Recovering Planar Lamé Moduli From A Single Traction Experiment

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Abstract: Under a simple nondegeneracy condition, the displacement and edge traction of a planar, isotropic, linearly elastic solid determine its Lamé moduli. When these moduli are constant, they can be recovered exactly; this is demonstrated by a specific traction satisfying the nondegeneracy condition. Spatially varying moduli can be computed numerically by considering the equations of linear elasticity as a hyperbolic system for the unknown moduli. A stable finite-difference scheme for solving this system is given; synthetic experiments demonstrate its efficacy.

1. Introduction

In the biaxial testing of planar tissue one seeks, see, e.g., Nielsen *et al.* [NHS], to determine constitutive behavior from measurements of displacement under prescribed edge traction. Where typically such experiments are used to determine a few parameters in a hypothetical strain energy density we wish to show here that one may in fact recover material heterogeneities.

We suppose that our square sample

$$\Omega = (0,a) \times (0,a)$$

is isotropic, linearly elastic, and subject only to traction. That is

$$\nabla \cdot \sigma = 0 \quad \text{in } \Omega \tag{1.1}$$

$$\sigma n = g \quad \text{on } \partial\Omega \tag{1.2}$$

where n denotes the outer unit normal and the stress and strain

$$\sigma = 2\mu\mathcal{E} + \lambda(\operatorname{tr}\mathcal{E})I \quad \text{and} \quad \mathcal{E} = (\nabla u + \nabla u^T)/2$$

are derived from the displacement u. The Lamé moduli, μ and λ, are assumed to be smooth functions of their coordinates and to everywhere satisfy the strong ellipticity condition

$$\mu > 0, \quad \mu + \lambda > 0. \tag{1.3}$$

We shall show, under the nondegeneracy condition $\mathcal{E}_{12}\operatorname{tr}\mathcal{E} \neq 0$, that u and g uniquely determine μ and λ. We also build and test a practical algorithm for the recovery of μ and λ from u and g. Our approach follows Richter's attack, see [R1] and [R2], on the associated scalar problem of recovering α in

$$-\nabla\cdot\alpha\nabla v = f \quad \text{in } \Omega, \qquad \alpha\nabla v\cdot n = h \quad \text{on } \partial\Omega, \tag{1.4}$$

from knowledge of v, f, and h. We remark that the question of whether μ and λ are determined by knowledge of the associated Dirichlet to Neumann map has been answered in the affirmative by Nakamura and Uhlmann [NU].

2. The Uniqueness Theorem

The nondegeneracy condition is already apparent in the edge condition, (1.2). In particular, on the bottom edge we find

$$-2\mu\mathcal{E}_{12} = g_1 \quad \text{and} \quad -2\mu\mathcal{E}_{22} - \lambda \mathrm{tr}\,\mathcal{E} = g_2,$$

and so μ is determined there so long as $\mathcal{E}_{12}$ does not vanish while the calculation of λ there requires the nonvanishing of $\mathrm{tr}\,\mathcal{E}$. Similarly, the right and left edges require

$$\pm\begin{pmatrix} 2\mu\mathcal{E}_{11} + \lambda\mathrm{tr}\,\mathcal{E} \\ 2\mu\mathcal{E}_{12}\end{pmatrix} = g$$

respectively. Regarding the interior, we evaluate the divergence in (1.1) and so arrive at a first order system for $\ell \equiv (\mu,\lambda)$,

$$A\partial_y\ell + B\partial_x\ell + C\ell = 0 \tag{2.1}$$

where

$$A = \begin{pmatrix} 2\mathcal{E}_{12} & 0 \\ 2\mathcal{E}_{22} & \mathrm{tr}\,\mathcal{E}\end{pmatrix},\quad B = \begin{pmatrix} 2\mathcal{E}_{11} & \mathrm{tr}\,\mathcal{E} \\ 2\mathcal{E}_{12} & 0\end{pmatrix},\quad C = \begin{pmatrix} 2(\partial_x\mathcal{E}_{11} + \partial_y\mathcal{E}_{12}) & \partial_x\mathrm{tr}\,\mathcal{E} \\ 2(\partial_x\mathcal{E}_{12} + \partial_y\mathcal{E}_{22}) & \partial_y\mathrm{tr}\,\mathcal{E}\end{pmatrix}.$$

