Two-Parameter Process Limits for Infinite-Server Queues with Dependent Service Times via Chaining Bounds

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ABSTRACT. We prove two-parameter process limits for infinite-server queues with weakly dependent service times satisfying the ρ-mixing condition. The two-parameter processes keep track of the elapsed or residual service times of customers in the system. We use the new methodology developed in Pang and Zhou (2017) to prove weak convergence of two-parameter stochastic processes. Specifically, we employ the maximal inequalities for two-parameter queueing processes resulting from the method of chaining. This new methodology requires a weaker mixing condition on the service times than the ϕ-mixing condition in Pang and Whitt (2013), as well as less regularity conditions on the service time distribution function.

1. Introduction

In this paper we continue the study on infinite-server queues with weakly dependent service times in Pang and Whitt [24, 22, 23]. When the consecutive service times satisfy the ϕ-mixing or S-mixing conditions, two-parameter process limits are established for the system dynamics tracking the amount of elapsed and residual service times in [24]. In the decomposition of the diffusion-scaled two-parameter processes (Lemma 3.1), one component is handled by applying the continuous mapping theorem (see Lemma 6.1 in [21] and Lemmas 6.2 and 6.3 in [26]). For the second component, the approach to prove its weak convergence employs the convergence criterion by showing convergence of finite dimensional distributions and tightness of the associated two-parameter processes (see the proofs of Theorem 3.2 and Lemmas 4.1–4.2 in [24]). To prove the convergence of the finite dimensional distributions, the representation of the limit via a mean-square integral with respect to a generalized Kiefer process is used (see the proof of Lemma 4.2 in [24]), which relies on the established invariance principles for sequential empirical processes driven by the dependent random variables [1, 2]. (It is worth noting that under the ϕ-mixing condition, the convergence of finite dimensional distributions of the total count process is also studied with a different approach in Section 2.6 of [6].) To prove the tightness, the martingale difference sequences constructed from the sequence of service times and then the associated Doob's martingale inequalities play a key role in verifying the tightness criterion in the space $\mathbb{D}_{\mathbb{D}} \equiv \mathbb{D}([0,\infty), \mathbb{D}([0,\infty),\mathbb{R}))$ endowed with Skorohod J_1 topology (see the proof of Lemma 4.1 in [24]).

That approach has mostly followed the "machinery" developed in Krichagina and Puhalskii [15], with a generalization to weakly dependent service times under the above two mixing conditions. That methodology has been mostly useful for many-server queues with i.i.d.

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service times [27, 28, 21, 19, 17, 18, 8]. It relies on the semimartingale decomposition of sequential empirical processes driven by i.i.d. random variables. As a result, the associated queueing processes have the corresponding decomposition so that the tightness criteria can be verified by taking advantage of the martingale properties, in particular, Doob's maximal inequalities for martingales. However, for many stochastic systems, the assumption of i.i.d. service times may not hold; for example, the service times may be correlated themselves or time-varying (e.g., depending on the arrival times [26]). As a consequence, the semimartingale decomposition may not be obtained, and new methodology must be developed to prove weak convergence of the queueing processes, especially new tools to establish maximal inequalities in the lack of convenient martingale property.

In our recent work [26], we have developed a new methodology to prove weak convergence of two-parameter processes for infinite-server queues with arrival dependent (time-varying) service times (which includes i.i.d. service times as a special case). That approach differs from the "machinery" in [15] in many aspects. First, a new auxiliary two-parameter process is introduced, and it is shown that weak convergence of the two-parameter queueing processes tracking either elapsed or residual service times follows from that of the new auxiliary process in a straightforward way (see the proof of Lemma 6.4 in [26]). Second, in order to prove the weak convergence of the new auxiliary two-parameter process, we employ a sufficient weak convergence criterion in $\mathbb{D}_{\mathbb{D}}$ (Theorem 4.1 in [26] and Theorem 6.1 below), which is adapted from Theorem 13.5 in [5]. For the convergence of finite dimensional distributions, although we do not represent the limit process as a mean-square integral with respect to some "generalized" Kiefer process, the limiting two-parameter process is clearly a Gaussian process (two-parameter Gaussian random field; of course, the existence and continuity of the limit process are part of the proof), and we have used the standard approach with characteristic functions. What is more important is that for the proof of the probability bound in the criterion (see equation (6.4) below), we employ the maximal inequalities for two-parameter processes arising from the method of chaining. As discussed in Section 8 of [26], in the special case of i.i.d. service times, this new approach does not require any assumption on the service time distribution function.

The method of chaining is an important technique to prove maximal inequalities in probability for many interesting stochastic processes [30, 31, 32]. It results in many useful and important bounds for the expectation of the supremum of a process over a domain given some moment bound conditions on its increments. To our best knowledge, the maximal inequalities resulting from the method of chaining were first used to prove weak convergence in queueing theory in [26]. It provides an extremely powerful tool to verify weak convergence criteria associated with tightness, bridging the gap between moment bounds on the increments of the process of interest and the associated maximal inequalities, which are necessary for tightness. It is expected that this new methodology may turn out to be useful for heavy-traffic analysis of many queueing systems and stochastic networks.

The objective of this paper is to further develop the methodology in [26] in the study of the infinite-server queueing model with weakly dependent service times. We highlight the distinguishing contributions from the work in [24].

(i) We relax the conditions imposed on the service times. We assume that the sequence of service times satisfies a weaker mixing condition, ρ-mixing, than the ϕ-mixing condition. In [24] it is assumed that the marginal distribution function is continuous and has a density, but the new approach here does not impose any assumption on the distribution function. See also Remark 2.1.

(ii) We do not represent the limit process for the second component in the decomposition as a mean-square integral with respect to a generalized Kiefer process. Instead, the limit process is characterized as a continuous Gaussian process (two-parameter Gaussian random field). We prove the existence and continuity of the limit process (see Definition 2.2 and Lemma 6.2).

(iii) To prove the weak convergence, we employ a sufficient convergence criterion as used in [26] (see Theorem 6.1). We introduce an auxiliary two-parameter process as in [26]; see $\hat{V}^n$ defined in (3.1) and their relationship with the two-parameter queueing processes tracking elapsed and residual times in Lemma 3.4. Since the procedure to prove the weak convergence of the two-parameter queueing processes from that of $\hat{V}^n$ follows exactly the same argument as in [26], we focus on the proof of $\hat{V}^n$ under the ρ-mixing condition. In the proof of the convergence of finite dimensional distributions, under the ρ-mixing condition, we employ a general criterion for proving CLT for dependent random variables, see Theorem 5.1 and [33]. To prove the criterion on the probability bound, we establish the maximal inequalities for the limiting two-parameter process $\hat{V}$ and the prelimit two-parameter processes $\hat{V}^n$ (see Propositions 4.2 and 4.3). It is worth noting that the maximal inequalities hold in the same way (except the constant coefficients) for the limit and prelimit processes; see also Remark 4.1.

For the applications of infinite-server queues with dependent service times, we refer the interested readers to [22, 23].

1.1. **Organization of the paper.** We first summarize the notation used in this paper in the next subsection. In Section 2, we describe the model and assumptions in detail and present our main results. In Section 3, we introduce the auxiliary two-parameter process $\hat{V}^n$ and the limit Gaussian random field $\hat{V}$. Based on the weak convergence of $\hat{V}^n \Rightarrow \hat{V}$ in $\mathbb{D}_{\mathbb{D}}$ (Theorem 3.1), we prove our main results in Section 3. We review the maximal inequalities for two-parameter processes resulting from the method of chaining, and prove maximal inequalities for both the limiting two-parameter Gaussian process $\hat{V}$ and the auxiliary two-parameter processes $\hat{V}^n$ in Section 4. The proof for the weak convergence of finite-dimensional distributions of $\hat{V}^n$ (Lemma 4.2) is given in Section 5. We then prove the weak convergence of $\hat{V}^n$ in $\mathbb{D}_{\mathbb{D}}$ (Theorem 3.1) in Section 6. We make some concluding remarks in Section 7.

