Beaded Strings
Forward and Inverse Problems

Hunter Gilbert, Walter Kelm, and Brian Leake

Physics of Strings PFUG

11 June 2008

Frequencies \rightarrow \text{Masses} + \text{Lengths}

\text{Masses} + \text{Lengths} \rightarrow \text{Frequencies}
Introduction

Why Study Strings?

- Consists of a simple physical system
- Governed by a system of differential equations
- Interesting math for special conditions
- Experimental results easy to acquire
Classes of problems we have worked on:

- Dirac Damping of String (Sean Hardesty’s Thesis)
- Viscous constant damping
- Magnetic damping (Last summer)
- Network of strings (Jesse Chan)
  - ‘http://cnx.org/content/m16177/latest/’
- Physical lab for CAAM 335
Forward Problem
- Given: masses, lengths, and initial input
- Find: the string’s motion

Inverse Problem
- Given: the string’s motion for any input
- Find: masses and lengths that uniquely describe the system
A Beaded String is uniquely defined by masses and lengths:

- \( m_k \) = Mass of a particular bead.
- \( \ell_k \) = Distance between the \( k \) and \( k+1 \) beads.

We will use these expressions to get the equations of motion:

- \( T(y') \) = Total Kinetic Energy as a function of time.
- \( V(y) \) = Total Potential Energy as a function of time.
- \( y_k(t) \) = Vertical Displacement of a particular bead.
To find frequencies of vibration given masses and positions of the beads, we will use a system of differential equations. Kinetic energy is the focus of one of the equations.

\[ T(y') = \frac{1}{2} \sum_{k=1}^{n} m_k (y'_k(t))^2 \]

Symbol meanings:
- \( T(y') \) = Kinetic energy as a function of time.
- \( m_k \) = Mass of the given bead.
- \( y'_k \) = Velocity of given bead.
Dynamics of a Beaded String

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Gives only part of the required system.
Potential Energy is the work done by stretching the string

\[ \text{Work} = \text{Force} \times \text{Distance} = \text{Tension} \times \text{String Elongation} \]

Assume constant tension \( \sigma \), and measure elongation by:

\[
\sqrt{\ell_k^2 + (y_{k+1} - y_k)^2} - \ell_k = \ell_k \sqrt{1 + \frac{(y_{k+1} - y_k)^2}{\ell_k^2}} - \ell_k
\]

\[
\approx \ell_k \left(1 + \frac{1}{2} \frac{(y_{k+1} - y_k)^2}{\ell_k^2}\right) - \ell_k
\]

\[
\approx \frac{(y_{k+1} - y_k)^2}{2\ell_k^2}.
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Dynamics of a Beaded String

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\]

\[
V(y) = \frac{\sigma}{2} \sum_{k=0}^{n} \frac{(y_{k+1} - y_k)^2}{\ell_k}
\]
We can use the Euler-Lagrange Equation to generate a unified system of differential equations.

\[
\frac{d}{dt} \frac{\partial T}{\partial y'_j} + \frac{\partial V}{\partial y_j} = 0, \quad j = 1, \ldots, n
\]

\[
\frac{d}{dt} \frac{\partial T}{\partial y'_j}(t) = m_j y''_j(t)
\]

\[
\frac{\partial V}{\partial y_j} = \left( -\frac{\sigma}{\ell_{j-1}} \right) y_{j-1} + \left( \frac{\sigma}{\ell_{j-1}} + \frac{\sigma}{\ell_j} \right) y_j + \left( -\frac{\sigma}{\ell_j} \right) y_{j+1}
\]
Doing the necessary algebra provides \( n \) equations:

\[
\frac{u_k - u_{k+1}}{\ell_k} + \frac{u_k - u_{k-1}}{\ell_{k-1}} - m_k \lambda^2 u_k = 0, \quad k = 1, 2, ..., n
\]

We can write this system in matrix form

\[
My''(t) = -Ky(t)
\]

- Here, \( M \) is a diagonal matrix holding the masses of the beads.
- \( K \) is a tridiagonal matrix that gives the elongation of the string.
Dynamics of a Beaded String

Essential Linear Algebra

- Inverse: \( M^{-1}M = I \)
- Eigenvalues (\( \lambda \)) and eigenvectors (\( u \))
  - Defined for Matrix \( A \) by:
    \[
    Au = \lambda u
    \]

- They are fundamental properties that describe the matrix behavior
- For a beaded string, they correspond to vibration frequency (eigenvalue) and vibration mode shape (eigenvector)
If we can solve that particular differential equation, we will be able to know the frequencies of vibration.

