

On the Functional and Local Limit Theorems for Markov Modulated Compound Poisson Processes

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ABSTRACT. We study a class of Markov-modulated compound Poisson processes whose arrival rates and the compound random variables are both modulated by a stationary finite-state Markov process. The compound random variables are i.i.d. in each state of the Markov process, while having a distribution depending on the state of the Markov process. We prove a functional central limit theorem and local limit theorems under appropriate scalings of the arrival process, compound random variables and underlying Markov process.

1. INTRODUCTION

We consider a Markov-modulated compound Poisson (MMCP) process $X := \{X(t) : t \geq 0\}$ described as follows. Let $Y := \{Y(t) : t \geq 0\}$ be a finite-state Markov process with state space $S := \{1, \dots, I\}$ and transition rate matrix $Q := (q_{ij})_{i,j \in S}$. Let $A := \{A(t) : t \geq 0\}$ be a Markov modulated Poisson process with an arrival rate λ_i when Y is in state i , for each $i \in S$. Then it is standard to write

$$A(t) = A_* \left(\int_0^t \lambda_{Y(s)} ds \right), \quad t \geq 0, \quad (1.1)$$

where $A_* = \{A_*(t) : t \geq 0\}$ is a unit-rate Poisson process. Let $\{\tau_i : i \in \mathbb{N}\}$ be the sequence of arrival times for the process A . Let $\{Z_i : i \in \mathbb{N}\}$ be a sequence of conditionally independent random variables given the Markov process Y , such that at each arrival time τ_i , $i \in \mathbb{N}$, the conditional distribution of Z_i given the state of the Markov process $Y = k$ is

$$P(Z_i \leq x | Y(\tau_i) = k) = F_k(x), \quad x \in \mathbb{R},$$

for each $k \in S$. We write the sequence $\{Z_i\}$ as $\{Z_i(Y(\tau_i))\}$ to indicate the dependence on $\{Y(\tau_i)\}$ explicitly. We assume that for each $i \in S$, the distribution F_i has finite variance, and let m_i and σ_i^2 be its mean and variance. We now define the process X as

$$X(t) := \sum_{i=1}^{A(t)} Z_i(Y(\tau_i)) = \int_0^t Z_{A(s)}(Y(s)) dA(s), \quad t \geq 0. \quad (1.2)$$

We also assume that the Markov process Y starts from stationarity at time zero. Let $\pi := (\pi_1, \dots, \pi_I)$ be the stationary distribution of Y , and Π be a matrix with each row being the steady-state vector π . Let $\Upsilon = (\Upsilon_{ij})_{i,j=1,\dots,I}$ be the fundamental matrix, given by

$$\Upsilon_{ij} = \int_0^\infty (P_{ij}(t) - \pi_j) dt.$$

It is known that $\Upsilon = (\Pi - Q)^{-1} - \Pi$.

We consider a sequence of the MMCP processes X^n , indexed by a superscript n , and let $n \rightarrow \infty$. Similarly for the processes A^n and Y^n , and the compound random variables $\{Z_i^n\}$.

We index the corresponding parameters λ_i^n , m_i^n and σ_i^n and the transition rate matrix Q^n by a superscript n . Denote $\boldsymbol{\lambda}^n := (\lambda_1^n, \dots, \lambda_I^n)$, $\mathbf{m}^n := (m_1^n, \dots, m_I^n)$ and $\boldsymbol{\sigma}^n := (\sigma_1^n, \dots, \sigma_I^n)$, which are all in \mathbb{R}_+^I . We make the following assumptions on these parameters.

Assumption 1. As $n \rightarrow \infty$,

$$\frac{\boldsymbol{\lambda}^n}{n} \rightarrow \boldsymbol{\lambda}, \quad \mathbf{m}^n \rightarrow \mathbf{m} \quad \text{and} \quad \boldsymbol{\sigma}^n \rightarrow \boldsymbol{\sigma},$$

where $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_I)$, $\mathbf{m} := (m_1, \dots, m_I)$ and $\boldsymbol{\sigma} := (\sigma_1, \dots, \sigma_I)$ are all in \mathbb{R}_+^I . The transition rate matrix $Q^n = n^\alpha Q$ for some $\alpha > 0$.

It is evident that under the assumption that $Q^n = n^\alpha Q$, the Markov process Y^n is stationary and has the same stationary distribution π as the process Y . The value of $\alpha > 1$ or $\alpha < 1$ indicates the speeding or slowing effect of the modulating process, respectively.

Define the diffusion-scaled process $\hat{X}^n := \{\hat{X}^n(t) : t \geq 0\}$ by

$$\hat{X}^n(t) := \frac{1}{n^\delta} \left(X^n(t) - \sum_{i=1}^I \lambda_i^n m_i^n \pi_i t \right), \quad \text{for } \frac{1}{2} \leq \delta < 1 \quad \text{and} \quad t \geq 0.$$

Such a scaling for counting processes is studied in [1, 19] for infinite-server queueing systems. We prove the following functional central limit theorem (FCLT) for \hat{X}^n .

Theorem 1.1. Under Assumptions 1,

$$\hat{X}^n \Rightarrow \hat{X} \quad \text{in } (\mathbb{D}, J_1) \quad \text{as } n \rightarrow \infty,$$

where the limit process \hat{X} is a driftless Brownian motion with variance coefficient

$$\varrho := \begin{cases} \bar{\sigma}^2 + \bar{\nu}, & \text{if } \delta = 1/2, \alpha > 1, \\ \bar{\sigma}^2 + \bar{\nu} + \bar{\beta}, & \text{if } \delta = 1/2, \alpha = 1, \\ \bar{\beta}, & \text{if } \delta = 1 - \alpha/2, 0 < \alpha < 1, \end{cases}$$

and

$$\bar{\sigma}^2 := \sum_{i=1}^I \lambda_i \sigma_i^2 \pi_i, \quad \bar{\nu} := \sum_{i=1}^I \lambda_i m_i^2 \pi_i, \quad \bar{\beta} := 2 \sum_{i=1}^I \sum_{j=1}^I \lambda_i \lambda_j m_i m_j \pi_i \Upsilon_{ij}. \quad (1.3)$$

Note that the parameters $\bar{\nu}, \bar{\beta}, \bar{\sigma}$ represent the variabilities from the arrival process, the underlying Markov process and the compound random variables, respectively. This result captures the effects of these variabilities under different scaling parameter values.