1.2. **Notation.** Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}^k$ ($\mathbb{R}^k_+$) denotes the space of real-valued (nonnegative) k-dimensional vectors, and we write $\mathbb{R}$ ($\mathbb{R}_+$) for $k = 1$. Let $\mathbb{D}^k = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^k)$ denote $\mathbb{R}^k$-valued function space of all cádlág functions on $\mathbb{R}_+$. Denote $\mathbb{D} \equiv \mathbb{D}^1$. $(\mathbb{D},\ J_1)$ denotes space $\mathbb{D}$ equipped with Skorohod J_1 topology with the metric d_{J_1} [5, 7, 34]. Note that the space $(\mathbb{D},\ J_1)$ is complete and separable. Let $\mathbb{D}_{\mathbb{D}} = \mathbb{D}(\mathbb{R}_+,\ \mathbb{D})$ denote the $\mathbb{D}$-valued function space of all cádlág functions on $\mathbb{R}_+$ with both $\mathbb{D}$ spaces equipped with J_1 topology. Let $\mathbb{C}$ be the subset of $\mathbb{D}$ for continuous functions, and similarly for $\mathbb{C}^k$ and $\mathbb{C}_{\mathbb{C}}$. $\mathbb{D}_2 \equiv \mathbb{D}(\mathbb{R}^2_+,\ \mathbb{R})$ denotes the space of all "continuous from above with limits from below" functions on $\mathbb{R}^2_+$, and is endowed with the same metric $d_{\mathbb{D}_2}$ as in [4]. Let $\mathbb{C}_2$ be the subset of $\mathbb{D}_2$ for continuous functions. It is worth noting that $\mathbb{D}_2 \subset \mathbb{D}_{\mathbb{D}}$ (see Example 4.1 in [9] and discussions in Remark 3.3 in [21]), and $\mathbb{D}_2 \equiv \mathbb{D}_{\mathbb{D}}$ provided the second $\mathbb{D}$ in $\mathbb{D}_\mathbb{D}$ is equipped with uniform norm [4], and thus, we have $\mathbb{C}_2 \equiv \mathbb{C}_\mathbb{C}$. When considering functions defined on finite intervals, we write $\mathbb{D}([0,T],\ \mathbb{R})$, $\mathbb{D}([0,T],\mathbb{D}([0,T'],\ \mathbb{R}))$ and $\mathbb{D}([0,T]\times[0,T'],\ \mathbb{R})$ for $T,T'>0$. We refer the readers to [14, 4, 20, 29, 9] for the general theory of two-parameter (and multi-parameter) stochastic processes and their weak convergence. For any two complete separable metric spaces S_1 and S_2, we denote $S_1\times S_2$ as their product space equipped with the product topology (Section 11.4 in [34]).

2. Model and Results

Consider an infinite-server queueing model with dependent service times, denoted as "$G_t/G^D/\infty$". The arrival process A is a general non-homogeneous counting process, with arrival times $\{\tau_i : i\in\mathbb{N}\}$. The i^{th} customer has a service time η_i, for $i\in\mathbb{N}$. The consecutive service times $\{\eta_i : i\in\mathbb{N}\}$ are a sequence of weakly dependent nonnegative random variables satisfying the ρ-mixing condition (see Assumption 2), with a general distribution function F. Denote $F^c := 1-F$ as the complement of F. We assume that the system starts from empty at time 0 and the arrival process is independent from the service process.

Let $X^e := \{X^e(t,y) : t,y\geq 0\}$ and $X^r := \{X^r(t,y) : t,y\geq 0\}$ be two-parameter stochastic processes tracking the elapsed and residual service times respectively. Specifically, $X^e(t,y)$ and $X^r(t,y)$ represent the number of customers in the system at time t that have received an amount of service less than or equal to y, and whose residual service is strictly greater than y, respectively. By definition, the two-parameter processes $X^{n,e}(t,y)$ and $X^{n,r}(t,y)$ can be written as

$$X^e(t,y) = \sum_{i=A((t-y)-)+1}^{A(t)} \mathbf{1}(\tau_i+\eta_i>t),$$

$$X^r(t,y) = \sum_{i=1}^{A(t)} \mathbf{1}(\tau_i+\eta_i>t+y),$$

for each $t,y\geq 0$, where $A(t-)$ is the left limit of A at $t>0$. Note that the sample paths of the processes $X^e(t,y)$ and $X^r(t,y)$ are in space $\mathbb{D}_\mathbb{D}$ but not in $\mathbb{D}_2$ (see Remark 3.3 in [21] for a detailed discussion). Let $X := \{X(t) : t\geq 0\}$ be the process counting the total number of customers in the system. By definition, we have $X(t) = X^e(t,t) = X^r(t,0)$ for each $t\geq 0$. Let $D := \{D(t) : t\geq 0\}$ be the departure process, defined by $D(t) := A(t)-X(t)$ for $t\geq 0$.

We consider a sequence of such $G_t/G^D/\infty$ queueing models indexed by n and let $n\to\infty$, in which the service times are unscaled, independent of n. We denote the processes A^n, $X^{n,e}$, $X^{n,r}$ and D^n with a superscript n, and similarly for the other relevant processes.

We make the following assumption on the arrival processes A^n.

Assumption 1. (*Arrival Process*) *The sequence of arrival processes A^n satisfies an FCLT:*

$$\hat{A}^n := \sqrt{n}\big(\bar{A}^n-\Lambda\big) \Rightarrow \hat{A} \quad \text{in} \quad (\mathbb{D},\ J_1) \quad \text{as} \quad n\to\infty$$

where $\bar{A}^n := n^{-1}A^n$, $\Lambda := \{\Lambda(t) : t\geq 0\}$ is a deterministic strictly increasing continuous and unbounded above function with $\Lambda(0)=0$, and $\hat{A}$ is a stochastic process with continuous sample paths.

The ρ-mixing condition on the service times $\{\eta_i : i\in\mathbb{N}\}$ requires the following assumption.

Assumption 2. (*Weakly Dependent Service Times*) *The successive service times $\{\eta_i : i \in \mathbb{N}\}$ are weakly dependent and constitute a one-sided stationary sequence, satisfying the ρ-mixing condition: $\rho_k \to 0$ as $k \to \infty$, where*

$$\rho_k := \sup\left\{\frac{|E[\xi\zeta] - E[\xi]E[\zeta]|}{\|\xi\|_2\|\zeta\|_2} : \xi \in \mathcal{F}_m, \zeta \in \mathcal{G}_{m+k}, m \geq 1\right\},$$

with $\mathcal{F}_k := \sigma\{\eta_i : 1 \leq i \leq k\}$, $\mathcal{G}_k := \sigma\{\eta_i : i \geq k\}$ and $\|\xi\|_2 := (E[\xi^2])^{1/2}$. Let $C_\rho := \sum_{k=1}^{\infty} \rho_k < \infty$.

Remark 2.1. There are various mixing coefficients characterizing dependence between random variables (see [3] for a thorough review). The uniformly strong mixing (ϕ-mixing) condition considered in [24] requires that $\phi_k \to 0$ as $k \to \infty$ and $\sum_{k=1}^{\infty} \phi_k < \infty$, where

$$\phi_k := \sup\{P(B|A) - P(B)| : A \in \mathcal{F}_m, P(A) > 0, B \in \mathcal{G}_{m+k}, m \geq 1\}.$$

It is shown that $\rho_k \leq 2\sqrt{\phi_k}$ for $k \geq 1$ (see [3] and [5]). Thus, we impose a weaker mixing condition here. In terms of regularity conditions, in [24], it also requires that $E[\eta_1^2] < \infty$ and $\sum_{i=1}^{\infty} \left(E[(E[\eta_{i+k}|\mathcal{F}_k])^2]\right)^{1/2} < \infty$ for each $k = 1, 2, \ldots$, which are not needed in this paper. In addition, we do not require the continuity and existence of its density on the distribution function F and $F(0) = 0$ as in [24].

Define the fluid-scaled processes $\bar{X}^{n,e} := \{\bar{X}^{n,e}(t,y) : t, y \geq 0\}$, $\bar{X}^{n,r} := \{\bar{X}^{n,r}(t,y) : t, y \geq 0\}$, $\bar{X}^n := \{\bar{X}^n(t) : t \geq 0\}$ and $\bar{D}^n := \{\bar{D}^n(t) : t \geq 0\}$ by

$$\bar{X}^{n,e} := n^{-1}X^{n,e}, \quad \bar{X}^{n,r} := n^{-1}X^{n,r}, \quad \bar{X}^n := n^{-1}X^n, \quad \bar{D}^n := n^{-1}D^n.$$

The functional weak law of large numbers (FWLLN) is stated as follows.

Theorem 2.1. (*FWLLN*) *Under Assumptions 1–2,*

$$\left(\bar{A}^n, \bar{X}^{n,e}, \bar{X}^{n,r}, \bar{X}^n, \bar{D}^n\right) \Rightarrow \left(\Lambda, \bar{X}^e, \bar{X}^r, \bar{X}, \bar{D}\right) \quad \text{in} \quad \mathbb{D} \times (\mathbb{D}_{\mathbb{D}})^2 \times \mathbb{D}^2 \quad \text{as} \quad n \to \infty,$$

where Λ is given in Assumption 1, the fluid limits $\bar{X}^e := \{\bar{X}^e(t,y) : t, y \geq 0\}$, $\bar{X}^r := \{\bar{X}^r(t,y) : t, y \geq 0\}$, $\bar{X} := \{\bar{X}(t) : t \geq 0\}$ and $\bar{D} := \{\bar{D}(t) : t \geq 0\}$ are continuous deterministic functions given by

$$\bar{X}^e(t,y) := \int_{(t-y)^+}^{t} F^c(t-s)d\Lambda(s), \quad t, y \geq 0,$$

$$\bar{X}^r(t,y) := \int_0^t F^c(t+y-s)d\Lambda(s), \quad t, y \geq 0,$$

$$\bar{X}(t) := \bar{X}^e(t,t) = \bar{X}^r(t,0), \quad t \geq 0,$$

and

$$\bar{D}(t) := \int_0^t F(t-s)d\Lambda(s), \quad t \geq 0.$$

Define the diffusion-scaled processes $\hat{X}^{n,e} := \{\hat{X}^{n,e}(t,y) : t, y \geq 0\}$, $\hat{X}^{n,r} := \{\hat{X}^{n,r}(t,y) : t, y \geq 0\}$, $\hat{X}^n := \{\hat{X}^n(t) : t \geq 0\}$ and $\hat{D}^n := \{\hat{D}^n(t) : t \geq 0\}$ by

$$\hat{X}^{n,e} := \sqrt{n}\big(\bar{X}^{n,e} - \bar{X}^e\big), \ \hat{X}^{n,r} := \sqrt{n}\big(\bar{X}^{n,r} - \bar{X}^r\big), \ \hat{X}^n := \sqrt{n}\big(\bar{X}^n - \bar{X}\big), \ \hat{D}^n := \sqrt{n}\big(\bar{D}^n - \bar{D}\big),$$

where $\bar{X}^e$, $\bar{X}^r$, $\bar{X}$ and $\bar{D}$ are given in Theorem 2.1.