It remains to transform (2.1) into a form in which we can establish the existence of a solution. When $\mathcal{E}_{12}\mathrm{tr}\,\mathcal{E} \neq 0$ we multiply (2.1) on the left by

$$A^{-1} = \frac{1}{2\mathcal{E}_{12}\mathrm{tr}\,\mathcal{E}}\begin{pmatrix} \mathrm{tr}\,\mathcal{E} & 0 \\ -2\mathcal{E}_{22} & 2\mathcal{E}_{12}\end{pmatrix}$$

and find

$$\partial_y\ell + A^{-1}B\partial_x\ell + A^{-1}C\ell = 0. \tag{2.2}$$

Though

$$A^{-1}B = \frac{1}{2\mathcal{E}_{12}}\begin{pmatrix} 2\mathcal{E}_{11} & \mathrm{tr}\,\mathcal{E} \\ \frac{-4\det\mathcal{E}}{\mathrm{tr}\,\mathcal{E}} & -2\mathcal{E}_{22}\end{pmatrix}$$

is not symmetric it is diagonalizable. Its eigenvalues, in terms of the principle stretches $e_\pm$, are

$$\gamma_\pm = \frac{(\mathcal{E}_{11}-\mathcal{E}_{22}) \pm \sqrt{(\mathcal{E}_{11}-\mathcal{E}_{22})^2 + 4\mathcal{E}_{12}^2}}{2\mathcal{E}_{12}} = \frac{e_\pm - \mathcal{E}_{22}}{\mathcal{E}_{12}}.$$

These are two real numbers of opposite sign, in particular, $\mathrm{sgn}(\gamma_\pm) = \pm\mathrm{sgn}(\mathcal{E}_{12})$. We collect $\gamma_\pm$ in the diagonal matrix Γ and record in

$$P = \begin{pmatrix} \mathrm{tr}\,\mathcal{E} & \mathrm{tr}\,\mathcal{E} \\ -2e_- & -2e_+ \end{pmatrix}$$

the corresponding matrix of eigenvectors. Using $A^{-1}B = P\Gamma P^{-1}$ and setting $\ell = P\xi$ brings (2.2) into the form

$$\partial_y(P\xi) - P\Gamma P^{-1}\partial_x(P\xi) + A^{-1}CP\xi = 0.$$

Evaluating the derivatives and then multiplying through by P^{-1} yields the final form

$$\partial_y\xi + \Gamma\partial_x\xi + \mathcal{A}\xi = 0 \tag{2.3}$$

where

$$\mathcal{A} = (AP)^{-1}CP + P^{-1}\partial_y P + \Gamma P^{-1}\partial_x P.$$

As Γ is diagonal with a real, distinct diagonal it follows that (2.3) is a hyperbolic system. We now determine the side conditions that render its solution unique. Let us call $\gamma_1 = \gamma_+$ and $\gamma_2 = \gamma_-$ and suppose, without loss, that $\gamma_2 < 0 < \gamma_1$. They define characteristic curves in Ω via

$$\frac{dx}{dy} = \gamma_k, \quad k = 1, 2. \tag{2.4}$$

We shall denote by

$$s = X_k(t; x, y) \tag{2.5}$$

the solution to (2.4) that passes through the point $(x, y) \in \Omega$ and by $(\overline{x}_k, \overline{y}_k)$ the point on $\partial\Omega$ at which the characteristic enters ($\overline{y}_k < y$) Ω. Under the convention that $\gamma_2 < 0 < \gamma_1$ it follows that the characteristics defined by X_1 enter on the left and bottom while those defined by X_2 enter at the right and bottom. This said, we find (2.3) equivalent to

$$\xi_k(x, y) = \xi_k(\overline{x}_k, \overline{y}_k) - \int_{\overline{y}_k}^{y} [\mathcal{A}(X_k(t; x, y), t)\xi(X_k(t; x, y), t)]_k \, dt. \tag{2.6}$$

By the classical method of successive approximations, see, e.g., Petrovskii [P,§10], this integral equation possesses a unique C^1 solution so long as $\mathcal{A}$ and X are C^1, ξ_1

is given and C^1 on the left and bottom edges and ξ_2 is given and C^1 on the right and bottom. We remark that $\mathcal{A}$ and X are C^1 when the displacement, u, is C^3 and $\mathcal{E}_{12}\mathrm{tr}\,\mathcal{E} \neq 0$. Regarding the edge data, we have seen that ℓ may be determined on the left, right, and bottom edges from g and $\mathcal{E}$. Applying P^{-1} to ℓ we arrive at the required boundary values. They will be C^1 so long as $u \in C^2(\overline{\Omega})$ and g is C^1. Moreover, the linearity of (2.6), allows us to deduce continuous dependence from the proof of uniqueness. In particular,