Instead of scaling the parameters, we can also scale the process X in the conventional approach, that is, define the diffusion-scaled process

$$\tilde{X}^n(t) := \frac{1}{\sqrt{n}} (X(nt) - n\bar{\lambda}t), \quad t \geq 0, \quad (1.4)$$

where $\bar{\lambda} := \sum_{i=1}^I \lambda_i m_i \pi_i$. We observe that this scaling is equivalent to our scaling above with $\delta = 1/2$ and $\alpha = 1$. As a consequence of Theorem 1.1, we have the weak convergence of \tilde{X}^n :

$$\tilde{X}^n \Rightarrow \tilde{X} \quad \text{in } (\mathbb{D}, J_1) \quad \text{as } n \rightarrow \infty,$$

where \tilde{X} is a Brownian motion with mean zero and variance coefficient $\bar{\sigma}^2 + \bar{\nu} + \bar{\beta}$.

We next consider local limit theorems for the process X^n . We distinguish two cases in which the compound variables have either lattice (integer-valued) or non-lattice distributions.

Assumption 2. For $i \in \mathbb{N}$, $k \in S$ and fixed u ,

$$\left| E \left[e^{iuZ_i^n(Y(\tau_i))} \middle| Y(\tau_i) = k \right] \right| \rightarrow \left| E \left[e^{iu\hat{Z}_i(Y(\tau_i))} \middle| Y(\tau_i) = k \right] \right| \quad \text{as } n \rightarrow \infty, \quad (1.5)$$

where the sequences $\{Z_i^n(Y(\tau_i)) : i \in \mathbb{N}\}$ and $\{\hat{Z}_i(Y(\tau_i)) : i \in \mathbb{N}\}$ have integer values given the Markov process Y .

Define the process $\{\check{X}^n(t) : t \geq 0\}$ as

$$\check{X}^n(t) := X^n(t) - \bar{\lambda}nt, \quad t \geq 0,$$

where $\bar{\lambda}$ is defined in (1.4).

Theorem 1.2. Under Assumption 1 and 2, for each $t \geq 0$,

$$\sup_{x \in \check{\mathcal{X}}_t^n} \left| n^\delta P(\check{X}^n(t) = x) - \psi_t(x) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (1.6)$$

where $\check{\mathcal{X}}_t^n := \{-\bar{\lambda}nt + z : z \in \mathbb{Z}\}$ and $\psi_t(x)$ is defined by

$$\psi_t(x) := (2\pi \varrho t)^{-1/2} \exp(-x^2/2\varrho t), \quad (1.7)$$

with ϱ given in (1.3).

Assumption 3. For $i \in \mathbb{N}$, $k \in S$ and fixed u , (1.5) holds, where the sequences $\{Z_i^n(Y(\tau_i)) : i \in \mathbb{N}\}$ and $\{\hat{Z}_i(Y(\tau_i)) : i \in \mathbb{N}\}$ have nonlattice distribution given the Markov Process Y .

Theorem 1.3. Under Assumption 1 and 3. If $x_n/n^\delta \rightarrow x$ as $n \rightarrow \infty$ and $a < b$, then for each $t \geq 0$,

$$n^\delta P(\check{X}^n(t) \in (x_n + a, x_n + b)) \rightarrow (b - a)\psi_t(x) \quad \text{as } n \rightarrow \infty, \quad (1.8)$$

where $\psi_t(x)$ is given in (1.7).

These results can be applied in several contexts. First, when the compound variables are positive integers (e.g., geometrically distributed), the process X becomes a *Markov-modulated Poisson batch arrival process*. It can be used to model queueing systems with batch arrivals (see, e.g., [24]). Second, in insurance risk theory, the compound variables represent the claim sizes and the process X is the cumulative amount claims with both the arrivals and claims modulated by a Markov process (see, e.g., [2, 4, 20]). Although some asymptotic results have been derived for the associated ruin probability, the FCLT and local limit theorem for the process X under the scaling regime in our paper have not been studied in the literature. Third, in the Markov-modulated $M/G/1$ queue studied in [26, 16, 17, 3, 5], the compound variables represent the service times for each arrival, and the process X is the cumulative-input process. By Theorem 1.1, we can thus obtain new diffusion approximations correspondingly for the net-input process and the workload process with the Skorohod mapping (see, e.g., Section 13.5 in [28]). Local limit theorems are of particular interest to study local behavior of non-stationary stochastic systems, see, e.g., [13, 30]. Our results here on the MMCP processes can be used to study the local behaviors of the Markov-modulated risk reserve processes and Markov-modulated $M/G/1$ queues.

1.1. Literature review. MMCP processes have been studied to some extent in various applications. In [2], the MMCP process was first introduced to the insurance risk theory in random environments, where some asymptotic results on the associated ruin probability are derived. Many interesting results are subsequently obtained for the ruin problems with the MMCP processes; see, e.g., [4, 31, 6, 10] and references therein. In [21], the MMCP processes are also extended to Markov modulated Poisson shot noise processes for ruin problems. As mentioned above, the MMCP processes appear in Markov-modulated $M/G/1$ queueing models [26, 16, 17, 3, 5]. However, these studies have focused on exact analytical methods and associated asymptotic results.

There are very limited results on diffusion approximations of the MMCP processes. In [2], an approximation of the ruin probability based on a diffusion approximation of the Markov modulated risk reserve process is provided. In [7], an FCLT is established for a Markov-modulated risk reserve process, where the limit process becomes a Markov modulated diffusion risk model under certain conditions on the claim sizes and the arrival rates of the claims. However, our result in the FCLT shows a Brownian motion limit under suitable scalings of the arrivals, compound variables and the underlying Markov process. We have particularly studied the effects of different scalings of the transition matrix of the underlying Markov process. In the heavy-traffic scaling regime, the effects of such different scalings have been studied in infinite-server queueing models in [1, 19]. It is worth noting that in [5], the effect of speeding up the modulating process is also studied for the tail of the waiting time in a Markov-modulated $M/G/1$ queue.