We next define the following limit processes.

Definition 2.1. Define the two-parameter processes $\hat{X}_1^e := \{\hat{X}_1^e(t,y) : t,y \geq 0\}$ and $\hat{X}_1^r := \{\hat{X}_1^r(t,y) : t,y \geq 0\}$ by

$$\hat{X}_1^e(t,y) := \int_{(t-y)^+}^t F^c(t-u)d\hat{A}(u),$$

$$\hat{X}_1^r(t,y) := \int_0^t F^c(t+y-u)d\hat{A}(u),$$

for each $t \geq 0$ and $y \geq 0$. They are well-defined as stochastic integrals with "integration by parts" (that is, a pathwise construction via integration by parts), and it is easy to verify that they have continuous sample paths.

Definition 2.2. Define the processes $\hat{X}_2^e := \{\hat{X}_2^e(t,y) : t,y \geq 0\}$ and $\hat{X}_2^r := \{\hat{X}_2^r(t,y) : t,y \geq 0\}$ to be two-parameter Gaussian processes with mean zero and covariance functions: for $t,s \geq 0$ and $y,x \geq 0$,

$$\begin{aligned} &Cov\big(\hat{X}_2^e(t,y), \hat{X}_2^e(s,x)\big) \\ &\quad= \int_{(t-y)^+ \vee (s-x)^+}^{t\wedge s} \big(F(t\wedge s-u) - F(t-u)F(s-u) + \Gamma(t-u,s-u)\big)d\Lambda(u), \end{aligned}$$

and

$$\begin{aligned} &Cov\big(\hat{X}_2^r(t,y), \hat{X}_2^r(s,x)\big) \\ &\quad= \int_0^{t\wedge s} \big(F^c((t+y)\wedge(s+x)-u) - F^c(t+y-u)F^c(s+x-u) \\ &\qquad\qquad +\Gamma(t+y-u, s+x-u)\big)d\Lambda(u), \end{aligned}$$

where

$$\Gamma(x,y) := \sum_{k=2}^{\infty} \big(E\big[\tilde{\gamma}_1(x)\tilde{\gamma}_k(y)\big] + E\big[\tilde{\gamma}_1(y)\tilde{\gamma}_k(x)\big]\big)$$

with

$$\tilde{\gamma}_k(x) := \mathbf{1}(\eta_k \leq x) - F(x), \quad k \in \mathbb{N}. \tag{2.1}$$

We prove the following FCLT for the two-parameter processes $\hat{X}^{n,e}$ and $\hat{X}^{n,r}$.

Theorem 2.2. (*FCLT*) *Under Assumptions 1–2,*

$$(\hat{A}^n, \hat{X}^{n,e}, \hat{X}^{n,r}, \hat{X}^n) \Rightarrow (\hat{A}, \hat{X}^e, \hat{X}^r, \hat{X}) \quad in \quad \mathbb{D}\times(\mathbb{D}_{\mathbb{D}})^2\times\mathbb{D} \quad as \quad n\to\infty, \tag{2.2}$$

where $\hat{A}$ is given in Assumption 1, and

$$\hat{X}^e = \hat{X}_1^e + \hat{X}_2^e, \quad \hat{X}^r = \hat{X}_1^r + \hat{X}_2^r,$$

with $\hat{X}_1^e$ and $\hat{X}_1^r$ given in Definition 2.1 and $\hat{X}_2^e$ and $\hat{X}_2^r$ given in Definition 2.2, and $\hat{X}(t) = \hat{X}^e(t,t) = \hat{X}^r(t,0)$ for $t \geq 0$. The limit departure process $\hat{D} := \{\hat{D}(t) : t \geq 0\}$ is given by $\hat{D}(t) = \hat{A}(t) - \hat{X}(t)$ for $t \geq 0$. All the limit processes have continuous sample paths.

3. Proof of Theorem 2.2

In this section we prove Theorem 2.2. Theorem 2.1 follows from Theorem 2.2 and its proof is omitted. By simple algebra, we obtain the following representations for the diffusion-scaled processes $\hat{X}^{n,e}$ and $\hat{X}^{n,r}$.

Lemma 3.1. *The diffusion-scaled processes $\hat{X}^{n,e}$ and $\hat{X}^{n,r}$ can be represented as*

$$\hat{X}^{n,e}(t,y) = \hat{X}_1^{n,e}(t,y) + \hat{X}_2^{n,e}(t,y), \quad t, y \geq 0,$$
$$\hat{X}^{n,r}(t,y) = \hat{X}_1^{n,r}(t,y) + \hat{X}_2^{n,r}(t,y), \quad t, y \geq 0,$$

where

$$\hat{X}_1^{n,e}(t,y) := \int_{(t-y)^+}^t F^c(t-u)d\hat{A}^n(u), \quad \hat{X}_1^{n,r}(t,y) := \int_0^t F^c(t+y-u)d\hat{A}^n(u),$$

$$\hat{X}_2^{n,e}(t,y) := -\frac{1}{\sqrt{n}} \sum_{i=A^n((t-y)-)+1}^{A^n(t)} \left(\mathbf{1}(\eta_i \leq t - \tau_i^n) - F(t-\tau_i^n)\right),$$

$$\hat{X}_2^{n,r}(t,y) := -\frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} \left(\mathbf{1}(\eta_i \leq t + y - \tau_i^n) - F(t+y-\tau_i^n)\right).$$

The weak convergence of $\hat{X}_1^{n,e}$ and $\hat{X}_1^{n,r}$ follows the same proof of Lemmas 6.2 and 6.3 in [26] by employing continuous mapping theorem (without additional assumptions on F since the service time distribution is not time-varying). We quote it here for completeness.

Lemma 3.2. *Under Assumptions 1–2,*

$$\left(\hat{X}_1^{n,e}, \hat{X}_1^{n,r}\right) \Rightarrow \left(\hat{X}_1^e, \hat{X}_1^r\right) \quad in \quad (\mathbb{D}_\mathbb{D})^2 \quad as \quad n \to \infty.$$

We focus on the weak convergence of $\hat{X}_2^{n,e}$ and $\hat{X}_2^{n,r}$, as stated in the following lemma.

Lemma 3.3. *Under Assumptions 1–2,*

$$\left(\hat{X}_2^{n,e}, \hat{X}_2^{n,r}\right) \Rightarrow \left(\hat{X}_2^e, \hat{X}_2^r\right) \quad in \quad (\mathbb{D}_\mathbb{D})^2 \quad as \quad n \to \infty.$$

Both $\hat{X}_2^e$ and $\hat{X}_2^r$ have continuous sample paths.

Define an auxiliary two-parameter process $\hat{V}^n := \{\hat{V}^n(t,x) : t \geq 0\}$ by

$$\hat{V}^n(t,x) := -\frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} \left(\mathbf{1}(\eta_i \leq x - \tau_i^n) - F(x-\tau_i^n)\right) \tag{3.1}$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} \left(\mathbf{1}(\eta_i > x - \tau_i^n) - F^c(x-\tau_i^n)\right), \qquad t, x \geq 0.$$

By direct observation, we obtain the following relationships of $\hat{X}_2^{n,e}$ and $\hat{X}_2^{n,r}$ with $\hat{V}^n$.

Lemma 3.4. *For each $t, y \geq 0$ and $n \geq 1$, we have*

$$\hat{X}_2^{n,r}(t,y) = \hat{V}^n(t, t+y), \quad a.s.,$$

and

$$\hat{X}_2^{n,e}(t,y) = \hat{V}^n(t,t) - \hat{V}^n((t-y)-, t), \quad a.s.$$

We state the following theorem for the weak convergence of $\hat{V}^n$ and its proof is given in Section 6.

Theorem 3.1. *Under Assumptions 1–2,*

$$\hat{V}^n \Rightarrow \hat{V} \quad in \quad \mathbb{D}_{\mathbb{D}} \quad as \quad n \to \infty,$$

where $\hat{V} := \{\hat{V}(t,x) : t, x \geq 0\}$ *is a continuous two-parameter Gaussian process with mean zero and covariance function*

$$\begin{aligned} &Cov\big(\hat{V}(t,y), \hat{V}(s,x)\big) \\ &\quad = \int_0^{t\wedge s} \big(F(x\wedge y - u)F^c(x \vee y - u) + \Gamma(x-u, y-u)\big) d\Lambda(u), \end{aligned} \tag{3.2}$$

for $t, s \geq 0$ *and* $x, y \geq 0$.