With the inverse, we can say:

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With the inverse, we can say:

$$y''(t) = -M^{-1}Ky(t)$$

With the eigenvalues of $M^{-1}K$, we can rewrite the equation as:

$$y''(t) = -V\Lambda V^{-1}y(t)$$

$V$ is the matrix where each column is an eigenvector of $M^{-1}K$, and $\Lambda$ is a diagonal matrix of its eigenvalues.
Since Λ is a diagonal matrix, the $n \times n$ system reduces to $n$ independent scalar equations.

$\gamma_j''(t) = -\omega_j^2 \gamma_j(t)$
Since $\Lambda$ is a diagonal matrix, the $n \times n$ system reduces to $n$ independent scalar equations.

$\gamma''_j(t) = -\omega^2_j \gamma_j(t)$

If we presume that the string begins with no initial velocity, we can state

$\gamma_j(t) = \gamma_j(0) \cos(\omega_j t)$

We then have:

$y(t) = \sum_{j=1}^{n} \gamma_j(0) \cos(\omega_j t) v_j$
The masses are displaced as the superposition of $n$ independent vectors vibrating at distinct frequencies. These frequencies are in turn given by the eigenvalues of the matrix $M^{-1}K$. 
We are able to go from masses and positions to frequencies of vibration.

The natural question is now to ask if we can hear the positions and masses of beads on a string from the vibrations it undergoes.
Hearing the Composition of a String

- We are able to go from masses and positions to frequencies of vibration.
- The natural question is now to ask if we can hear the positions and masses of beads on a string from the vibrations it undergoes.
- Using the techniques put forward by Gantmacher and Krein, we can solve this inverse problem.
We wish to find values of $M$ and $K$ to solve $Ku = \lambda Mu$.

By simple matrix multiplication, we can show:

$$
\left( -\frac{\sigma}{\ell_{j-1}} \right) u_{j-1} + \left( \frac{\sigma}{\ell_{j-1}} + \frac{\sigma}{\ell_j} \right) u_j + \left( -\frac{\sigma}{\ell_j} \right) u_{j+1} = \lambda m_j u_j
$$
Recurrence

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  \]
- Having a fixed left & right end implies $u_0 = 0$ and $u_{n+1} = 0$
- Rearranging the above, we get:
  \[
  u_{j+1} = \left( -\frac{\ell_j}{\ell_{j-1}} \right) u_{j-1} + \left( 1 + \frac{\ell_j}{\ell_{j-1}} - \frac{\lambda \ell_j m_j}{\sigma} \right) u_j
  \]
- Since we know $u_{n+1} = 0$ when the conditions for a fixed-fixed string are met...
Recurrence

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  \]
- Since we know $u_{n+1} = 0$ when the conditions for a fixed-fixed string are met...
- We can use this condition to make sure $\lambda$ is an eigenvalue.
When $j = 1$, we can say:

$$u_2 = \left( -\frac{\ell_1}{\ell_0} \right) u_0 + \left( 1 + \frac{\ell_1}{\ell_0} - \frac{\lambda \ell_1 m_1}{\sigma} \right) u_1$$

We now create a polynomial $p$ of linear degree, and define it such that:

$$u_2 \equiv p_1(\lambda) u_1$$
Polynomial Construction

- When \( j = 1 \), we can say:
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  \[
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  \]

- Recalling the formula for a general element of the eigenvector,
  \[
  u_{j+1} = \left( -\frac{\ell_j}{\ell_{j-1}} \right) u_{j-1} + \left( 1 + \frac{\ell_j}{\ell_{j-1}} - \frac{\lambda \ell_j m_j}{\sigma} \right) u_j
  \]

- We can reuse the same trick from above to create a polynomial of degree \( j \).
  \[
  u_{j+1} \equiv p_j(\lambda) u_1
  \]
We can build $p_n$ by following the recurrence, which requires knowledge of lengths & masses.

Or, if we have knowledge of the roots of the polynomial beforehand, we can construct a polynomial of degree $n$ multiplied by a real coefficient that will provide the same behavior.
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Or, if we have knowledge of the roots of the polynomial beforehand, we can construct a polynomial of degree $n$ multiplied by a real coefficient that will provide the same behavior.