Local limit theorems have not been established for the MMCP processes, to the best of our knowledge. The early work on local limit theorems for Markov processes was marked by [23], and more recent results are shown in, e.g., [18, 14] and references therein. Local limit theorems for random walks are well studied; see, e.g., [12, 25, 11, 22] and Chapter 8 in [9]. Some local limit theorems are shown for certain random walks in random environments (see, e.g., [15, 27]). Here we prove the local limit theorems for the MMCP processes in the cases of lattice and nonlattice distributed compound variables under the scaling regime in Assumption 1.

1.2. Notation. The following notations will be used throughout the paper. \mathbb{N} denotes the set of positive integers. For $k \in \mathbb{N}$, \mathbb{R}^k (\mathbb{R}_+^k) denotes the space of real-valued (nonnegative) k -dimensional vectors, and we write \mathbb{R} (\mathbb{R}_+) for $k = 1$. For $a, b \in \mathbb{R}$, denote $a \wedge b := \min(a, b)$. For any $x \in \mathbb{R}_+$, $\lfloor x \rfloor$ denotes the largest integer not greater than x . $\mathbf{1}(A)$ denotes the indicator function of a set A . We use ‘ i ’ to denote the imaginary unit. For two real-valued functions f and g (non-zero), we write $f(x) = o(g(x))$ if $\limsup_{x \rightarrow \infty} |f(x)/g(x)| = 0$, and $f(x) = O(g(x))$ if $\limsup_{x \rightarrow \infty} |f(x)/g(x)| < \infty$.

All random variables are defined in a common complete probability space (Ω, \mathcal{F}, P) . Notations \rightarrow and \Rightarrow mean convergence of real numbers and convergence in distribution, respectively. The abbreviation a.s. means almost surely, i.i.d. means independent and identically distributed and WLLN represents the weak law of large number. Let $\mathbb{D} := \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ denote real-valued function space of all càdlàg functions on \mathbb{R}_+ . Note that \mathbb{D} is complete and separable. We endow the space \mathbb{D} with the Skorohod J_1 topology (see, e.g., [8, 28]) throughout the paper. For any two complete and separable metric spaces S_1 and S_2 , $S_1 \times S_2$ is used to denote their product space, endowed with the maximum metric, that is, the maximum of two metrics on S_1 and S_2 (see, e.g., [28]). For any complete and separable

space S , S^k denotes the k -fold product space with the maximum metric, for $k \in \mathbb{N}$. (\mathbb{D}^k, J_1) denotes the k -fold product of (\mathbb{D}, J_1) with the product topology.

2. PROOF OF THEOREM 1.1

We first give a decomposition of the process \hat{X}^n .

Lemma 2.1. *The diffusion-scaled process \hat{X}^n can be decomposed into the following three processes:*

$$\hat{X}^n(t) = \hat{X}_1^n(t) + \hat{X}_2^n(t) + \hat{X}_3^n(t), \quad t \geq 0,$$

where

$$\hat{X}_1^n(t) := \frac{1}{n^\delta} \sum_{i=1}^{A^n(t)} (Z_i^n(Y^n(\tau_i^n)) - m_{Y^n(\tau_i^n)}^n),$$

$$\hat{X}_2^n(t) := \frac{1}{n^\delta} \left(\sum_{i=1}^{A^n(t)} m_{Y^n(\tau_i^n)}^n - \int_0^t m_{Y^n(s)}^n \lambda_{Y^n(s)}^n ds \right),$$

and

$$\hat{X}_3^n(t) := \frac{1}{n^\delta} \left(\int_0^t m_{Y^n(s)}^n \lambda_{Y^n(s)}^n ds - \sum_{i=1}^I \lambda_i^n m_i^n \pi_i t \right).$$

For each $n \in \mathbb{N}$, define $m_*^n := \max_{i \in S} m_i^n$, $\lambda_*^n := \max_{i \in S} \lambda_i^n$ and $\sigma_*^n := \max_{i \in S} \sigma_i^n$. By Assumption 1, we can obtain that

$$\frac{1}{n} \lambda_*^n \rightarrow \lambda_*, \quad m_*^n \rightarrow m_* \quad \text{and} \quad \sigma_*^n \rightarrow \sigma_* \quad (2.1)$$

in \mathbb{R} as $n \rightarrow \infty$. Then we can find $n_1 > 0$ and $\Delta > 0$ such that, for any $n > n_1$,

$$\max \left\{ \frac{1}{n} \lambda_*^n, m_*^n, \sigma_*^n \right\} < \Delta. \quad (2.2)$$

We fix the n_1 and Δ throughout the proof.

We start proving the convergence of \hat{X}_1^n . We quote the following two lemmas from [12].

Lemma 2.2. *Let z_1, \dots, z_n and w_1, \dots, w_n be complex numbers of modulus $\leq b$. Then*

$$\left| \prod_{i=1}^n z_i - \prod_{i=1}^n w_i \right| \leq b^{n-1} \sum_{i=1}^n |z_i - w_i|.$$

Lemma 2.3. *If b is a complex number with $|b| \leq 1$, then $|e^b - (1+b)| \leq |b|^2$.*

We now prove two lemmas in order to prove the convergence of \hat{X}_1^n .

Lemma 2.4. *The finite-dimensional distributions of \hat{X}_1^n converge to those of \hat{X}_1 , where $\hat{X}_1 := \{\hat{X}_1(t) : t \geq 0\}$ is given by*

$$\hat{X}_1 := \begin{cases} \bar{\sigma} B_1, & \text{if } \delta = 1/2, \alpha \geq 1, \\ 0, & \text{if } \delta = 1 - \alpha/2, \alpha \in (0, 1), \end{cases} \quad (2.3)$$

with B_1 being a standard Brownian motion, and $\bar{\sigma}$ defined in (1.3).

Proof. We need to prove

$$(\hat{X}_1^n(t_1), \dots, \hat{X}_1^n(t_k)) \Rightarrow (\hat{X}_1(t_1), \dots, \hat{X}_1(t_k)) \quad \text{in } \mathbb{R}^k \quad \text{as } n \rightarrow \infty, \quad (2.4)$$

for any $0 \leq t_1 \leq \dots \leq t_k \leq T$ and $k \geq 1$.