Theorem 1. *If $u \in C^3(\overline{\Omega})$ and $g \in C^1(\partial\Omega)$ and $|\mathcal{E}_{12}\mathrm{tr}\,\mathcal{E}| \geq c > 0$ then there exists a unique $\ell \in C^1(\overline{\Omega})$ such that (u, g, ℓ) satisfies (1.1)–(1.2). If $(u^{(1)}, g^{(1)}, \ell^{(1)})$ and $(u^{(2)}, g^{(2)}, \ell^{(2)})$ are two such triples then there exists a constant M such that*

$$\|\ell^{(1)} - \ell^{(2)}\|_{C^0(\overline{\Omega})} \leq M(\|g^{(1)} - g^{(2)}\|_{C^0(\partial\Omega)} + \|u^{(1)} - u^{(2)}\|_{C^2(\overline{\Omega})})$$

As a simple illustration let us show how this apparatus may be used to recover constant Lamé moduli. The key issue, of course, is to devise a traction g under which neither the shear strain, $\mathcal{E}_{12}$, nor the sum of the principal stretches, $\mathrm{tr}\,\mathcal{E}$, changes sign. We have found

$$g = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} n \tag{2.7}$$

to be one such traction. This choice, with constant Lamé moduli, produces the linear (up to an additive constant) displacement

$$u(x, y) = \frac{1}{2\mu(\mu+\lambda)} \begin{pmatrix} x\mu + y(\mu+\lambda) \\ y\mu + x(\mu+\lambda) \end{pmatrix}$$

and corresponding constant strains

$$\mathcal{E}_{11} = \mathcal{E}_{22} = \frac{1}{2\mu(\mu+\lambda)} \quad \text{and} \quad \mathcal{E}_{12} = \frac{1}{2\mu}.$$

As μ and λ satisfy (1.3) we find that our choice of g leads to a trivial satisfaction of the nondegeneracy condition. Moreover, we find $\mathcal{A} = 0,\ \gamma_1 = 1,\ \gamma_2 = -1$ and so (2.3) takes the simple form

$$(\partial_y + \partial_x)\xi_1 = 0 \quad \text{and} \quad (\partial_y - \partial_x)\xi_2 = 0,$$

and hence $\xi_k(x, y) = \xi_k(\overline{x}_k, \overline{y}_k)$. As both ℓ and P are constant on $\partial\Omega$ it follows that so too is ξ. Hence ξ is constant throughout Ω. Applying P to ξ one recovers the constant Lamé moduli, ℓ.

In the more general case of nonconstant strains one may not solve (2.3) by hand and so is compelled to seek numerical alternatives.

3. Numerical Solution of the Inverse Problem

We now describe a convergent finite-difference scheme for solving equation (2.3). The scheme is a first-order, explicit upwind scheme that uses adaptive step-lengths to ensure stability. It is a modification of the method found in Garabedian [G], pages 469-472.

It will be convenient to label the known boundary values of ξ by

$$\begin{aligned} \psi_1(y) &\equiv \left[P^{-1}(0,y)\ell(0,y)\right]_1, \quad \psi_2(x) \equiv \left[P^{-1}(x,0)\ell(x,0)\right]_1, \\ \psi_3(x) &\equiv \left[P^{-1}(x,0)\ell(x,0)\right]_2, \quad \psi_4(y) \equiv \left[P^{-1}(1,y)\ell(1,y)\right]_2. \end{aligned} \tag{3.1}$$

We shall assume that the values of the strain $\mathcal{E}$, and its partial derivatives, $\partial_x\mathcal{E}$ and $\partial_y\mathcal{E}$, are given on a regular grid of h by h squares, thus allowing computation of the coefficients in (2.3) on the same grid. We set $h = 1/(N-1)$ and write

$$x_j = (j-1)h,\ y_i = (i-1)h, \quad \hat{\xi}(i,j) = \xi(x_j, y_i), \quad i = 1, \ldots, N\ j = 1, \ldots, N.$$