Note that the covariance function of the two-parameter process $\hat{V}$ defined in (3.2) is continuous, which is evidently implied by the continuity of Λ, the definition of Γ, and the ρ-mixing condition.

Given the weak convergence of $\hat{V}^n$ in Theorem 3.1, and the relationship of $\hat{X}_2^{n,e}$ and $\hat{X}_2^{n,r}$ with $\hat{V}^n$ in Lemma 3.4, the weak convergence of $\hat{X}_2^{n,e}$ and $\hat{X}_2^{n,r}$ should follow in a straightforward manner as shown in [26]. The existence, continuity and Gaussian properties of the limits $\hat{X}_2^e$ and $\hat{X}_2^r$ also follow directly from those of $\hat{V}$.

Proof of Lemma 3.3. The proof follows from the same argument as that of Lemma 6.4 in [26], by applying Lemma 3.4 and Theorem 3.1. □

Proof of Theorem 2.2. The weak convergence in (2.2) follows from Lemmas 3.2–3.3 and the continuous mapping theorem. □

4. The Method of Chaining and Maximal Inequalities

4.1. **A Maximal Inequality for Two-Parameter Stochastic Processes.** We review an important maximal inequality for two-parameter stochastic processes first introduced in Pang and Zhou [26]. The inequality provides a useful bound for the moments of the supremum norm of two-parameter processes in any finite time interval provided some moment conditions on their increments. That is derived from the maximal inequalities for general stochastic processes resulting from the *method of chaining* (see a good review in [30, 31]). The maximal inequalities will provide useful bounds for the two-parameter Gaussian limit processes and for the proof of weak convergence of two-parameter processes.

Recall a semimetric satisfies all conditions of a metric except (possibly) the triangle inequality. For a semimetric space $(\mathbb{T}, \mathrm{d})$, define the covering number $N(\epsilon, \mathrm{d})$ as the minimal number of balls of radius ϵ needed to cover $\mathbb{T}$. In this paper, we use $\mathbb{T} = [0,T]$ for $T > 0$. We state the following proposition in [26].

Proposition 4.1. *Let* $X(t,y)$ *be a real-valued, separable two-parameter stochastic process on* $[0,T] \times [0,T']$. *For* $0 \leq s < t \leq T$, *define* $Z_{s,t}(y) := X(t,y) - X(s,y)$ *for* $y \in [0,T']$. *Suppose that*

$$E\big[|Z_{s,t}(y) - Z_{s,t}(x)|^p\big] \leq C_1 \big(\mathrm{d}_{s,t}(x,y)\big)^p, \quad for \quad x, y \in [0,T'] \quad and \quad p > 1,$$

where C_1 is a positive constant, $\mathrm{d}_{s,t}(x,y)$ is a semimetric on $[0,T']$ such that the diameter $\mathrm{d}_{s,t}(T')$ of $[0,T']$ under this semimetric is equal to $\mathrm{d}_{s,t}(0,T')$, and the covering number

$$N(\epsilon, \mathrm{d}_{s,t}) \le \left\lceil \frac{\mathrm{d}_{s,t}(0,T')}{2\epsilon} \right\rceil + 1. \tag{4.1}$$

Then,

$$E\bigg[\sup_{x,y\in[0,T']} |Z_{s,t}(y) - Z_{s,t}(x)|^p\bigg] \le C_2\big(\mathrm{d}_{s,t}(0,T')\big)^p,$$

for some constant $C_2 > 0$ depending only on p and C_1. The same bound holds for $0 \le t < s \le T$ by defining a semimetric $\mathrm{d}_{t,s}$ symmetrically.

4.2. **A Maximal Inequality for the Two-Parameter Gaussian Process $\hat{V}$.** We apply Proposition 4.1 to obtain the following maximal inequality for the two-parameter Gaussian process $\hat{V}$ introduced in Theorem 3.1.

Definition 4.1. For any $0 \le s < t \le T$, define a semimetric $\mathrm{d}_{s,t}(x,y)$ on $[0,T']$ as follows: for $0 \le x \le y \le T'$, let

$$\mathrm{d}_{s,t}(x,y) := \left((t-s)\wedge(y-x) + (1+2C_\rho)\int_s^t \big[F(y-u) - F(x-u)\big]d\Lambda(u)\right)^{1/2}, \tag{4.2}$$

and for $0 \le y \le x \le T'$, by symmetry, let

$$\mathrm{d}_{s,t}(x,y) := \mathrm{d}_{s,t}(y,x). \tag{4.3}$$

It is easy to check that $\mathrm{d}_{s,t}(x,y)$ defined in (4.2)–(4.3) is indeed a semimetric on $[0,T']$ for any $T' > 0$. The diameter of $[0,T']$ under $\mathrm{d}_{s,t}$ is equal to

$$\mathrm{d}_{s,t}(T') = \mathrm{d}_{s,t}(0,T') = \left((t-s)\wedge T' + (1+2C_\rho)\int_s^t F(T'-u)d\Lambda(u)\right)^{1/2},$$

and the covering number satisfies (4.1). The nature of the semimetric is similar to the one used to prove the convergence of $\hat{V}^n$ in the cases of i.i.d. service times and of arrival dependent (time-varying) service times studied in [26].

Lemma 4.1. *For $0 \le s < t \le T$ and $y \in [0,T']$, define*

$$Z_{s,t}[\hat{V}](y) := \hat{V}(t,y) - \hat{V}(s,y). \tag{4.4}$$

Then for $x, y \in [0,T']$,

$$E\left[\big|Z_{s,t}[\hat{V}](y) - Z_{s,t}[\hat{V}](x)\big|^4\right] \le 3\big(\mathrm{d}_{s,t}(x,y)\big)^4.$$

Proof. By direct calculation, we obtain that

$$\begin{aligned}
& E\left[\big|Z_{s,t}[\hat{V}](y) - Z_{s,t}[\hat{V}](x)\big|^4\right] \\
&= 3\bigg(\int_s^t \big([F(y-u) - F(x-u)][1 - F(y-u) + F(x-u)] \\
&\qquad + \Gamma(y-u,y-u) + \Gamma(x-u,x-u) - 2\Gamma(x-u,y-u)\big)d\Lambda(u)\bigg)^2 \\
&= 3\bigg(\int_s^t \Big([F(y-u) - F(x-u)][1 - F(y-u) + F(x-u)]
\end{aligned}$$

$$
\begin{aligned}
&\quad +2\sum_{k=2}^{\infty} E[\tilde\gamma_1(y-u)-\tilde\gamma_1(x-u)][\tilde\gamma_k(y-u)-\tilde\gamma_k(x-u)]\Big) d\Lambda(u)\Big)^2 \\
&\le 3\bigg(\int_s^t \bigg([F(y-u)-F(x-u)] + 2[F(y-u)-F(x-u)]\sum_{k=1}^{\infty}\rho_k\bigg) d\Lambda(u)\bigg)^2 \\
&= 3\bigg(\int_s^t (1+2C_\rho)[F(y-u)-F(x-u)] d\Lambda(u)\bigg)^2 \\
&\le 3\bigg((t-s)\wedge(y-x) + (1+2C_\rho)\int_s^t \big[F(y-u)-F(x-u)\big] d\Lambda(u)\bigg)^2 \\
&= 3\big(\mathrm{d}_{s,t}(x,y)\big)^4. \qquad (4.5)
\end{aligned}
$$

This completes the proof. □

Proposition 4.2. *The two-parameter Gaussian process $\hat V$ satisfies: for $0 \le s,t \le T$,*

$$E\bigg[\sup_{x\in[0,T']}\big|\hat V(t,x)-\hat V(s,x)\big|^p\bigg] \;\le\; \hat K_1\big|t-s+(\Lambda(t)-\Lambda(s))\big|^{p/2},$$

for $p = 2,\ 4$, and some constant $\hat K_1 > 0$.

Proof. We prove the case when $p = 4$. The case when $p = 2$ follows from a similar argument. Without loss of generality, we only prove the bound for $0 \le s < t \le T$. Recall $Z_{s,t}[\hat V](y)$ defined in (4.4). By Proposition 4.1, Lemma 4.1, and the fact that $\hat V(t,0) = 0$ a.s., we obtain

$$
\begin{aligned}
&E\bigg[\sup_{x\in[0,T']}\big|\hat V(t,x)-\hat V(s,x)\big|^4\bigg] \\
&\le \breve K\big(\mathrm{d}_{s,t}(0,T')\big)^4 \\
&\le \hat K_1\big(t-s+(\Lambda(t)-\Lambda(s))\big)^2,
\end{aligned}
$$

for some $\breve K > 0$ and $\hat K_1 > 0$. This completes the proof. □

4.3. **A Maximal Inequality for the Two-Parameter Process $\hat V^n$.** We first state a lemma on the finite dimensional distributions of $\hat V^n$ and $\hat V$. The proof is postponed to Section 5.