We first consider one-dimensional case: to prove that for each $t \geq 0$,

$$\hat{X}_1^n(t) \Rightarrow \hat{X}_1(t) \quad \text{in } \mathbb{R} \quad \text{as } n \rightarrow \infty. \quad (2.5)$$

By the continuity theorem in [12], it is sufficient to show that $\varphi_t(\theta)$ is continuous at 0 and $\varphi_t^n(\theta)$ converges pointwisely to $\varphi_t(\theta)$ for every θ , where the characteristic functions of $\hat{X}_1^n(t)$ and $\hat{X}_1(t)$ are denoted by $\varphi_t^n(\theta)$ and $\varphi_t(\theta)$, respectively. By the definition of \hat{X}_1 in (2.3), we obtain that

$$\varphi_t(\theta) = E[\exp(i\theta \hat{X}_1(t))] = \begin{cases} \exp\left(-\frac{1}{2}\theta^2 \bar{\sigma}^2 t\right), & \delta = 1/2, \alpha \geq 1, \\ 1, & \delta = 1 - \alpha/2, 0 < \alpha < 1, \end{cases} \quad (2.6)$$

which is continuous at $\theta = 0$.

Let $\mathcal{A}_t^n := \sigma\{A^n(s) : 0 \leq s \leq t\} \vee \sigma\{Y^n(s) : 0 \leq s \leq t\} \vee \mathcal{N}$, where \mathcal{N} is the collection of P -null sets. Then, by conditioning, we obtain

$$\begin{aligned} \varphi_t^n(\theta) &= E[\exp(i\theta \hat{X}_1^n(t))] = E[E[\exp(i\theta \hat{X}_1^n(t)) | \mathcal{A}_t^n]] \\ &= E\left[\prod_{i=1}^{A^n(t)} E\left[\exp\left(i\theta \frac{1}{n^\delta} (Z_i^n(Y^n(\tau_i^n)) - m_{Y^n(\tau_i^n)}^n)\right) \middle| \mathcal{A}_t^n\right]\right] \\ &= E\left[\prod_{i=1}^{A^n(t)} \left(1 - \frac{\theta^2}{2n^{2\delta}} (\sigma_{Y^n(\tau_i^n)}^n)^2 + o(n^{-2\delta})\right)\right]. \end{aligned} \quad (2.7)$$

Under Assumption 1, by (2.1), we can find n_2 such that for any $n > n_2$,

$$0 < \max_{1 \leq i \leq A^n(t)} \left\{ \frac{\theta^2}{2n^{2\delta}} (\sigma_{Y^n(\tau_i^n)}^n)^2 - o(n^{-2\delta}) \right\} < 1.$$

Recall n_1 in (2.2). Thus, for $\delta = 1/2$, $\alpha \geq 1$ and for any

$$n > n_3 := \max\{n_1, n_2\}, \quad (2.8)$$

we have

$$\begin{aligned} \left| \varphi_t^n(\theta) - \varphi_t(\theta) \right| &\leq E\left[\left|\prod_{i=1}^{A^n(t)} \left(1 - \frac{\theta^2}{2n} (\sigma_{Y^n(\tau_i^n)}^n)^2 + o(n^{-1})\right) - \prod_{i=1}^{A^n(t)} \exp\left(-\frac{\theta^2}{2n} (\sigma_{Y^n(\tau_i^n)}^n)^2\right)\right|\right] \\ &\quad + \left| E\left[\exp\left(-\frac{\theta^2}{2n} \sum_{i=1}^{A^n(t)} (\sigma_{Y^n(\tau_i^n)}^n)^2\right)\right] - \exp\left(-\frac{\theta^2}{2} \bar{\sigma}^2 t\right) \right| \\ &\leq \frac{\theta^4}{4n^2} E\left[\sum_{i=1}^{A^n(t)} (\sigma_{Y^n(\tau_i^n)}^n)^4\right] + o(1) \\ &\quad + \left| E\left[\exp\left(-\frac{\theta^2}{2n} \sum_{i=1}^{A^n(t)} (\sigma_{Y^n(\tau_i^n)}^n)^2\right)\right] - \exp\left(-\frac{\theta^2}{2} \bar{\sigma}^2 t\right) \right| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (2.9)$$

The first inequality is implied by the triangle inequality. The second inequality follows by Lemmas 2.2 and 2.3. By (2.2), for large enough n defined above, we have

$$E \left[\frac{1}{n} \sum_{i=1}^{A^n(t)} (\sigma_{Y^n(\tau_i^n)}^n)^4 \right] \leq \Delta^5 t, \quad t \geq 0.$$

So the first two terms in the last equation converge to 0. For the last term convergence, since $\{ \exp(-\frac{\theta^2}{2n} \sum_{i=1}^{A^n(t)} (\sigma_{Y^n(\tau_i^n)}^n)^2) : n \geq 1 \}$ is uniformly integrable for each $t \geq 0$, it suffices to show that

$$\frac{1}{n} \sum_{i=1}^{A^n(t)} (\sigma_{Y^n(\tau_i^n)}^n)^2 \Rightarrow \bar{\sigma}^2 t \quad \text{in } \mathbb{R} \quad \text{as } n \rightarrow \infty. \quad (2.10)$$

This follows from the convergences:

$$\sum_{i=1}^I \frac{\lambda_i^n}{n} \int_0^t \mathbf{1}(Y^n(s) = i) ds \rightarrow \sum_{i=1}^I \lambda_i \pi_i t \quad a.s., \quad (2.11)$$

and

$$\sum_{i=1}^I \frac{1}{n} \lambda_i^n (\sigma_i^n)^2 \int_0^t \mathbf{1}(Y^n(s) = i) ds \rightarrow \sum_{i=1}^I \lambda_i \sigma_i^2 \pi_i t \quad a.s. \quad (2.12)$$

by the claim in (4) in [1] and Assumption 1, and the WLLN of Poisson processes and random change of time Lemma in [8] (pp.151).