Under our assumption on the signs of γ_1, γ_2, the characteristics of ξ_1 have positive slope in the xy-plane, while those of ξ_2 have negative slope. We integrate along these characteristics, advancing one row in the grid (i.e. one step in y) by computing ξ_1 from left to right and ξ_2 from right to left. The scheme for advancing ξ_1 is based on the discretization

$$\begin{aligned} \frac{\hat{\xi}_1(i+1,j) - \hat{\xi}_1(i,j)}{\Delta y} &+ \hat{\gamma}_1(i,j)\frac{\hat{\xi}_1(i,j) - \hat{\xi}_1(i,j-1)}{\Delta x} \\ &+ \hat{\mathcal{A}}_{11}(i,j)\hat{\xi}_1(i,j) + \hat{\mathcal{A}}_{12}(i,j)\hat{\xi}_2(i,j) = 0. \end{aligned}$$

Solving this for $\hat{\xi}_1(i+1,j)$ we obtain

$$\begin{aligned} \hat{\xi}_1(i+1,j) = (1 - \frac{\Delta y}{\Delta x}\hat{\gamma}_1(i,j))\hat{\xi}_1(i,j) &+ \frac{\Delta y}{\Delta x}\hat{\gamma}_1(i,j)\hat{\xi}_1(i,j-1) \\ - \Delta y &\left(\hat{\mathcal{A}}_{11}(i,j)\hat{\xi}_1(i,j) + \hat{\mathcal{A}}_{12}(i,j)\hat{\xi}_2(i,j)\right), \end{aligned} \tag{3.2}$$

where both Δx and Δy initially equal h.

Applying similar considerations to ξ_2, we derive the following formula:

$$\begin{aligned} \hat{\xi}_2(i+1,j) = (1 + \frac{\Delta y}{\Delta x}\hat{\gamma}_2(i,j))\hat{\xi}_2(i,j) &- \frac{\Delta y}{\Delta x}\hat{\gamma}_2(i,j)\hat{\xi}_2(i,j+1) \\ - \Delta y &\left(\hat{\mathcal{A}}_{21}(i,j)\hat{\xi}_1(i,j) + \hat{\mathcal{A}}_{22}(i,j)\hat{\xi}_2(i,j)\right). \end{aligned} \tag{3.3}$$

Stability requires that the von Neumann conditions

$$\frac{\Delta y}{\Delta x}\hat{\gamma}_1(i,j) \leq 1, \quad \frac{\Delta y}{\Delta x}\hat{\gamma}_2(i,j) \leq 1$$

be satisfied. If this requirement is violated, then we reduce Δy appropriately, and take several intermediate steps to reach the row representing $y = y_{i+1}$. In the unlikely event that $\hat{\gamma}_1$ or $\hat{\gamma}_2$ grows sufficiently that this smaller value of Δy is not small enough to advance the integration from row i to row $i+1$ of the original grid, this idea is applied recursively. To produce the needed coefficients at these intermediate steps, we simply use linear interpolation on the original data (which, we recall, was given only on the original, square grid).

We now define the algorithm in detail. In what follows, note that $\hat{\xi}$ refers to the computed values of ξ on the original grid, while we use $\tilde{\xi}$ to refer to these values together with intermediate values arising from reducing the step size to retain stability. Thus $\tilde{\xi}(x_j, y_i) = \hat{\xi}(i,j)$.

ALGORITHM

0. Let N be a positive integer, and assume that $\hat{\Gamma}$ and $\hat{\mathcal{A}}$ are samples, on the regular N by N grid described above, of the continuously differentiable functions Γ and $\mathcal{A}$ defined on Ω. In addition, assume that $\{\hat{\psi}_k\}_{k=1}^4$ are samples of the continuously differentiable functions $\{\psi_k\}_{k=1}^4$, defined, per (3.1), on the unit interval. The following algorithm computes estimates $\hat{\xi}(i,j)$ of $\xi(x_j, y_i)$, $i,j = 1,2,\ldots,N$, where ξ solves (2.3).
1. Set the boundary values: $\hat{\xi}_1(i,1) = \hat{\psi}_1(i)$, $\hat{\xi}_1(1,j) = \hat{\psi}_2(j)$, $\hat{\xi}_2(1,j) = \hat{\psi}_3(j)$, and $\hat{\xi}_2(i,N) = \hat{\psi}_4(i)$, $i,j = 1,2,\ldots,N$.
2. For $i = 1,2,\ldots,N-1$:
   Execute **STEP** with $y = (i-1)h$ and $\Delta y = h$ to compute $\hat{\xi}_1(i+1,j)$, $j = 2,3,\ldots,N$ and $\hat{\xi}_2(i+1,j)$, $j = N-1, N-2, \ldots, 1$ from $\hat{\xi}(i,j)$, $j = 1,2,\ldots,N$ and $\hat{\xi}_1(i+1,1)$, $\hat{\xi}_2(i+1,N)$.