We already have this knowledge.

The string is fixed at both ends, requiring $u_0 = 0$ and $u_{n+1} = 0$

Therefore, we can say

$$p_n(\lambda) = \gamma \prod_{j=1}^{n} (\lambda - \lambda_j)$$
In order to make the solution unique, we need to find more information.

We will break our assumption that the string is fixed at both ends.

Allow the string to move at one end, but require it to have 0 slope at its last node.
In order to make the solution unique, we need to find more information.

We will break our assumption that the string is fixed at both ends.

Allow the string to move at one end, but require it to have 0 slope at its last node.

There is now another set of eigenvalues, $\lambda'$, that represent the system.

The system is no longer underdetermined.
We can develop recurrence relations for the slope of the string, like was done for the positions of the beads.

Rearranging our equation developed from matrix multiplication:

\[
\frac{u_{j+1} - u_j}{\ell_j} = \frac{u_j - u_{j-1}}{\ell_{j-1}} - \left( \frac{\lambda m_j}{\sigma} \right) u_j
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Rearranging our equation developed from matrix multiplication:

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\]

Using the polynomials already developed, we can rewrite the equation as:

\[
\frac{u_{j+1} - u_j}{\ell_j} = p_j(\lambda)u_1 - p_{j-1}(\lambda)u_1
\]
Using the same semantic trick as earlier, we define a new polynomial $q_n$ that characterizes the system in fixed-flat operation.

$$q_j(\lambda) = \frac{1}{\ell_j}(p_j(\lambda) - p_{j-1}(\lambda))$$
Using the same semantic trick as earlier, we define a new polynomial $q_n$ that characterizes the system in fixed-flat operation.

\[ q_j(\lambda) = \frac{1}{\ell_j}(p_j(\lambda) - p_{j-1}(\lambda)) \]

Equating expressions for the left and right sides of the equation used to define $q_j$ gives us:

\[ q_j(\lambda)u_1 = q_{j-1}(\lambda) - \left( \frac{\lambda m_j}{\sigma} \right) p_{j-1}(\lambda)u_1 \]
Continued Fractions

- With these characteristic polynomials, we can find masses & displacements.
- Using recurrence relationships of the polynomials:
Continued Fractions

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\frac{p_n(\lambda)}{q_n(\lambda)} = \frac{\ell_n q_n(\lambda) + p_{n-1}(\lambda)}{q_n(\lambda)}
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\[
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\]
Practical Problems

- With the continued fraction, we have values for masses & lengths.
- The process works from a theoretical standpoint...
- Fixed-flat is hard to implement.
Practical Problems

- With the continued fraction, we have values for masses & lengths.
- The process works from a theoretical standpoint...
- Fixed-flat is hard to implement.
- Fortunately, there is a workaround.
- It can be shown that if the beads are symmetric about the midpoint of the string, we can find the fixed-flat eigenvalues without any extra work.
Experimentally measuring eigenvalues on a symmetric beaded string gives us a new set of eigenvalues, termed $\Lambda$.

$\Lambda$ has some beautiful properties.
Experimentally measuring eigenvalues on a symmetric beaded string gives us a new set of eigenvalues, termed $\Lambda$.

$\Lambda$ has some beautiful properties.

It can be shown that:

- The odd indexed eigenvalues of $\Lambda$ are all symmetric about the middle of the string.
- The even indexed eigenvalues of $\Lambda$ are antisymmetric about the midpoint.

This relationship holds for all symmetric strings.
The N eigenvalues of a symmetric beaded string fixed at both ends exactly match the N/2 fixed-fixed and N/2 fixed-flat eigenvalues associated with half of the string.

Figure: Eigenvalues of Symmetric Beaded Strings
The Inverse Algorithm

- Record the eigenvalues for the whole string.
- Use the eigenvalues to generate characteristic polynomials $p_n$ and $q_n$.
- Use the characteristic polynomials to find the set of masses and lengths.