For $\delta = 1 - \alpha/2$ and $0 < \alpha < 1$, we follow the similar arguments and prove

$$\left| E \left[\exp \left(-\frac{\theta^2}{2n^{2\delta}} \sum_{i=1}^{A^n(t)} (\sigma_{Y^n(\tau_i^n)}^n)^2 \right) \right] - 1 \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.13)$$

Therefore, we have shown (2.5).

To show the convergence of the finite-dimensional distributions, it is sufficient to prove that for any $(\theta_1, \dots, \theta_k) \in \mathbb{R}^k$ and $0 \leq t_1 < \dots < t_k \leq T$,

$$E \left[\exp \left(i \sum_{i=1}^k \theta_i \hat{X}_1^n(t_i) \right) \right] \rightarrow E \left[\exp \left(i \sum_{i=1}^k \theta_i \hat{X}_1(t_i) \right) \right] \quad \text{as } n \rightarrow \infty,$$

and the limit is continuous at $(0, \dots, 0) \in \mathbb{R}^k$. By the definition of \hat{X}_1 , we have

$$\begin{aligned} & E \left[\exp \left(i \sum_{i=1}^k \theta_i \hat{X}_1(t_i) \right) \right] \\ &= \begin{cases} \exp \left(-\frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k \theta_i \theta_j \bar{\sigma}^2 (t_i \wedge t_j) \right), & \delta = 1/2, \quad \alpha \geq 1, \\ 1, & \delta = 1 - \alpha/2, \quad 0 < \alpha < 1, \end{cases} \end{aligned}$$

and it is continuous at $(0, \dots, 0) \in \mathbb{R}^k$. Let $t_0 := 0$. By conditioning and direct calculation as in (2.7), we have

$$E \left[\exp \left(i \sum_{i=1}^k \theta_i \hat{X}_i^n(t_i) \right) \right] = E \left[\prod_{j=1}^k \exp \left(i \frac{1}{n^\delta} \sum_{i=j}^k \theta_i \sum_{h=A^n(t_{j-1})+1}^{A^n(t_j)} \left(Z_h^n(Y^n(\tau_h^n)) - m_{Y^n(\tau_h^n)}^n \right) \right) \right]$$

$$\rightarrow \begin{cases} \prod_{j=1}^k \exp\left(-\frac{1}{2}\left(\sum_{i=j}^k \theta_i\right)^2 \bar{\sigma}^2(t_j - t_{j-1})\right), & \delta = 1/2, \alpha \geq 1, \\ 1, & \delta = 1 - \alpha/2, 0 < \alpha < 1, \end{cases}$$

as $n \rightarrow \infty$, and

$$\prod_{j=1}^k \exp\left(-\frac{1}{2}\left(\sum_{i=j}^k \theta_i\right)^2 \bar{\sigma}^2(t_j - t_{j-1})\right) = \exp\left(-\frac{1}{2}\sum_{i=1}^k \sum_{j=1}^k \theta_i \theta_j \bar{\sigma}^2(t_i \wedge t_j)\right).$$

Applying the continuous mapping theorem, the convergence can be shown in the similar way in (2.7) and (2.9). Therefore, we have proved the convergence of finite-dimensional distributions. \square

Lemma 2.5. For $0 \leq r \leq s \leq t \leq T$ and $n > n_3$ with n_3 in (2.8),

$$E\left[(\hat{X}_1^n(r) - \hat{X}_1^n(s))^2 (\hat{X}_1^n(s) - \hat{X}_1^n(t))^2\right] \leq K_1(t - r)^2, \quad (2.14)$$

for some constant $K_1 > 0$.

Proof. For any $0 \leq r \leq t$, define $\mathcal{G}_{r,t}^n := \sigma\{Y^n(u) : r \leq u \leq t\} \vee \mathcal{N}$ with \mathcal{N} being the collection of P -null sets. It is evident that the inequality holds when $r = s$, $s = t$ or $r = s = t$. For $r < s < t$, we notice that

$$\begin{aligned} E\left[(\hat{X}_1^n(s) - \hat{X}_1^n(r))^2 | \mathcal{G}_{r,t}^n\right] &= \frac{1}{n^{2\delta}} E\left[\left(\sum_{i=A^n(r)+1}^{A^n(s)} (Z_i^n(Y^n(\tau_i^n)) - m_{Y^n(\tau_i^n)}^n)\right)^2 | \mathcal{G}_{r,t}^n\right] \\ &= \frac{1}{n^{2\delta}} E\left[\sum_{i=A^n(r)+1}^{A^n(s)} (\sigma_{Y^n(\tau_i^n)}^n)^2 | \mathcal{G}_{r,t}^n\right], \end{aligned}$$

where the first central moment of $\{Z_i^n : i \in \mathbb{N}\}$ is zero given the Markov process Y^n . Then, by the conditional independence property, we have, for $n > n_3$,

$$\begin{aligned} &E\left[(\hat{X}_1^n(s) - \hat{X}_1^n(r))^2 (\hat{X}_1^n(t) - \hat{X}_1^n(s))^2\right] \\ &= E\left[E\left[(\hat{X}_1^n(s) - \hat{X}_1^n(r))^2 | \mathcal{G}_{r,t}^n\right] E\left[(\hat{X}_1^n(t) - \hat{X}_1^n(s))^2 | \mathcal{G}_{r,t}^n\right]\right] \\ &\leq \frac{\Delta^4}{n^{4\delta}} E\left[E\left[A^n(s) - A^n(r) | \mathcal{G}_{r,t}^n\right] E\left[A^n(t) - A^n(s) | \mathcal{G}_{r,t}^n\right]\right] \\ &\leq \Delta^4 n^{2-4\delta} \left(\frac{\lambda_*^n}{n}\right)^2 (s-r)(t-s) \leq \Delta^6 (t-r)^2. \end{aligned} \quad (2.15)$$

The inequalities are implied by (2.2). Thus, we have completed the proof. \square

Lemma 2.6. $\hat{X}_1^n \Rightarrow \hat{X}_1$ in \mathbb{D} as $n \rightarrow \infty$, where \hat{X}_1 is given in (2.3).