STEP

0. The purpose of the algorithm is to take values of $\tilde{\xi}(x_j, y)$, $j = 1,2,\ldots,N$, along with $\tilde{\xi}_1(0, y+\Delta y)$ and $\tilde{\xi}_2(1, y+\Delta y)$, and produce the values of $\tilde{\xi}(x_j, y+\Delta y)$, $j = 1,2,\ldots,N$.
1. Set $\gamma_{max} = \max\{|\gamma_k(x_j, y)| : k = 1,2,\ j = 1,2,\ldots,N\}$
2. If $\gamma_{max} \leq 1$
   i. For $j = 2,3,\ldots,N$, compute $\tilde{\xi}_1(x_j, y+\Delta y)$ by formula (3.2). NOTE: The value of $\tilde{\xi}_1(0, y+\Delta y)$ is given.
   ii. For $j = N-1, N-2, \ldots, 1$, compute $\tilde{\xi}_2(x_j, y+\Delta y)$ by formula (3.3). NOTE: The value of $\tilde{\xi}_2(1, y+\Delta y)$ is given.
2. Else
   i. Set $k = $ceil$(\gamma_{max})$
   ii. Set $\Delta y = \Delta y / k$
   iii. For $l = 1,2,\ldots,k$
   a. Call **STEP** recursively to compute the values of $\tilde{\xi}(x_j, dy + l\Delta y)$, $j = 1,2,\ldots,N$. The values of $\tilde{\xi}_1(0, y+l\Delta y)$ and $\tilde{\xi}_2(1, y+l\Delta y)$ are estimated by linear interpolation.

iv. Return the values $\tilde{\xi}(x_j, y + k\Delta y),\ j = 1, 2, \ldots, N$.

Since $|\gamma_1|$ and $|\gamma_2|$ are bounded on Ω, the recursion "bottoms out" when Δy is sufficiently small, and thus the algorithm is well-defined.

The above algorithm is clearly first-order consistent; therefore, to prove that it converges, we need only prove stability. Let us define

$$M_b = \max_k \|\hat{\psi}_k\|_\infty \quad \text{and} \quad M_{\mathcal{A}} = \|\hat{\mathcal{A}}\|_\infty,$$

where $\|\cdot\|_\infty$ denotes a maximum over components and indices.

Theorem 2. *The values $\hat{\xi}(i, j),\ i, j = 1, 2, \ldots, N$ produced by the algorithm satisfy*

$$\|\hat{\xi}\|_\infty \leq M_b e^{M_{\mathcal{A}}}.$$

Proof: Let $\Delta y_1, \Delta y_2, \ldots, \Delta y_t$ be the step sizes encountered in the above algorithm, and define $z_1 = 0$,

$$z_i = y + \Delta y_1 + \Delta y_2 + \ldots + \Delta y_{i-1},\ i > 1.$$

We shall show that the above inequality is satisfied by the values $\tilde{\xi}(x_j, z_i)$ computed in the course of the algorithm (note that there is a sequence $l_1, l_2, \ldots, l_N$ such that $\hat{\xi}(i, j) = \tilde{\xi}(x_j, z_{l_i})$). Define

$$M_l = \max_j \|\tilde{\xi}(x_j, z_l)\|_\infty$$

and note that $M_1 \leq M_b$. From equations (3.2) and (3.3), we see that

$$M_{l+1} \leq (1 + \Delta y_l M_{\mathcal{A}}) M_l.$$

Therefore,

$$\begin{aligned}
&M_2 \leq (1 + \Delta y_1 M_{\mathcal{A}}) M_b, \\
&M_3 \leq (1 + \Delta y_2 M_{\mathcal{A}}) M_1 \leq (1 + \Delta y_2 M_{\mathcal{A}})(1 + \Delta y_1 M_{\mathcal{A}}) M_b, \\
&M_4 \leq (1 + \Delta y_3 M_{\mathcal{A}}) M_3 \leq (1 + \Delta y_3 M_{\mathcal{A}})(1 + \Delta y_2 M_{\mathcal{A}})(1 + \Delta y_1 M_{\mathcal{A}}) M_b
\end{aligned}$$

