observe that $\xi_- < 0$ and that $\xi_+ > 0$ and that

$$\xi_- \to -\infty \quad \text{and} \quad \xi_+ \to \frac{\lambda}{\lambda+1} \quad \text{as} \quad \gamma \to 2.$$

In addition, it is a simple matter to verify that $\xi_+ \le \lambda$. As such, the two conditions of (3.5) reduce to $\xi < \xi_+$, i.e.,

$$C + \delta < \frac{2\gamma\sqrt{(\gamma+2\lambda)^2 + \lambda(2-\gamma)^2} - 2\gamma(\gamma+2\lambda)}{(2-\gamma)^2}.$$

This completes our proof of Theorem 3.1. Regarding its Corollary we need only look on (3.6) as an inequality for a quadratic in $\gamma$. In particular, (3.6) brings

$$\gamma^2(\xi^2 + 4\xi - 4\lambda) + \gamma 4\xi(2\lambda - \xi) + 4\xi^2 < 0. \quad (3.7)$$

which of course, as $2\lambda > \xi$, may only be satisfied when the coefficient of $\gamma^2$ is negative. As a result we must require

$$\xi + \xi^2/4 < \lambda. \tag{3.8}$$

With this, (3.7) is satisfied so long as $\gamma$ exceeds the quadratic's greatest root, namely when

$$\gamma > \frac{2\xi}{4\lambda - \xi^2 - 4\xi}\left(2\sqrt{\lambda^2 + \lambda(1-\xi) - \xi} + 2\lambda - \xi\right).$$

This inequality, together with (3.8), establishes Corollary 3.2.

## 4. **The Mechanical Transmission Line**

As an application of Corollary 3.2 we shall determine the amount of damping necessary to render stable the trivial solution of the mechanical transmission line depicted in figure 1. This line has been the subject of frequent study, see, e.g., Remoissenet (1994), since its introduction by Scott (1969). It consists of a series of $n$ pendula connected by a spring and supported horizontally on a taut one meter length of wire.

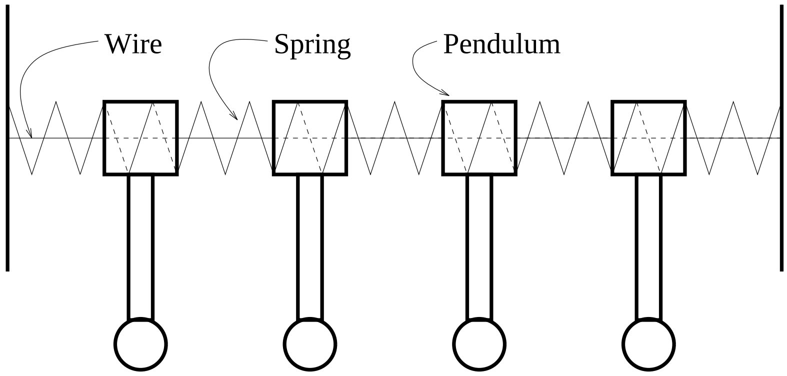


Figure 1. The Mechanical Transmission Line

With $u$ denoting the angle of rotation of the pendula the equation of motion of the $i$th pendulum takes the form

$$Ju_i'' = Q(u_{i+1} - 2u_i + u_{i-1}) - G\sin u_i - Ru_i', \quad i = 1, ..., n, \quad u_0 = u_{n+1} = 0. \tag{4.1}$$

Here $J$ is the moment of inertia of a single pendulum (kg m$^2$), $Q$ is the torque constant of that part of the spring separating two pendula (N m/rad), $G\sin u_i$ is the gravitational restoring torque (N m), and $Ru_i'$ is the dissipative torque (N m) associated with the introduction of rotational damping. Casting this as in (1.1) we find

$$K = (Q/J)T + (G/J)I, \quad D = (R/J)I, \quad \text{and} \quad f = (G/J)(u - \sin(u)),$$

where $T$ is the $n$-by-$n$ symmetric tridiagonal matrix with 2 on the diagonal and $-1$ on the superdiagonal and $I$ is the $n$-by-$n$ identity matrix. Our interest is in stating simple conditions on $J, Q, G$, and $R$ that guarantee the stability of the zero solution of (1.1) for such $K$, $D$, and $f$.