Proof. By the continuity of \hat{X}_1 , Lemma 2.4 and Lemma 2.5, we can apply Theorem 13.5 in [8] to conclude the convergence of \hat{X}_1^n . \square

Lemma 2.7. $\hat{X}_2^n \Rightarrow \hat{X}_2$ in \mathbb{D} as $n \rightarrow \infty$, where \hat{X}_2 is given by

$$\hat{X}_2 := \begin{cases} \bar{\nu}^{1/2} B_2, & \text{if } \delta = 1/2, \alpha \geq 1, \\ 0, & \text{if } \delta = 1 - \alpha/2, \alpha \in (0, 1), \end{cases} \quad (2.16)$$

with B_2 being a standard Brownian motion, independent of B_1 , and $\bar{\nu}$ defined in (1.3).

Proof. For each $n \in \mathbb{N}$, since $\{A^n(t) - \int_0^t \lambda_{Y^n(u)}^n du : t \geq 0\}$ is a Martingale, \hat{X}_2^n is also a Martingale. The maximum jump for \hat{X}_2^n is m_*^n/n^δ . By (2.1), we obtain the expected value of the maximum jump is asymptotically negligible, i.e.

$$\frac{1}{n^\delta} E[m_*^n] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.17)$$

For $n \in \mathbb{N}$, let $\{[\hat{X}_2^n, \hat{X}_2^n](t) : t \geq 0\}$ be the quadratic-variation process of \hat{X}_2^n . Then, for each t , we have

$$\begin{aligned} [\hat{X}_2^n, \hat{X}_2^n](t) &= \frac{1}{n^{2\delta}} \sum_{i=1}^{A^n(t)} (m_{Y^n(\tau_i)}^n)^2 \\ &\Rightarrow \begin{cases} \bar{\nu}t, & \delta = 1/2, \alpha \geq 1, \\ 0, & \delta = 1 - \alpha/2, 0 < \alpha < 1, \end{cases} \quad \text{in } \mathbb{R} \quad \text{as } n \rightarrow \infty \end{aligned} \quad (2.18)$$

where the convergence can be proved in the same way as (2.10). Applying Theorem 2.1 in [29], we have shown the convergence of \hat{X}_2^n . \square

Lemma 2.8. $\hat{X}_3^n \Rightarrow \hat{X}_3$ in \mathbb{D} as $n \rightarrow \infty$, where the limit process $\hat{X}_3 = \{\hat{X}_3(t) : t \geq 0\}$ is given by

$$\hat{X}_3 := \begin{cases} 0, & \text{if } \delta = 1/2, \alpha > 1, \\ \bar{\beta}^{1/2} B_3, & \text{if } \delta = 1 - \alpha/2, 0 < \alpha \leq 1, \end{cases} \quad (2.19)$$

with B_3 being a standard Brownian motion, independent of B_1 and B_2 , and $\bar{\beta}$ defined in (1.3).

Proof. By Proposition 3.2 in [1] and Assumption 1, we have, as $n \rightarrow \infty$,

$$\frac{1}{n^\delta} \left(\sum_{i=1}^I m_i^n \lambda_i^n \int_0^t \mathbf{1}(Y^n(s) = i) ds - \sum_{i=1}^I m_i^n \lambda_i^n \pi_i t \right) \Rightarrow \begin{cases} 0, & \delta = 1/2, \alpha > 1, \\ B_3(\bar{\beta}t), & \delta = 1 - \alpha/2, 0 < \alpha \leq 1, \end{cases}$$

where B_3 is a standard Brownian motion and $\bar{\beta}$ is defined in (1.3). Therefore, we have shown the convergence of \hat{X}_3^n . \square

Completing the Proof of Theorem 1.1. We first prove the joint convergence of the three processes \hat{X}_1^n , \hat{X}_2^n and \hat{X}_3^n . For $k = 1, \dots, I$, let A_k^n be the Poisson process with rate λ_k^n , and we assume $\{A_k^n : k = 1, \dots, I\}$ are mutually independent. Define the random processes $\{R_k^n : k = 1, \dots, I\}$ by

$$R_k^n(t) := \int_0^t \mathbf{1}(Y^n(s) = k) ds, \quad t \geq 0.$$

Note that $\sum_{k=1}^I A_k^n(R_k^n(\cdot)) \stackrel{d}{=} A^n(\cdot)$. Then, we can rewrite the following processes (equivalent in distribution). For each n and $t \geq 0$, X^n can be written as

$$X^n(t) = \sum_{k=1}^I \int_0^t Z_{k, A_k^n(R_k^n(s))}^n dA_k^n(R_k^n(s)) = \sum_{k=1}^I \sum_{i=1}^{A_k^n(R_k^n(t))} Z_{k,i}^n, \quad t \geq 0,$$

where $\{Z_{k,i}^n : i \in \mathbb{N}\}$ is a sequence of independent random variables following the distribution F_k^n , for each $k = 1, \dots, I$. \hat{X}_1^n and \hat{X}_2^n can be written as

$$\hat{X}_1^n(t) = \frac{1}{n^\delta} \sum_{k=1}^I \sum_{i=1}^{A_k^n(R_k^n(t))} (Z_{k,i}^n - m_k^n), \quad \hat{X}_2^n(t) = \frac{1}{n^\delta} \sum_{k=1}^I m_k^n \left(A_k^n(R_k^n(t)) - \lambda_k^n R_k^n(t) \right),$$

for $t \geq 0$, respectively. Define $\{\tilde{X}_{1,k}^n, \tilde{X}_{2,k}^n : k = 1, \dots, I\}$ by

$$\tilde{X}_{1,k}^n(t) := \frac{1}{n^\delta} \sum_{i=1}^{A_k^n(t)} (Z_{k,i}^n - m_k^n), \quad \tilde{X}_{2,k}^n(t) := \frac{1}{n^\delta} m_k^n \left(A_k^n(t) - \lambda_k^n t \right), \quad t \geq 0,$$

respectively. It is evident that

$$\hat{X}_1^n = \sum_{k=1}^I \tilde{X}_{1,k}^n \circ R_k^n, \quad \hat{X}_2^n = \sum_{k=1}^I \tilde{X}_{2,k}^n \circ R_k^n.$$

Then, we approximate $\tilde{X}_{1,k}^n$ by $\check{X}_{1,k}^n$:

$$\check{X}_{1,k}^n(t) := \frac{1}{n^\delta} \sum_{i=1}^{\lfloor \lambda_k^n t \rfloor} (Z_{k,i}^n - m_k^n).$$