and thus, by induction,

$$M_i \leq M_b \prod_{l=1}^{i-1} (1 + \Delta y_l M_{\mathcal{A}})$$

We now use the inequality $(1 + s) \leq e^s$ to simplify the upper bound. We have

$$\prod_{l=1}^{i-1} (1 + \Delta y_l M_{\mathcal{A}}) \leq e^{M_{\mathcal{A}} \sum_{l=1}^{i-1} \Delta y_l}$$

Since we are integrating y from 0 to 1, we have

$$\sum_{l=1}^{i-1} \Delta y_l \leq 1.$$

Thus we have $M_i \leq M_b e^{M_{\mathcal{A}}}$ as desired. ∎

4. **Numerical Results**

We shall apply the algorithm of the previous section to three distinct choices of Lamé moduli. Though we intend to test our methods against experimental data we shall content ourselves here with synthetic samples. We use the Matlab PDE Toolbox to compute u and $\mathcal{E}$ on the unit square under the traction specified in (2.7) for three representative choices of Lamé moduli. As the PDE Toolbox expects Young's Modulus, E, and Poisson's Ratio, ν, rather than μ and λ we list our choices as

Sample 1: $E(x,y) = 1000$ and $\nu(x,y) = 0.3$
Sample 2: $E(x,y) = 1000 + 100(x+y)$ and $\nu(x,y) = 0.3 + 0.1(x-y)$
Sample 3: $E(x,y) = 1000 + 100\sin(2\pi x)$ and $\nu(x,y) = 0.3 + 0.1\sin(2\pi x)$

and recall that

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{\nu E}{(1-\nu)^2}.$$

The PDE Toolbox solves (1.1)–(1.2) for a particular sample by triangulating the unit square and applying the finite element method with u assumed linear on each triangle. For each sample we adopted the default (irregular) triangulation of 185 (irregularly spaced) vertices. Under laboratory conditions the displacements are read by an overhead camera and then digitized. We have attempted to simulate this environment by tainting the computed displacements with camera blur and round–off error. In particular, we independently considered both

$$u^{(1)}(x,y) = u(x,y) + \rho\beta(x,y) \quad \text{and} \quad u^{(2)}(x,y) = u(x,y) + \rho\omega(x,y) \tag{4.1}$$

where ρ is a simple weighting parameter, ω is a zero–mean, unit–variance, Gaussian random variable that is intended to simulate round–off error, and

$$\beta(x,y) \equiv \begin{pmatrix} (x-.5)^2 \mathrm{sgn}\,(x-.5) \\ (y-.5)^2 \mathrm{sgn}\,(y-.5) \end{pmatrix}$$

is intended to account for measurement errors stemming from a camera whose accuracy diminishes as one travels away from the center of the sample. For $k = 1$

and $k = 2$ we (i) calculate the strain of the tainted displacement, $u^{(k)}$, at the finite element vertices, (ii) interpolate the strain onto the finite difference grid, (iii) calculate $\hat{P}^{(k)}$, $\hat{\Gamma}^{(k)}$, $\hat{\mathcal{A}}^{(k)}$, and $\{\hat{\psi}_j^{(k)}\}_{j=1}^4$ there, (iv) compute $\hat{\xi}^{(k)}$ via the algorithm of the previous section, and finally, (v) set $\hat{\ell}^{(k)} = \hat{P}^{(k)}\hat{\xi}^{(k)}$.