Lemma 4.2. *The finite dimensional distributions of $\hat V^n$ converge weakly to those of $\hat V$ as $n \to \infty$.*

Next we state a moment condition for truncated processes $\hat V^n$. Observe that for any $K \in \mathbb{N}$, $K = \bar A^n(\tau^n_{nK})$. Fix $T > 0$ below, choose $K \in \mathbb{N}$ such that $K > \Lambda(T)$. Also, fix this K for the next lemma and proposition.

Lemma 4.3. *For $0 \le s \le t \le T$ and $x \in [0,T']$, define*

$$Z^n_{s,t}[\hat V^n](x) := \hat V^n(t\wedge\tau^n_{nK},x) - \hat V^n(s\wedge\tau^n_{nK},x).$$

There exists some constant $\hat K_2$ such that for $n \ge 1$ and for $x,y \in [0,T']$,

$$E\Big[\big|Z^n_{s,t}[\hat V^n](y) - Z^n_{s,t}[\hat V^n](x)\big|^4\Big] \le \hat K_2\big(\mathrm{d}_{s,t}(x,y)\big)^4. \qquad (4.6)$$

Proof. Notice that $\tau^n_{nK} = (\bar{A}^n)^{-1}(K)$ and continuity of first passage time (Theorem 13.6.4 in [34]), Assumption 1 implies that

$$\tau^n_{nK} \Rightarrow \Lambda^{-1}(K) \quad \text{as} \quad n \to \infty.$$

Then, by random time change (lemma on page 151, [5]) and Lemma 4.2, the finite dimensional distributions of $\hat{V}^n(t \wedge \tau^n_{nK}, y)$ converge weakly to those of $\hat{V}(t \wedge \Lambda^{-1}(K), y)$ as $n \to \infty$.

Since the truncated processes $\hat{V}^n(t \wedge \tau^n_{nK}, y)$ are uniformly integrable, we obtain that

$$\begin{aligned} & E\left[\left|Z^n_{s,t}[\hat{V}^n](y) - Z^n_{s,t}[\hat{V}^n](x)\right|^4\right] \\ = \;& E\left[\left|\hat{V}^n(t \wedge \tau^n_{nK}, y) - \hat{V}^n(s \wedge \tau^n_{nK}, y) - \hat{V}^n(t \wedge \tau^n_{nK}, x) + \hat{V}^n(s \wedge \tau^n_{nK}, x)\right|^4\right] \\ \to \;& E\left[\left|\hat{V}(t \wedge \Lambda^{-1}(K), y) - \hat{V}(s \wedge \Lambda^{-1}(K), y) - \hat{V}(t \wedge \Lambda^{-1}(K), x) + \hat{V}(s \wedge \Lambda^{-1}(K), x)\right|^4\right] \end{aligned}$$

as $n \to \infty$. By direct calculations, the right hand side is equal to

$$\begin{aligned} & 3\bigg(\int_{s\wedge\Lambda^{-1}(K)}^{t\wedge\Lambda^{-1}(K)} \big([F(y-u) - F(x-u)][1 - F(y-u) + F(x-u)] \\ & \qquad +\Gamma(y-u, y-u) + \Gamma(x-u, x-u) - 2\Gamma(x-u, y-u)\big) d\Lambda(u)\bigg)^2 \\ \leq \;& 3\left((1 + 2C_\rho)\int_s^t \big[F(y-u) - F(x-u)\big] d\Lambda(u)\right)^2 \\ \leq \;& 3\bigg((t-s) \wedge (y-x) + (1 + 2C_\rho)\int_s^t \big[F(y-u) - F(x-u)\big] d\Lambda(u)\bigg)^2 \\ = \;& 3\big(\mathrm{d}_{s,t}(x,y)\big)^4. \end{aligned}$$

Therefore, there exists a positive constant $\hat{K}_2$ such that (4.6) holds. This completes the proof. ☐

Proposition 4.3. *There exists some constant $\hat{K}_3 > 0$ such that for $n \geq 1$, the two-parameter process $\hat{V}^n$ satisfies: for $0 \leq s, t \leq T$,*

$$E\bigg[\sup_{x \in [0,T']} \left|\hat{V}^n(t \wedge \tau^n_{nK}, x) - \hat{V}^n(s \wedge \tau^n_{nK}, x)\right|^p\bigg] \leq \hat{K}_3\big|t - s + (\Lambda(t) - \Lambda(s))\big|^{p/2},$$

for $p = 2,\ 4$.

Proof. The proof follows exactly the same arguments as in the proof of Proposition 4.2 by applying Lemma 4.3 and Proposition 4.1. ☐

Remark 4.1. It is worth noting that the maximal inequalities for the limiting two-parameter process $\hat{V}$ and the corresponding prelimit process $\hat{V}^n$ hold in the same form except the different constants in the upper bounds. For the limit process $\hat{V}$ we have calculated the fourth moment of the process increments in Lemma 4.1. The same could be possibly done for the prelimit process $\hat{V}^n$, as in [26]. However, under the ρ-mixing condition, it seems quite challenging to directly compute the bound for the fourth moment of the process increments. Instead, we have taken a different approach here to prove the fourth moment bound in Lemma 4.3. We first prove the convergence of finite dimensional distributions of the processes $\hat{V}^n$ in Lemma 4.2, and then by truncating the arrival process (which is all that is needed for the proof of the convergence below), we can derive the fourth moment bound given the resulting uniform integrability property. In fact, this approach can be also used for the model with time-varying service times in [26] and the case of i.i.d. service times.

5. Proof of Lemma 4.2

To prove Lemma 4.2, we will apply the following theorem, which is adapted from Theorem 2.1 and Proposition 2.1(a) in [33].

Theorem 5.1. *Let $\{Z_{in} : i = 1, ..., \kappa_n, n = 1, 2, ...\}$ be a triangular array of bounded random variables with zero-mean and $\kappa_n \to \infty$ as $n \to \infty$. Define $S_n := \sum_{i=1}^{\kappa_n} Z_{in}$ for each $n \in \mathbb{N}$. Let*

$$\begin{aligned} S_n(a,b) &= \sum_{i=a+1}^{a+b} Z_{in}, \quad 0 \le a, 1 \le b \le n-a, \\ \tilde{c}_n(k) &= \sup_{|\tilde{l}-\tilde{m}|\ge k,\ 1\le \tilde{l},\tilde{m}\le n} \frac{|E[Z_{\tilde{l}n} Z_{\tilde{m}n}]|}{\|Z_{\tilde{l}n}\|_2 \|Z_{\tilde{m}n}\|_2}, \quad 0 \le k < n, \end{aligned}$$

and

$$\tilde{c}(k) = \max_{n:k<n} \tilde{c}_n(k).$$

Suppose that

(i)
$$\sup_{a,b,n} \frac{1}{b} E[S_n(a,b)^2] < \infty;$$

(ii)
$$\sum_{k=1}^{\infty} \tilde{c}(k) < +\infty;$$

(iii)
$$\sigma_n^2 := Var(S_n) \to \infty \quad as \quad n \to \infty.$$

Then

$$S_n/\sigma_n \Rightarrow N(0,1) \quad as \quad n \to \infty.$$

Remark 5.1. We have modified Theorem 2.1(A) in [33] by using the sufficient condition in Proposition 2.1(a) for the moment inequality (2.3) in that reference. The assumptions of Proposition 2.1(a) in [33] are satisfied because of boundedness of $\{Z_{in}\}$ and condition (i) above with $\epsilon = \gamma = 0$ there. The ℓ-mixing condition and condition (A) in Theorem 2.1 in [33] are implied by conditions (ii) and (iii) since ρ-mixing implies strong ℓ-mixing and Z_{in}'s are assumed to be bounded.

Proof of Lemma 4.2. By Cramér-Wold theorem, it is equivalent to show that for $0 \le t_1 < t_2 < \cdots < t_m \le T$, $0 \le y_1 < y_2 < \cdots < y_l \le T'$ and $\{a_{ij} \in \mathbb{R} : i = 1, ..., m, j = 1, ..., l\}$,

$$\sum_{i=1}^{m} \sum_{j=1}^{l} a_{ij} \hat{V}^n(t_i, y_j) \Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{l} a_{ij} \hat{V}(t_i, y_j) \quad \text{as} \quad n \to \infty.$$

Before proceeding to the proof, we fix the trajectory of $A^n(t)$ in the following way and thus omitting arguments involving conditional expectations. For each $n \ge 1$, let the set Υ^n be the collection of the trajectories of $\{A^n(t) : t \ge 0\}$ such that for each $T \ge 0$, $\sup_{0\le t\le T} |\hat{A}^n(t)| \le n^{1/4}$ and $\max_{1\le i\le A^n(T)} |\tau_{i+1}^n - \tau_i^n| \to 0$ as $n \to \infty$. It is evident that under Assumption 1, $P(\Upsilon^n) \to 1$ as $n \to \infty$ and $A^n(t)$ increases without limit and is of order $O(n)$. In the proof below, we consider the trajectories of A^n in the set Υ^n for each n.