Since the limits of $\{R_k^n : k = 1, \dots, I\}$ are deterministic functions by the claim in (4) of [1], by Donsker Theorem and the FCLT for Poisson processes, we obtain the joint weak convergence

$$\begin{aligned} & (\check{X}_{1,1}^n, \dots, \check{X}_{1,I}^n, \tilde{X}_{2,1}^n, \dots, \tilde{X}_{2,I}^n, R_1^n, \dots, R_I^n, \hat{X}_3^n) \\ & \Rightarrow (B_{1,1}, \dots, B_{1,I}, B_{2,1}, \dots, B_{2,I}, R_1, \dots, R_I, \hat{X}_3) \quad \text{in } (\mathbb{D}^{3I+1}, J_1) \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where $B_{1,k}$ is a Brownian motion with variance coefficient $\sigma_k^2 \lambda_k \mathbf{1}(\delta = 1/2, \alpha \geq 1)$, $B_{2,k}$ is a Brownian motion with variance coefficient $m_k^2 \lambda_k \mathbf{1}(\delta = 1/2, \alpha \geq 1)$, $R_k(t) \equiv \pi_k t$, for $k = 1, \dots, I$, and all limit processes are mutually independent. By Theorem 7.3.2 in [28], we have that $(\check{X}_{1,1}^n, \dots, \check{X}_{1,I}^n)$ and $(\tilde{X}_{1,1}^n, \dots, \tilde{X}_{1,I}^n)$ are asymptotically equivalent. Thus, by Theorem 3.1 in [8], we have the joint weak convergence

$$\begin{aligned} & (\tilde{X}_{1,1}^n, \dots, \tilde{X}_{1,I}^n, \tilde{X}_{2,1}^n, \dots, \tilde{X}_{2,I}^n, R_1^n, \dots, R_I^n, \hat{X}_3^n) \\ & \Rightarrow (B_{1,1}, \dots, B_{1,I}, B_{2,1}, \dots, B_{2,I}, R_1, \dots, R_I, \hat{X}_3) \quad \text{in } (\mathbb{D}^{3I+1}, J_1) \quad \text{as } n \rightarrow \infty, \end{aligned}$$

Applying the random change of time lemma in [8] and Theorem 13.2.2 in [28], we obtain the joint convergence

$$(\hat{X}_1^n, \hat{X}_2^n, \hat{X}_3^n) \Rightarrow (\hat{X}_1, \hat{X}_2, \hat{X}_3) \quad \text{in } (\mathbb{D}^3, J_1) \quad \text{as } n \rightarrow \infty.$$

By Lemmas 2.1, 2.6, 2.7 and 2.8, we complete our proof by applying the continuous mapping theorem. \square

3. PROOFS OF THEOREMS 1.2 AND 1.3

Proof of Theorem 1.2. By the Fourier inversion formula, for $x \in \check{\mathcal{X}}_t^n$, we have

$$P(\check{X}^n(t) = x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iux} E[e^{iu\check{X}^n(t)}] du = \frac{1}{2\pi n^\delta} \int_{-\pi n^\delta}^{\pi n^\delta} e^{-iun^{-\delta}x} \psi_t^n(u) du \quad (3.1)$$

where

$$\psi_t^n(u) := E[e^{iun^{-\delta}\check{X}^n(t)}] \quad \text{for } u \in \mathbb{R}. \quad (3.2)$$

Recall $\psi_t(x)$ defined in (1.7). Since $|e^{-iux}| \leq 1$ for any x , we obtain that

$$\begin{aligned} \left| 2\pi n^\delta P(\check{X}^n(t) = x) - 2\pi \psi_t(x) \right| &\leq \int_{-\pi n^\delta}^{\pi n^\delta} \left| \psi_t^n(u) - \exp(-\varrho tu^2/2) \right| du \\ &\quad + \int_{|u| > \pi n^\delta} \exp(-\varrho tu^2/2) du. \end{aligned} \quad (3.3)$$

It is evident that the second term at the right hand side converges to 0 as $n \rightarrow \infty$.

To prove the convergence of the first term, we split the integral into three parts. Applying Theorem 1.1 and the dominated convergence theorem, for any constant c , we have

$$\int_{-c}^c \left| \psi_t^n(u) - \exp(-\varrho tu^2/2) \right| du \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.4)$$

By Taylor's Theorem, recall that \mathcal{A}_t^n is defined in (2.7) and we can find small enough $\eta > 0$ such that, for $|u| \leq \eta n^\delta$ and large enough n ,

$$\begin{aligned} \left| E[e^{iun^{-\delta}X^n(t)}] \right| &= \left| E[E[e^{iun^{-\delta}X^n(t)} | \mathcal{A}_t^n]] \right| \leq \left| E \left[\prod_{i=1}^{A^n(t)} \left(1 - \frac{u^2}{4n^{2\delta}} (\sigma_{Y^n(\tau_i^n)}^n)^2 \right) \right] \right| \\ &\leq \left| E \left[\exp \left(-\frac{u^2}{4n^{2\delta}} \sum_{i=1}^{A^n(t)} (\sigma_{Y^n(\tau_i^n)}^n)^2 \right) \right] \right| \leq \exp(-K_2 u^2/4) \end{aligned} \quad (3.5)$$

where K_2 is a positive constant. The last inequality can be proved by using the convergence of the third term in (2.9), and (2.13). To prove the convergence for the second part of the integral, by the triangle inequality and (3.5), we can get

$$\begin{aligned} \int_{c \leq |u| \leq \eta n^\delta} \left| \psi_t^n(u) - \exp(-\varrho tu^2/2) \right| du \\ \leq \int_{c \leq |u| \leq \eta n^\delta} (\exp(-K_2 u^2/4) + \exp(-\varrho tu^2/2)) du, \end{aligned} \quad (3.6)$$

which becomes arbitrarily small when c is arbitrarily large.