As $D$ is simply a multiple of the identity our system is obviously classically damped. This produces $\delta = 0$ and one also finds

$$\lambda = (Q\mu + G)/J, \quad \gamma = R/J, \quad \text{and} \quad C = 2G/J$$

where

$$\mu = 4\sin^2\left(\frac{\pi}{2(n+1)}\right)$$

is the least eigenvalue of $T$. A direct application of Corollary 3.2 now guarantees stability of the trivial solution when

$$G + G^2/J < Q\mu \quad \text{and} \quad R > \frac{2G}{Q\mu - G - G^2/J}\left(Q\mu + \sqrt{(Q\mu + G)(Q\mu - G + J) - 2G}\right), \tag{4.2}$$

that is, when the stiffness dominates the inertial components of the pendulum and when the rotational damping is on the order of the reciprocal of their difference. However, as $f$ is free of terms involving the velocity, $u'$, one may suspect that sharper inequalities may be found. This is indeed the case though, rather than deriving them at this stage we shall find it instructive to first pass to the limit of a continuum of pendula. More precisely, we let the distance, $h$, between the pendula approach zero. We rewrite (4.1) as

$$(J/h)u_i'' = (Qh)\frac{[(u_{i+1} - u_i)/h] - [(u_i - u_{i-1})/h]}{h} - (G/h)\sin u_i - (R/h)u_i'.$$

As $h$ tends to zero we find that $J/h$ tends to $\mathcal{J}$, the moment of inertia per meter of line (kg m$^2$/m), that $Qh$ tends to $\mathcal{Q}$, the spring torque per meter of line (N m/rad m), that $(G/h)\sin u_i$ tends to $\mathcal{G}\sin u$, the gravitational torque per meter (N m/m), and that $(R/h)u_i'$ tends to $\mathcal{R}\partial u/\partial t$, the dissipative torque per meter (N m/m). The limiting differential equation is hence

$$\mathcal{J}u_{tt}(x,t) - \mathcal{Q}u_{xx}(x,t) + \mathcal{R}u_t(x,t) = -\mathcal{G}\sin u(x,t), \quad 0 < x < 1, \ 0 < t. \tag{4.3}$$

It is subject to the end conditions,

$$u(0,t) = u(1,t) = 0, \quad 0 < t.$$

Freely adopting the Lyapunov function of (3.4) brings

$$\begin{aligned} L_0(u, u_t) &= \int_0^1 \{u[-(\mathcal{Q}/\mathcal{R})\partial_{xx} + (\mathcal{R}/2\mathcal{J})]u + (\mathcal{J}/\mathcal{R})u_t^2 + uu_t\}\,dx \\ &= \int_0^1 \{(1/\mathcal{R})(\mathcal{Q}u_x^2 + \mathcal{J}u_t^2) + (\mathcal{R}/2\mathcal{J})u^2 + uu_t\}\,dx. \end{aligned}$$

We note that the Euclidean inner product on $\mathbf{R}^n$ has given rise to the $L^2(0,1)$ inner product on $H_0^1(0,1)$, the class of square integrable functions that vanish at zero and one and possess square integrable derivatives. We first verify that $L_0$ is indeed positive definite. Invoking Hölder's inequality brings

$$\begin{aligned} L_0(u,u_t) &\geq (\mathcal{Q}/\mathcal{R})\|u_x\|^2 + (\mathcal{J}/\mathcal{R})\|u_t\|^2 - \|u\|\|u_t\| + (\mathcal{R}/2\mathcal{J})\|u\|^2 \\ &\geq (\mathcal{Q}/\mathcal{R})\|u_x\|^2 + \omega(\|u_t\|^2 + \|u\|^2) \end{aligned}$$

where

$$\omega = \frac{(\mathcal{J}/\mathcal{R}) + (\mathcal{R}/2\mathcal{J}) - \sqrt{(\mathcal{J}/\mathcal{R})^2 + (\mathcal{R}/2\mathcal{J})^2}}{2} > 0.$$

Finally we demonstrate that the time derivative of $L_0$ is negative definite along solutions to (4.3). For this we make use of Rayleigh's Principle

$$\pi^2 \leq \frac{\int_0^1 w_x(x)^2\,dx}{\int_0^1 w(x)^2\,dx} \quad \forall\, w \in H_0^1(0,1).$$