**Figure:** The Symmetric Beaded String
Non-symmetric Problem

Introduction

- Masses and lengths may vary arbitrarily
- Spectra of entire string no longer sufficient
- Clamp string at some interior point between two masses
  - Leads to three problems with fixed-fixed boundary conditions
Non-symmetric Problem

Setup

For \( n_1 \) masses on the left and \( n_2 \) masses on the right:

\[
\frac{u_k - u_{k+1}}{\ell_k} + \frac{u_k - u_{k-1}}{\ell_{k-1}} - m_k \lambda^2 u_k = 0, \quad k = 1, 2, \ldots, n_1
\]

\[
\frac{\tilde{u}_k - \tilde{u}_{k+1}}{\tilde{\ell}_k} + \frac{\tilde{u}_k - \tilde{u}_{k-1}}{\tilde{\ell}_{k-1}} - \tilde{m}_k \lambda^2 \tilde{u}_k = 0, \quad k = 1, 2, \ldots, n_2
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Whole String Boundary Conditions

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u_{n_1+1} = \tilde{u}_{n_2+1}
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- Whole String Boundary Conditions
  - \( u_{n_1+1} = \tilde{u}_{n_2+1} \)
  - \( u_0 = 0, \tilde{u}_0 = 0 \)
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Non-symmetric Problem

Setup

- For $n_1$ masses on the left and $n_2$ masses on the right:

  $\frac{u_k - u_{k+1}}{\ell_k} + \frac{u_k - u_{k-1}}{\ell_{k-1}} - m_k \lambda^2 u_k = 0, \ k = 1, 2, \ldots, n_1$

  $\frac{\tilde{u}_k - \tilde{u}_{k+1}}{\tilde{\ell}_k} + \frac{\tilde{u}_k - \tilde{u}_{k-1}}{\tilde{\ell}_{k-1}} - \tilde{m}_k \lambda^2 \tilde{u}_k = 0, \ k = 1, 2, \ldots, n_2$

- Whole String Boundary Conditions

  $u_{n_1+1} = \tilde{u}_{n_2+1}$

  $u_0 = 0, \ \tilde{u}_0 = 0$

  $\frac{u_{n_1+1} - u_{n_1}}{\ell_{n_1}} + \frac{\tilde{u}_{n_2+1} - \tilde{u}_{n_2}}{\tilde{\ell}_{n_2}} = 0$

- Clamped String Boundary Conditions

  $u_{n_1+1} = 0, \ \tilde{u}_{n_2+1} = 0$
Non-symmetric Problem

Setup

For $n_1$ masses on the left and $n_2$ masses on the right:

- $\frac{u_k - u_{k+1}}{\ell_k} + \frac{u_k - u_{k-1}}{\ell_{k-1}} - m_k \lambda^2 u_k = 0, \ k = 1, 2, \ldots, n_1$

- $\frac{\tilde{u}_k - \tilde{u}_{k+1}}{\tilde{\ell}_k} + \frac{\tilde{u}_k - \tilde{u}_{k-1}}{\tilde{\ell}_{k-1}} - \tilde{m}_k \lambda^2 \tilde{u}_k = 0, \ k = 1, 2, \ldots, n_2$

Whole String Boundary Conditions

- $u_{n_1+1} = \tilde{u}_{n_2+1}$
- $u_0 = 0, \tilde{u}_0 = 0$

- $\frac{u_{n_1+1} - u_{n_1}}{\ell_{n_1}} + \frac{\tilde{u}_{n_2+1} - \tilde{u}_{n_2}}{\tilde{\ell}_{n_2}} = 0$

Clamped String Boundary Conditions

- $u_{n_1+1} = 0, \tilde{u}_{n_2+1} = 0$
- $u_0 = 0, \tilde{u}_0 = 0$
Three spectra Inverse Problem

The Problem

- Relation between characteristic equations

\[ p_{\text{whole}}(\lambda) = p_{\text{left}}(\lambda)q_{\text{right}}(\lambda) + p_{\text{right}}(\lambda)q_{\text{left}}(\lambda) \]

- The roots of \( p_{\text{left}} \) and \( p_{\text{right}} \) are the eigenvalues of the fixed-fixed problems for the left and right segments.

- The roots of \( q_{\text{left}} \) and \( q_{\text{right}} \) are the eigenvalues of the fixed-flat problems.
Three spectra Inverse Problem

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- The roots of \( q_{\text{left}} \) and \( q_{\text{right}} \) are the eigenvalues of the fixed-flat problems

Problem: we can construct \( p_{\text{left}} \) and \( p_{\text{right}} \) and \( p_{\text{whole}} \) up to a scaling factor from measured roots, but we cannot measure the roots of \( q_{\text{left}} \) and \( q_{\text{right}} \), preventing us from forming the continued fractions for the left and right segments.
Through the clever use of polynomials, Boyko and Pivovarchik show that we do not need to know the roots of $q_{\text{left}}$ and $q_{\text{right}}$ to construct them.
Three spectra Inverse Problem

Definitions

- Through the clever use of polynomials, Boyko and Pivovarchik show that we do not need to know the roots of $q_{\text{left}}$ and $q_{\text{right}}$ to construct them.