We first consider the case $m = l = 1$, i.e., $\hat{V}^n(t_1, y_1) \Rightarrow \hat{V}(t_1, y_1)$ in $\mathbb{R}$ as $n \to \infty$. For brevity, we omit the subscripts and simply write $\hat{V}^n(t, y)$ and $\hat{V}(t, y)$.

Fix t and y. Define for $i = 1, ..., A^n(t)$ and $n \geq 1$,

$$Z_{in} := -\left(\mathbf{1}(\eta_i \leq y - \tau_i^n) - F(y - \tau_i^n)\right).$$

(The explicit dependence on t and y is omitted for brevity.) Then

$$S_n := \sum_{i=1}^{A^n(t)} Z_{in} = \sqrt{n}\hat{V}^n(t, y). \tag{5.1}$$

We apply Theorem 5.1 to this sequence $\{S_n\}$ with the associated variables $\{Z_{in}\}$.

By definition of Z_{in}, it is easy to see that $\sup_{i,n} |Z_{in}| \leq 1$, a.s. By Assumption 2, direct calculations yield

$$\begin{aligned}
\frac{1}{b}E\Big[\Big(\sum_{i=a+1}^{a+b} Z_{in}\Big)^2\Big] &= \frac{1}{b}\sum_{i=a+1}^{a+b} E\big[Z_{in}^2\big] + \frac{2}{b}\Big(\sum_{i,j=a+1,\ i<j}^{a+b} E\big[Z_{in}Z_{jn}\big]\Big) \\
&\leq 1 + \frac{2}{b}\sum_{i,j=a+1,\ i<j}^{a+b} \rho_{|j-i|} \\
&\leq 1 + 2\sum_{k=1}^{b}\rho_k \leq 1 + 2C_\rho < \infty.
\end{aligned}$$

Thus, condition (i) above is satisfied by Assumption 2.

The ρ-mixing condition satisfied by $\{\eta_i : i \in \mathbb{N}\}$ is naturally inherited by $\{Z_{in}\}$ through its definition and thus by Assumption 2, for each $n \geq 1$,

$$\sup_{|\tilde{l}-\tilde{m}|\geq k,\ 1\leq \tilde{l},\tilde{m}\leq n} \frac{|E[Z_{\tilde{l}n}Z_{\tilde{m}n}]|}{\|Z_{\tilde{l}n}\|_2\|Z_{\tilde{m}n}\|_2} \leq \rho_k.$$

Therefore, the inequality above implies that $\tilde{c}(k) = \max_{n:k<n} \tilde{c}_n(k) \leq \rho_k$ and condition (ii) is satisfied.

We next check condition (iii). Recall that $\tilde{\gamma}_i(x) = \mathbf{1}(\eta_i \leq x) - F(x)$ in (2.1). We have

$$\begin{aligned}
\frac{\sigma_n^2}{n} &= \frac{1}{n}Var\Big(\sum_{i=1}^{A^n(t)} Z_{in}\Big) \\
&= \frac{1}{n}\sum_{i=1}^{A^n(t)} E\big[\big(\tilde{\gamma}_i(y - \tau_i^n)\big)^2\big] + \frac{2}{n}\sum_{i<j}^{A^n(t)} E\left[\tilde{\gamma}_i(y - \tau_i^n)\tilde{\gamma}_j(y - \tau_j^n)\right] \\
&= \frac{1}{n}\sum_{i=1}^{A^n(t)} F(y - \tau_i^n)F^c(y - \tau_i^n) + \frac{2}{n}\sum_{i<j}^{A^n(t)} r_{i,j}^n,
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
r_{i,j}^n &:= E[\tilde{\gamma}_i(y - \tau_i^n)\tilde{\gamma}_j(y - \tau_j^n)] \\
&= P(\eta_i \leq y - \tau_i^n, \eta_j \leq y - \tau_j^n) - P(\eta_i \leq y - \tau_i^n)P(\eta_j \leq y - \tau_j^n).
\end{aligned}$$

By stationarity of $\{\eta_i : i \geq 1\}$, for fixed k, we consider

$$\begin{aligned} R_k^n := \sum_{i=1}^{A^n(t)-k} r_{i,i+k}^n &= \sum_{i=1}^{A^n(t)-k} P(\eta_1 \leq y - \tau_i^n, \eta_{1+k} \leq y - \tau_i^n) - P(\eta_1 \leq y - \tau_i^n)^2 \\ &\quad + \sum_{i=1}^{A^n(t)-k} \Delta_{i,k}^{n,(1)} + \sum_{i=1}^{A^n(t)-k} \Delta_{i,k}^{n,(2)}, \end{aligned}$$

where

$$\begin{aligned} \Delta_{i,k}^{n,(1)} &:= P(\eta_1 \leq y - \tau_i^n, \eta_{1+k} \leq y - \tau_{i+k}^n) - P(\eta_1 \leq y - \tau_i^n, \eta_{1+k} \leq y - \tau_i^n), \\ \Delta_{i,k}^{n,(2)} &:= P(\eta_1 \leq y - \tau_i^n)^2 - P(\eta_1 \leq y - \tau_i^n) P(\eta_1 \leq y - \tau_{i+k}^n). \end{aligned}$$

Given the trajectories of A^n in Υ^n, for each fixed k and $t \geq 0$, it is easy to check that

$$\max_{1 \leq i \leq A^n(t)-k} \Delta_{i,k}^{n,(1)} \to 0 \quad \text{and} \quad \max_{1 \leq i \leq A^n(t)-k} \Delta_{i,k}^{n,(2)} \to 0 \quad \text{as} \quad n \to \infty.$$

Note that this holds under general distribution function F and joint distribution function $F_{1,k}$ for (η_1, η_k) for any $k \geq 2$, since these functions are right continuous with left limits and the convergence is from the left. Thus, we have

$$R_k^n = \int_0^t E[\tilde{\gamma}_1(y-u)\tilde{\gamma}_{1+k}(y-u)] dA^n(u) + o(n).$$

Therefore, as $n \to \infty$,

$$\begin{aligned} \frac{2}{n} \sum_{i<j}^{A^n(t)} r_{i,j}^n = \frac{2}{n} \sum_{k=1}^{A^n(t)-1} R_k^n &\to 2 \sum_{k=1}^{\infty} \int_0^t E[\tilde{\gamma}_1(y-u)\tilde{\gamma}_{1+k}(y-u)] d\Lambda(u) \\ &= \int_0^t \Gamma(y-u, y-u) d\Lambda(u). \end{aligned}$$

Thus, by (5.2), we obtain

$$\frac{\sigma_n^2}{n} \to \sigma^2 := \int_0^t [F(y-u)F^c(y-u) + \Gamma(y-u, y-u)] d\Lambda(u) \quad \text{as} \quad n \to \infty.$$

We have verified condition (iii).

Therefore, by Theorem 5.1, we have

$$\left(\sum_{i=1}^{A^n(t)} Z_{in} \right) \Big/ \sigma_n \Rightarrow N(0,1) \quad \text{as} \quad n \to \infty.$$

Since $\sigma_n/\sqrt{n} \to \sigma$ as $n \to \infty$, by (5.1), we obtain that for the fixed t and y,

$$\hat{V}^n(t,y) \Rightarrow N(0, \sigma^2) \stackrel{\mathrm{d}}{=} \hat{V}(t,y) \quad \text{as} \quad n \to \infty.$$

Here "$\stackrel{\mathrm{d}}{=}$" denotes "equal in distribution".

Now we consider the case when $m = 2$ and $l = 2$. By simple algebra, we can write

$$-\sqrt{n} \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \hat{V}^n(t_i, y_j) = a_{11} \sum_{i=1}^{A^n(t_1)} \tilde{\gamma}_i(y_1 - \tau_i^n) + a_{12} \sum_{i=1}^{A^n(t_1)} \tilde{\gamma}_i(y_2 - \tau_i^n)$$

$$+a_{21}\sum_{i=1}^{A^n(t_2)}\tilde{\gamma}_i(y_1-\tau_i^n)+a_{22}\sum_{i=1}^{A^n(t_2)}\tilde{\gamma}_i(y_2-\tau_i^n)$$
$$=\sum_{i=1}^{A^n(t_2)}\gamma_i^{n,*},$$

where

$$\gamma_i^{n,*}:=\begin{cases}(a_{11}+a_{21})\tilde{\gamma}_i(y_1-\tau_i^n)+(a_{12}+a_{22})\tilde{\gamma}_i(y_2-\tau_i^n) & \text{for} \quad 1\le i\le A^n(t_1),\\ a_{21}\tilde{\gamma}_i(y_1-\tau_i^n)+a_{22}\tilde{\gamma}_i(y_2-\tau_i^n) & \text{for} \quad A^n(t_1)+1\le i\le A^n(t_2).\end{cases}$$

Since the randomness of $\gamma_i^{n,*}$ comes only from η_i, the dependence between $\gamma_i^{n,*}$ is the same between $\tilde{\gamma}_i(y-\tau_i^n)$. Therefore, the similar arguments for the first case apply. It is clear that this argument can be extended for any general $m>2$ and $l>2$. This completes the proof. □

6. Proof of Theorem 3.1

We first prove that the sample path of $\hat{V}$ is continuous, i.e., $\hat{V}\in\mathbb{C}_{\mathbb{C}}$ and $\mathbb{C}_2$. We quote the following lemma for the continuity of two-parameter stochastic processes in [26].