Now

$$\begin{aligned} \frac{d}{dt}L_0(u,u_t) &= \int_0^1 \{(2/\mathcal{R})(\mathcal{Q}u_x u_{xt} + \mathcal{J}u_t u_{tt}) + (\mathcal{R}/\mathcal{J})uu_t + uu_{tt} + u_t^2\}\,dx \\ &= \int_0^1 \{(2/\mathcal{R})(-\mathcal{Q}u_{xx} + \mathcal{J}u_{tt})u_t + (1/\mathcal{J})(\mathcal{R}u_t + \mathcal{J}u_{tt})u + u_t^2\}\,dx \\ &= \int_0^1 \{(2/\mathcal{R})(-\mathcal{R}u_t - \mathcal{G}\sin u)u_t + (1/\mathcal{J})(\mathcal{Q}u_{xx} - \mathcal{G}\sin u)u + u_t^2\}\,dx \\ &= \int_0^1 \{(2\mathcal{G}/\mathcal{R})u_t \sin u - u_t^2 + (\mathcal{G}/\mathcal{J})u\sin u - (\mathcal{Q}/\mathcal{J})u_x^2\}\,dx \\ &\leq (2\mathcal{G}/\mathcal{R})\|u_t\|\|u\| - \|u_t\|^2 + (\mathcal{G}/\mathcal{J})\|u\|^2 - (\mathcal{Q}/\mathcal{J})\|u_x\|^2 \\ &\leq \{(\mathcal{G} - \mathcal{Q}\pi^2)/(\mathcal{J}\pi^2)\}\|u_x\|^2 + (2\mathcal{G}/\mathcal{R}\pi)\|u_x\|\|u_t\| - \|u_t\|^2 \\ &\leq \zeta(\|u_x\|^2 + \|u_t\|^2) \end{aligned}$$

where

$$\zeta = \frac{\mathcal{G} - \mathcal{Q}\pi^2 - \mathcal{J}\pi^2 + \sqrt{(\mathcal{G} - \mathcal{Q}\pi^2 - \mathcal{J}\pi^2)^2 - 4(\mathcal{Q}\pi^2 - \mathcal{G} - \mathcal{G}^2\mathcal{J}/\mathcal{R}^2)}}{2\mathcal{J}\pi^2}.$$

Now $\zeta$ is negative only when

$$\mathcal{Q}\pi^2 > \mathcal{G} \quad \text{and} \quad \mathcal{R}^2 > \frac{\mathcal{J}\mathcal{G}^2}{\mathcal{Q}\pi^2 - \mathcal{G}}. \tag{4.4}$$

In comparison to (4.2) we note that, as $h \to 0$, $Qh \to \mathcal{Q}$ and $(\mu/h^2) \to \pi^2$ and so $Q\mu/h \to \pi^2$. These final bounds, (4.4), are indeed sharper than (4.2). In particular, given prescribed inertial terms, $\mathcal{G}$ and $\mathcal{J}$, (4.4) affords greater leeway in the choice of torque and damping values, $\mathcal{Q}$ and $\mathcal{R}$. Finally we remark that our application has focused on systems with classically damped linear part. A more extensive application of Corollary 3.2 to either the discrete or continuous line would incorporate variations in the system parameters. In such a case $\delta$ would signify the distance from the variable coefficient line to the classically damped line.

## 5. References


Bellman, R., 1970, *Introduction to Matrix Analysis*, McGraw–Hill.

Caughey, T.K., and O'Kelly, M.E., 1965, "Classical Normal Modes in Damped Linear Dynamic Systems," *ASME J. of Appl. Mech.*, Vol. 32, pp. 583–588.

Cox, S.J., 1997, "Designing for Optimal Energy Absorption, I: Lumped Parameter Systems," *ASME J. of Vibration & Acoustics*, to appear.

Gajić, Z., and Qureshi, M.T.J., 1995, "Lyapunov Matrix Equation in System Stability and Control," Academic Press.

Liapunov, A.M., 1966, "Stability of Motion," Academic Press.

Miller, R.K., and Michel, A.N., 1982, "Ordinary Differential Equations," Academic Press.

Remoissenet, M., 1994, "Waves Called Solitons," Springer–Verlag.

Scott, A.C., 1969, "A Nonlinear Klein–Gordon Equation," American J. of Physics, Vol. 37, No. 1, pp. 52–61.



The work of the first author was supported by NSF Grant DMS-9258312.


Figure caption:

Figure 1. The Mechanical Transmission Line.