- Let $\lambda_k$, $k = 1, 2, \ldots, (n_1 + n_2)$, be the spectra of the whole string and let $\nu_{k,\ell}$, $k = 1, 2, \ldots, n_1$, and $\nu_{k,r}$, $k = 1, 2, \ldots, n_2$, be the spectra of the left and right parts, respectively.
Through the clever use of polynomials, Boyko and Pivovarchik show that we do not need to know the roots of \( q_{\text{left}} \) and \( q_{\text{right}} \) to construct them.

Let \( \lambda_k, \ k = 1, 2, \ldots, (n_1 + n_2), \) be the spectra of the whole string and let \( \nu_{k,\ell}, \ k = 1, 2, \ldots, n_1, \) and \( \nu_{k,r}, \ k = 1, 2, \ldots, n_2, \) be the spectra of the left and right parts, respectively.

Let \( L \) be the length of the whole string and let \( L_\ell \) and \( L_r \) be the lengths of the left and right segments of the clamped string.
Construct the polynomials we know, up to a scaling factor:

\[ p_{\text{whole}}(\lambda) = L \prod_{k=1}^{n_1+n_2} \left( 1 - \frac{\lambda}{\lambda_k} \right) \]

\[ p_{\text{left}}(\lambda) = L_\ell \prod_{k=1}^{n_1} \left( 1 - \frac{\lambda}{\nu_{k,\ell}} \right) \]

\[ p_{\text{right}}(\lambda) = L_r \prod_{k=1}^{n_2} \left( 1 - \frac{\lambda}{\nu_{k,r}} \right) \]
Recall the relationship of the characteristic equations:

\[ p_{\text{whole}}(\lambda) = p_{\text{left}}(\lambda)q_{\text{right}}(\lambda) + p_{\text{right}}(\lambda)q_{\text{left}}(\lambda) \]
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\[ p_{\text{whole}}(\lambda) = p_{\text{left}}(\lambda)q_{\text{right}}(\lambda) + p_{\text{right}}(\lambda)q_{\text{left}}(\lambda) \]

Notice that \( p_{\text{left}}(\nu_{k,\ell}) = 0 \) and \( p_{\text{right}}(\nu_{k,r}) = 0 \), as these are simply the roots we used to construct those polynomials.
Three spectra Inverse Problem

Finding $q_{\text{left}}$ and $q_{\text{right}}$

- Recall the relationship of the characteristic equations:
  \[ p_{\text{whole}}(\lambda) = p_{\text{left}}(\lambda)q_{\text{right}}(\lambda) + p_{\text{right}}(\lambda)q_{\text{left}}(\lambda) \]

- Notice that $p_{\text{left}}(\nu_{k,\ell}) = 0$ and $p_{\text{right}}(\nu_{k,r}) = 0$, as these are simply the roots we used to construct those polynomials.

- Plug in $\nu_{k,\ell}$ and $\nu_{k,r}$ into the relation between the characteristic equations to reveal:
  \[ q_{\text{left}}(\nu_{k,\ell}) = \frac{p_{\text{whole}}(\nu_{k,\ell})}{p_{\text{right}}(\nu_{k,\ell})}, \quad k = 1, 2, \ldots, n_1 \]
  \[ q_{\text{right}}(\nu_{k,r}) = \frac{p_{\text{whole}}(\nu_{k,r})}{p_{\text{left}}(\nu_{k,r})}, \quad k = 1, 2, \ldots, n_2 \]
Three spectra Inverse Problem
Finding $q_{left}$ and $q_{right}$ (continued)

- Recall the continued fraction equation:

\[
\frac{p_n(\lambda)}{q_n(\lambda)} = \ell_n + \frac{1}{-\frac{m_n}{\sigma} \lambda + \frac{1}{\ell_{n-1} + \frac{1}{-\frac{m_{n-1}}{\sigma} \lambda + \ldots + \frac{1}{\ell_1 + \frac{1}{-\frac{m_1}{\sigma} \lambda + \frac{1}{\ell_0}}}}}}
\]