In the subsequent figures we plot the relative error in the reconstructions,

$$\delta\ell(k) = \max_{i,j} \frac{|\mu(x_j, y_i) - \hat{\mu}^{(k)}(i,j)| + |\lambda(x_j, y_i) - \hat{\lambda}^{(k)}(i,j)|}{|\mu(x_j, y_i)| + |\lambda(x_j, y_i)|}$$

against the relative error in the measured strains as the weight, ρ, is increased. In particular, we track

$$\delta\mathcal{E}(k) = \max_{i,j} \frac{|\hat{\mathcal{E}}_{11}(i,j) - \hat{\mathcal{E}}_{11}^{(k)}(i,j)| + |\hat{\mathcal{E}}_{12}(i,j) - \hat{\mathcal{E}}_{12}^{(k)}(i,j)| + |\hat{\mathcal{E}}_{22}(i,j) - \hat{\mathcal{E}}_{22}^{(k)}(i,j)|}{|\hat{\mathcal{E}}_{11}(i,j)| + |\hat{\mathcal{E}}_{12}(i,j)| + |\hat{\mathcal{E}}_{22}(i,j)|}$$

where $\hat{\mathcal{E}}$ is the strain computed from the untainted displacement u and $\hat{\mathcal{E}}^{(k)}$ is the strain corresponding to the $u^{(k)}$ of (4.1).

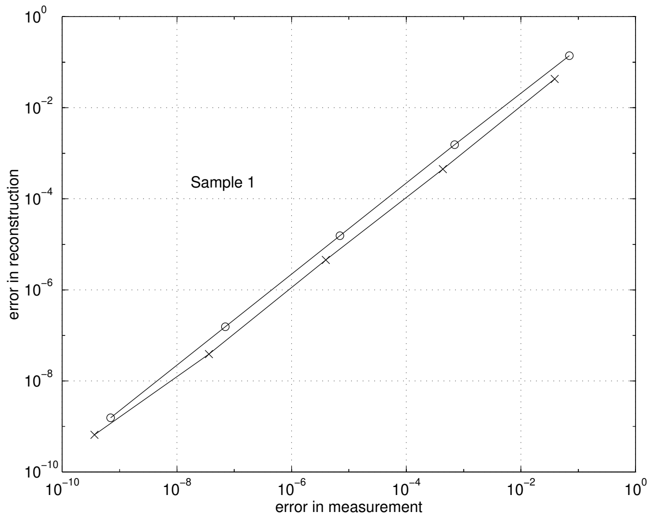

Figure 1. Recovery of Sample 1.

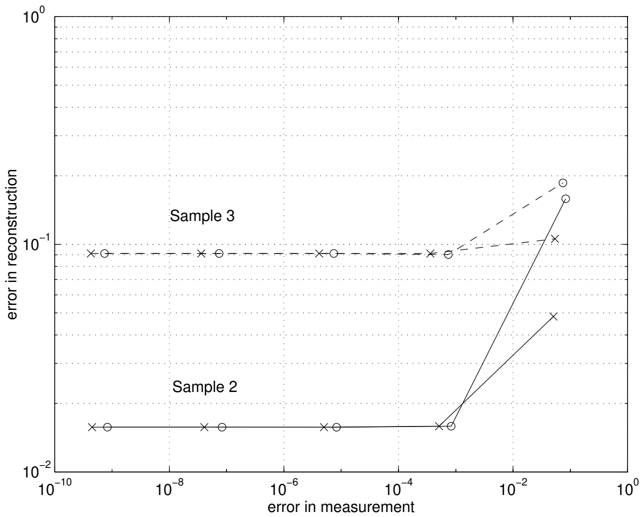

Figure 2. Recovery of Samples 2 and 3

The $\circ$ marks correspond to camera blur, $u^{(1)}$, and the $\times$ marks correspond to round–off error, $u^{(2)}$. In each case, the untainted strains, $\hat{\mathcal{E}}$, satisfied the nondegeneracy condition. As the tainted strains violated this condition when $\delta\hat{\mathcal{E}}(k)$ reached 1 our plots stop there.

With respect to Sample 1, where the actual strains are constant, the fairly coarse finite difference and finite element meshes are not an obstacle to accurate recovery. Fig. 2 indicates that this is not the case for Samples 2 and 3. In particular, the addition of measurement error is not felt until it exceeds roughly 1e-4.

Although these results indicate that our method of recovery can be successful even when the strains are mismeasured by as much as 10% we don't expect, given its reliance on the second derivatives of measured data, this approach to be of immediate practical significance. With respect to the scalar problem, (1.4), a sizable literature, see, e.g., Falk [F] and Kohn and Lowe [KL], has developed around circumventing this reliance on second derivatives. The suitability of their methods to systems remains to be determined.

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