Lemma 6.1. ([26, Lemma 4.2]) *Let X be a separable mean-zero Gaussian process with sample paths in $\mathbb{D}_2$. If X is continuous in quadratic mean, then it has sample paths in $\mathbb{C}_2$ a.s.*

Recall the following concepts for two-parameter processes defined in [4]. A block B in $[0,T]\times[0,T']$ is a subset of $[0,T]\times[0,T']$ of the form $(s,t]\times(x,y]$; two blocks B and C in $[0,T]\times[0,T']$ are said to be neighboring blocks if they share a common edge. Note that there are only two kinds of neighboring blocks in $[0,T]\times[0,T']$, (i) the first kind: $B=(s,t]\times(x,y]$ and $C=(s,t]\times(y,z]$ and (ii) the second kind: $B=(s,t]\times(x,y]$ and $C=(r,s]\times(x,y]$, for $r<s<t$ and $x<y<z$. For each block $B=(s,t]\times(x,y]$, define $X(B):=X(t,y)-X(t,x)-X(s,y)+X(s,x)$ be the increment of X around B for stochastic process X.

Lemma 6.2. *Under Assumptions 1–2, the two-parameter Gaussian process $\hat{V}$ is continuous, namely, it has continuous sample paths in $\mathbb{C}_2$, and thus in $\mathbb{C}_{\mathbb{C}}$.*

Proof. Since the covariance function of $\hat{V}$ is continuous, by Lemma 6.1, it suffices to show that the Gaussian process $\hat{V}\in\mathbb{D}([0,T]\times[0,T'],\mathbb{R})$ for $T,T'>0$. We apply Theorem 4 in [4] (see also Theorem 4.2 in [26]), which states a criterion for the existence of a stochastic process with sample paths in $\mathbb{D}([0,T]\times[0,T'],\mathbb{R})$ given its finite dimensional distributions. We check the required four conditions.

(i) The condition $P(\hat{V}(t,0)=0)=1$ and $P(\hat{V}(0,y)=0)=1$ for each $t\in[0,T]$ and $y\in[0,T']$ is satisfied since $Var(\hat{V}(t,0))=Var(\hat{V}(0,y))=0$ by the covariance function in (3.2).

The continuity of the covariance function implies that the following two conditions are satisfied:

(ii) for each $\epsilon>0$,

$$\lim_{h_1,h_2\to 0+}P\big(|\hat{V}(t+h_1,y+h_2)-\hat{V}(t,y)|\ge\epsilon\big)=0,\quad 0\le t<T,\ 0\le y\le T';$$

and

(iii) for each $\epsilon > 0$,

$$\lim_{t\to T-} P\big(|\hat{V}(t,y) - \hat{V}(T,y)| \geq \epsilon\big) = 0, \quad 0 \leq y \leq T',$$

and

$$\lim_{y\to T'-} P\big(|\hat{V}(t,y) - \hat{V}(t,T')| \geq \epsilon\big) = 0, \quad 0 \leq t \leq T.$$

The last condition (iv) requires that there exists a finite measure μ on $[0,T]\times[0,T']$ with continuous marginals such that

$$E\big[\hat{V}(B)^2\hat{V}(C)^2\big] \leq \mu(B)\mu(C), \tag{6.1}$$

for all pairs of neighboring blocks B and C in $[0,T]\times[0,T']$. Recall that there are only two kinds of neighboring blocks in $[0,T]\times[0,T']$. We consider the first kind with $B = (s,t]\times(x,y]$ and $C = (r,s]\times(x,y]$ for $r < s < t$ and $x < y$. By Cauchy-Schwarz inequality, it suffices to show that there exists some finite measure μ with continuous marginals on $[0,T]\times[0,T']$ such that

$$E\big[\hat{V}(B)^4\big] \leq \mu(B)^2. \tag{6.2}$$

By the same calculations in (4.5),

$$\begin{aligned} & E\big[\big|\hat{V}(t,y) - \hat{V}(s,y) - \hat{V}(t,x) + \hat{V}(s,x)\big|^4\big] \\ \leq & \left(\sqrt{3}(1+2C_\rho)\int_s^t \big[F(y-u) - F(x-u)\big]d\Lambda(u)\right)^2. \end{aligned}$$

It is easy to verify that the measure μ on $[0,T]\times[0,T']$ defined by

$$\mu(B) := \sqrt{3}(1+2C_\rho)\int_s^t \big[F(y-u) - F(x-u)\big]d\Lambda(u), \quad \forall B = (s,t]\times(x,y] \subset [0,T]\times[0,T'],$$

is finite and has continuous marginals. Thus, the condition (6.2) is verified for the first kind of neighboring blocks in $[0,T]\times[0,T']$. A similar argument also verifies it for the second kind of neighboring blocks. This completes the proof. □

For each fixed $t \geq 0$, we denote $\hat{V}^n(t) = \{\hat{V}^n(t,x) : x \geq 0\}$ and it is an element of $\mathbb{D}$. Similarly, for each fixed $t \geq 0$, we denote $\hat{V}(t) = \{\hat{V}(t,x) : x \geq 0\}$. Lemma 6.2 implies that for each $t \geq 0$, $\hat{V}(t)$ is also an element of $\mathbb{D}$ (actually $\mathbb{C}$). We next show the convergence of $\hat{V}^n(t)$ to $\hat{V}(t)$ in $\mathbb{D}$ for each $t \geq 0$ by employing the following theorem, which is a generalization of Theorem 13.5 in [5] from $\mathbb{D}([0,1],\mathbb{R})$ to $\mathbb{D}([0,T],\mathcal{S})$ for a metric space $\mathcal{S}$ (see also Theorem 4.1 in [26]).

Theorem 6.1. ([5, Theorem 13.5])
Let X^n and X be stochastic processes with sample paths in $\mathbb{D}([0,T],\mathcal{S})$ where $(\mathcal{S},\mathrm{m})$ is a metric space. Suppose that

(i) *for any $0 \leq t_1 \leq t_2 \leq ... \leq t_k \leq T$ and $k \geq 1$,*

$$\big(X^n(t_1),...,X^n(t_k)\big) \Rightarrow \big(X(t_1),...,X(t_k)\big) \quad in \quad \mathcal{S}^k \quad as \quad n\to\infty,$$

(ii)

$$\mathrm{m}(X(T), X(T-\delta)) \Rightarrow 0 \quad in \quad \mathbb{R} \quad as \quad \delta\to 0, \tag{6.3}$$

(iii) *for* $0 \le r \le s \le t \le T$, $n \ge 1$ *and* $\epsilon > 0$,

$$P\big(\mathrm{m}(X^n(r), X^n(s)) \wedge \mathrm{m}(X^n(s), X^n(t)) \ge \epsilon\big) \le \frac{1}{\epsilon^4}\big(H(t) - H(r)\big)^2, \qquad (6.4)$$

where H *is a nondecreasing and continuous function on* $[0, T]$.

Then $X^n \Rightarrow X$ *in* $\mathbb{D}([0,T], \mathcal{S})$ *as* $n \to \infty$.

Lemma 6.3. *Under Assumptions 1–2, for each fixed* $t \ge 0$, $\hat{V}^n(t) \Rightarrow \hat{V}(t)$ *in* $\mathbb{D}$ *as* $n \to \infty$.

Proof. It suffices to prove the convergence in $\mathbb{D}[0, T']$ for each $T' > 0$. We prove the theorem by verifying three conditions in Theorem 6.1. The first condition is implied by Lemma 4.2.

We next show that the limit process $\hat{V}(t) = \{\hat{V}(t, y) : y \ge 0\}$ for each $t \ge 0$ satisfies condition (6.3), that is,

$$\hat{V}(t, T') - \hat{V}(t, T' - \delta) \Rightarrow 0 \quad \text{as} \quad \delta \to 0.$$

It suffices to show that

$$E\left[\big|\hat{V}(t, T') - \hat{V}(t, T' - \delta)\big|^2\right] \to 0 \quad \text{as} \quad \delta \to 0.$$

This directly follows from the continuity of the covariance function of $\hat{V}$.