- \[
\frac{p_n(0)}{q_n(0)} = \ell_n + \ell_{n-1} + \ldots + \ell_1 + \ell_0 = L
\]

- These equations must hold for the two clamped string segments as well, giving us the last points, $q_{left}(0) = 1$ and $q_{right}(0) = 1$, needed to completely define the polynomials
Construct \( q_{\text{left}} \) and \( q_{\text{right}} \):

\[
q_{\text{left}}(\lambda) = \sum_{k=1}^{n_1} \frac{\lambda}{\nu_k,\ell} \frac{p_{\text{whole}}(\nu_k,\ell)}{p_{\text{right}}(\nu_k,\ell)} \prod_{j=1,j\neq k}^{n_1} \frac{(\lambda - \nu_j,\ell)}{(\nu_k,\ell - \nu_j,\ell)} + \prod_{k=1}^{n_1} \frac{\nu_k,\ell - \lambda}{\nu_k,\ell}
\]

\[
q_{\text{right}}(\lambda) = \sum_{k=1}^{n_2} \frac{\lambda}{\nu_k,r} \frac{p_{\text{whole}}(\nu_k,r)}{p_{\text{left}}(\nu_k,r)} \prod_{j=1,j\neq k}^{n_2} \frac{(\lambda - \nu_j,r)}{(\nu_k,r - \nu_j,r)} + \prod_{k=1}^{n_2} \frac{\nu_k,r - \lambda}{\nu_k,r}
\]

Notice:

\( q_{\text{left}}(\nu_k,\ell) = p_{\text{whole}}(\nu_k,\ell) \quad q_{\text{right}}(\nu_k,r) = p_{\text{whole}}(\nu_k,r) \)
Three spectra Inverse Problem

Constructing $q_{\text{left}}$ and $q_{\text{right}}$

- Construct $q_{\text{left}}$ and $q_{\text{right}}$:

\[
q_{\text{left}}(\lambda) = \sum_{k=1}^{n_1} \frac{\lambda p_{\text{whole}}(\nu_{k,\ell})}{\nu_{k,\ell} p_{\text{right}}(\nu_{k,\ell})} \prod_{j=1, j \neq k}^{n_1} \frac{(\lambda - \nu_{j,\ell})}{(\nu_{k,\ell} - \nu_{j,\ell})} + \prod_{k=1}^{n_1} \frac{\nu_{k,\ell} - \lambda}{\nu_{k,\ell}}
\]

\[
q_{\text{right}}(\lambda) = \sum_{k=1}^{n_2} \frac{\lambda p_{\text{whole}}(\nu_{k,r})}{\nu_{k,r} p_{\text{left}}(\nu_{k,r})} \prod_{j=1, j \neq k}^{n_2} \frac{(\lambda - \nu_{j,r})}{(\nu_{k,r} - \nu_{j,r})} + \prod_{k=1}^{n_2} \frac{\nu_{k,r} - \lambda}{\nu_{k,r}}
\]

- Notice:

\[
q_{\text{left}}(\nu_{k,\ell}) = \frac{p_{\text{whole}}(\nu_{k,\ell})}{p_{\text{right}}(\nu_{k,\ell})}, \quad q_{\text{right}}(\nu_{k,r}) = \frac{p_{\text{whole}}(\nu_{k,r})}{p_{\text{left}}(\nu_{k,r})}
\]

\[
q_{\text{left}}(0) = 1, \quad q_{\text{right}}(0) = 1
\]
Unique lengths and masses are then determined by continued fraction expansion of ratio of p’s and q’s, i.e.

\[
\frac{p_{left}(\lambda)}{q_{left}(\lambda)} = \ell_{n_1} + \frac{1}{\ell_{n_1-1} + \frac{1}{\ell_{n_1-2} + \frac{1}{\ddots + \frac{1}{\ell_1 + \frac{1}{-\frac{m_1}{\sigma} \lambda + \frac{1}{\ell_0}}}}}}
\]
Future Work

- Three spectra problem with damping at an interior point
  - No unique solution
- Damping at a mass
- Damping at the ends
Mentors

- Jeffrey Hokanson
- Dr. Steve Cox
- Dr. Mark Embree

References

- CAAM 335 Lab 9.
  http://www.caam.rice.edu/ caam335lab/lab9.pdf
- CAAM 335 Lab 10.
  http://www.caam.rice.edu/ caam335lab/lab10.pdf