We now verify condition (6.4) in Theorem 6.1. For $K \in \mathbb{N}$ such that $K > \Lambda(T)$ and $\epsilon > 0$,

$$\begin{aligned}
& P\big(\big|\hat{V}^n(t,x) - \hat{V}^n(t,y)\big| \wedge \big|\hat{V}^n(t,y) - \hat{V}^n(t,z)\big| \ge \epsilon\big) \\
\le\ & P\big(A^n(T) \ge nK\big) \\
& \quad + P\Big(A^n(T) \le nK, \big|\hat{V}^n(t,x) - \hat{V}^n(t,y)\big| \wedge \big|\hat{V}^n(t,y) - \hat{V}^n(t,z)\big| \ge \epsilon\Big) \\
\le\ & P\big(\bar{A}^n(T) \ge K\big) \\
& \quad + \frac{1}{\epsilon^4} E\left[\mathbf{1}(\bar{A}^n(T) \le K) \cdot \big|\hat{V}^n(t,x) - \hat{V}^n(t,y)\big|^2 \cdot \big|\hat{V}^n(t,y) - \hat{V}^n(t,z)\big|^2\right] \\
\le\ & P\big(\bar{A}^n(T) \ge K\big) \\
& \quad + \frac{1}{\epsilon^4}\left(E\left[\big|\hat{V}^n(t \wedge \tau^n_{nK}, x) - \hat{V}^n(t \wedge \tau^n_{nK}, y)\big|^4\right]\right)^{1/2} \\
& \qquad\qquad \times \left(E\left[\big|\hat{V}^n(t \wedge \tau^n_{nK}, y) - \hat{V}^n(t \wedge \tau^n_{nK}, z)\big|^4\right]\right)^{1/2} \\
\le\ & P\big(\bar{A}^n(T) \ge K\big) + \frac{3}{\epsilon^4}\left((1 + 2C_\rho)\int_0^t \big[F(z-u) - F(x-u)\big] d\Lambda(u)\right)^2
\end{aligned}$$

where the second last inequality is obtained from applying Cauchy-Schwartz inequality and the last one follows from Lemma 4.3 (with $s = 0$ in the lemma). Since $\bar{A}^n(T) \Rightarrow \Lambda(T)$ as $n \to \infty$ by Assumption 1, we have

$$P\big(\bar{A}^n(T) \ge K\big) \to P\big(\Lambda(T) \ge K\big) = 0 \quad \text{as} \quad n \to \infty \qquad (6.5)$$

for the chosen $K > \Lambda(T)$. Since for fixed t, as a function in x, $H^t(x) := \int_0^t F(x-u) d\Lambda(u)$ is nondecreasing and continuous in x, condition (iii) is satisfied. Thus, the proof is complete. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Consider $[0, T] \times [0, T']$. To prove this, it suffices to verify the following three conditions by Theorem 6.1 with $\mathcal{S} = \mathbb{D}$.

(i) for any $0 \le t_1 \le t_2 \le ... \le t_k \le T$ and $k \ge 1$,

$$\big(\hat{V}^n(t_1), ..., \hat{V}^n(t_k)\big) \Rightarrow \big(\hat{V}(t_1), ..., \hat{V}(t_k)\big) \quad \text{in} \quad \mathbb{D}^k \quad \text{as} \quad n \to \infty, \tag{6.6}$$

(ii)

$$d_{J_1}\big(\hat{V}(T), \hat{V}(T-\delta)\big) \Rightarrow 0 \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad \delta \to 0, \tag{6.7}$$

(iii) for $0 \le r \le s \le t \le T$ and $n \ge 1$,

$$P\left(d_{J_1}\big(\hat{V}^n(r), \hat{V}^n(s)\big) \wedge d_{J_1}\big(\hat{V}^n(s), \hat{V}^n(t)\big) \ge \epsilon\right) \le \frac{1}{\epsilon^4}\big(H(t) - H(r)\big)^2, \tag{6.8}$$

for some nondecreasing and continuous function H on $[0, T]$.

To prove (6.6), we show that for any $0 \le t_1 \le t_2 \le ... \le t_k \le T$ and $k \ge 1$,

$$\big(\hat{V}^n(t_1), ..., \hat{V}^n(t_k)\big) \Rightarrow \big(\hat{V}(t_1), ..., \hat{V}(t_k)\big) \quad \text{in} \quad (\mathbb{D}[0, T'])^k \quad \text{as} \quad n \to \infty. \tag{6.9}$$

Lemma 6.3 implies that the sequence $\{\hat{V}^n(t) : n \ge 1\}$ is tight for each $t \in [0, T]$, and thus, $\big(\hat{V}^n(t_1), \hat{V}^n(t_2), ..., \hat{V}^n(t_k)\big)$ is also tight for $0 \le t_1 < t_2 < ... < t_k \le T$. Then it suffices to show the finite dimensional distributions of $\big(\hat{V}^n(t_1), \hat{V}^n(t_2), ..., \hat{V}^n(t_k)\big)$ converge weakly to those of $\big(\hat{V}(t_1), \hat{V}(t_2), ..., \hat{V}(t_k)\big)$, which is implied again by Lemma 4.2.

Condition (6.7) is simply implied by the fact that $\hat{V} \in \mathbb{C}_{\mathbb{C}}$ proved in Lemma 6.2.

Now we focus on (6.8). For $K \in \mathbb{N}$ such that $K > \Lambda(T)$ and $\epsilon > 0$,

$$\begin{aligned}
& P\big(d_{J_1}\big(\hat{V}^n(r), \hat{V}^n(s)\big) \wedge d_{J_1}\big(\hat{V}^n(s), \hat{V}^n(t)\big) \ge \epsilon\big) \\
\le\ & P\bigg(\sup_{y \in [0,T']} \big|\hat{V}^n(r,y) - \hat{V}^n(s,y)\big| \wedge \sup_{y \in [0,T']} \big|\hat{V}^n(s,y) - \hat{V}^n(t,y)\big| \ge \epsilon\bigg) \\
\le\ & P\big(A^n(T) \ge nK\big) \\
& + P\bigg(A^n(T) \le nK, \sup_{y \in [0,T']} \big|\hat{V}^n(r,y) - \hat{V}^n(s,y)\big| \wedge \sup_{y \in [0,T']} \big|\hat{V}^n(s,y) - \hat{V}^n(t,y)\big| \ge \epsilon\bigg) \\
\le\ & P\big(\bar{A}^n(T) \ge K\big) \\
& + \frac{1}{\epsilon^4} E\bigg[\mathbf{1}(\bar{A}^n(T) \le K) \cdot \sup_{y \in [0,T']} \big|\hat{V}^n(r,y) - \hat{V}^n(s,y)\big|^2 \cdot \sup_{y \in [0,T']} \big|\hat{V}^n(s,y) - \hat{V}^n(t,y)\big|^2\bigg] \\
\le\ & P\big(\bar{A}^n(T) \ge K\big) \\
& + \frac{1}{\epsilon^4}\bigg(E\bigg[\sup_{x \in [0,T']} \big|\hat{V}^n(t \wedge \tau^n_{nK}, x) - \hat{V}^n(s \wedge \tau^n_{nK}, x)\big|^4\bigg]\bigg)^{1/2} \\
& \qquad \times \bigg(E\bigg[\sup_{x \in [0,T']} \big|\hat{V}^n(s \wedge \tau^n_{nK}, x) - \hat{V}^n(r \wedge \tau^n_{nK}, x)\big|^4\bigg]\bigg)^{1/2} \\
\le\ & P\big(\bar{A}^n(T) \ge K\big) + \frac{\hat{K}_3}{\epsilon^4}\big(t - r + (\Lambda(t) - \Lambda(r))\big)^2,
\end{aligned}$$

where the last inequality follows from Proposition 4.3. The first term on the right hand side vanishes as $n \to \infty$, as in (6.5). Since the function $H(t) = t + \Lambda(t)$ is nondecreasing and continuous, condition (iii) is verified. Therefore, the proof is now complete. ☐

7. Conclusion

We have proved the two-parameter process limits for infinite-server queues with ρ-mixing service times by employing the new methodology developed in Pang and Zhou [26]. The conditions required on the service times are much weaker than those in [24]. In the proof, as in [26], the auxiliary two-parameter process $\hat{V}^n$ plays a key role in bridging the two-parameter queueing processes tracking the elapsed and residual times. It is worth noting that as discussed in Section 8 of [26], we can also prove the convergence of the two-parameter processes $\hat{X}_2^{n,e}$ and $\hat{X}_2^{n,r}$ directly by deriving the corresponding maximal inequalities as in Propositions 4.2 and 4.3. However, that would require additional conditions on either the function Λ or the distribution function F. Specifically, for $\hat{X}_2^{n,e}$, it requires Λ to be Lipschitz continuous and for $\hat{X}_2^{n,r}$, it requires the service distribution function F to be Lipschitz continuous.

We conjecture that the new methodology in [26] can be further developed to prove two-parameter process limits for non-Markovian many-server queues. As mentioned earlier, the recent work on non-Markovian many-server queues with i.i.d. service times in [27, 28, 21, 19, 17, 18, 8] has adapted the "machinery" in Krichagina and Puhalskii [15]. We think the new methodology can be employed for these models and potentially much simplify the proofs as we have demonstrated in [26] and here. Also, the methodology using measure-valued processes has been recently developed to study non-Markovian many-server queues with i.i.d. service times in [35, 16, 17, 11, 12, 13, 36]. As shown in [10], it is equivalent to study measure-valued and two-parameter processes for many-server queues in the fluid level. Our approach may be potentially extended to study many-server queues with i.i.d. service times, and with time-varying and dependent service times.

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