We next show the convergence for the third part of the integral. By Assumption 2 and Theorem 3.5.1 in [12], we obtain that

$$\left| E[e^{iun^{-\delta}\hat{Z}_i(Y^n(\tau_i^n))} | Y^n(\tau_i^n) = k] \right| < 1 \quad \text{for } u \in (0, 2\pi n^\delta).$$

Since the characteristic function is uniformly continuous, by (1.5), for large enough n , there exists $r_1 > 0$ such that, for any $i \in \mathbb{N}$, $k \in \mathcal{S}$ and $u \in (\eta n^\delta, \pi n^\delta)$,

$$\left| E[e^{iun^{-\delta}Z_i^n(Y^n(\tau_i^n))} | Y^n(\tau_i^n) = k] \right| \leq \left| E[e^{iun^{-\delta}\hat{Z}_i(Y^n(\tau_i^n))} | Y^n(\tau_i^n) = k] \right| \leq r_1 < 1.$$

By Assumption 1, there exists $\tilde{\Delta}$ such that $\min_{i \in S} \{\lambda_i^n\}/n \geq \tilde{\Delta} > 0$ for large enough n . Then, by conditioning, we obtain that for large enough n and $u \in (\eta n^\delta, \pi n^\delta)$,

$$\begin{aligned} \left| E[e^{iun^{-\delta} X^n(t)}] \right| &= \left| E \left[\prod_{i=1}^{A^n(t)} E[e^{iun^{-\delta} Z_i^n(Y^n(\tau_i^n))} | \mathcal{A}_t^n] \right] \right| \leq E \left[\prod_{i=1}^{A^n(t)} \left| E[e^{iun^{-\delta} Z_i^n(Y^n(\tau_i^n))} | \mathcal{A}_t^n] \right| \right] \\ &\leq E[r_1^{A^n(t)}] = E \left[\exp \left((r_1 - 1) \int_0^t \lambda_{Y^n(s)}^n ds \right) \right] \\ &\leq \exp(\tilde{\Delta} n t (r_1 - 1)), \end{aligned} \quad (3.7)$$

Thus, we have

$$\begin{aligned} &\int_{\eta n^\delta < |u| < \pi n^\delta} \left| \psi_t^n(u) - \exp(-\varrho t u^2/2) \right| du \\ &\leq \int_{\eta n^\delta < |u| < \pi n^\delta} \left(\left| E[e^{iun^{-\delta} X^n(t)}] \right| \cdot |e^{-i\lambda n^{1-\delta} t}| + \exp(-\varrho t u^2/2) \right) du \\ &\leq \int_{\eta n^\delta < |u| < \pi n^\delta} \left(\exp(\tilde{\Delta} n t (r_1 - 1)) + \exp(-\varrho t u^2/2) \right) du \\ &\leq 2\pi n^\delta \exp(\tilde{\Delta} n t (r_1 - 1)) + 2 \int_{\eta n^\delta}^{+\infty} \exp(-\varrho t u^2/2) du. \end{aligned} \quad (3.8)$$

Since $r - 1 < 0$, it is easy to see (3.8) converges to 0 as $n \rightarrow \infty$. Therefore, by (3.4), (3.6) and (3.8), we have shown that (3.3) converges to 0 as $n \rightarrow \infty$ and this completes the proof. \square

Proof of Theorem 1.3. The proof follows closely from the proof of Theorem 3.5.3 in [12]. For some constant $d > 0$, let

$$p_0(y) = \frac{1 - \cos dy}{\pi dy^2}$$

be the density of the Polya's distribution and $p_\theta(y) = e^{i\theta y} p_0(y)$. Define

$$\hat{p}_0(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iuy} p_0(y) dy = \begin{cases} 1 - |u/d|, & |u| \leq d, \\ 0, & \text{otherwise,} \end{cases}$$

and we have $\hat{p}_\theta(u) = \hat{p}_0(u + \theta)$.

By the result following claim (a) in the proof of Theorem 3.5.3 in [12], to prove Theorem 1.3, it suffices to show that for each $t \geq 0$,

$$\left| n^\delta E[p_\theta(\check{X}^n(t) - x_n)] - \psi_t(x) \int_{-\infty}^{\infty} p_\theta(y) dy \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.9)$$

By the same calculation in the proof of Theorem 3.5.3 in [12], we have

$$E[p_\theta(\check{X}^n(t) - x_n)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} E[e^{-iu\check{X}^n(t)}] e^{iux_n} \hat{p}_\theta(u) du.$$

There is a constant M_1 such that $\hat{p}_\theta(u) = 0$ for $u \notin [-M_1, M_1]$. Then, changing the variable, we obtain that

$$n^\delta E[p_\theta(\check{X}^n(t) - x_n)] = \frac{1}{2\pi} \int_{-M_1 n^\delta}^{M_1 n^\delta} e^{iun^{-\delta} x_n} \psi_t^n(-u) \hat{p}_\theta(u/n^\delta) du.$$

By Assumption 3, Theorem 3.5.1 in [12] and the continuity of characteristic function again, for large enough n , we can find a constant $r_2 < 1$ such that for any $i \in \mathbb{N}$, $k \in S$ and $M_2 > 0$,

$$\left| E \left[e^{iun^{-\delta} Z_i^n(Y^n(\tau_i^n))} \middle| Y^n(\tau_i^n) = k \right] \right| \leq r_2 \quad \text{for } M_2 \leq |u| \leq M_1.$$

Since $|\hat{p}_\theta(u)| \leq 1$ for u in \mathbb{R} , taking similar steps in (3.3), (3.6) and (3.8), there exist constant $c > 0$, $K_3 > 0$ and small enough $M_2 > 0$ such that, for large enough n ,

$$\begin{aligned} & \left| n^\delta E[p_\theta(\check{X}^n(t) - x_n)] - \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\varrho tu^2/2) e^{iux} \hat{p}_\theta(0) du \right| \\ & \leq \frac{1}{2\pi} \int_{-c}^c \left| e^{iun^{-\delta} x_n} \psi_t^n(-u) \hat{p}_\theta(u/n^\delta) - \exp(-\varrho tu^2/2) e^{iux} \hat{p}_\theta(0) \right| du \\ & \quad + \frac{1}{2\pi} \int_{c \leq |u| \leq M_2 n^\delta} (\exp(-K_3 u^2/4) + \exp(-\varrho tu^2/2)) du \\ & \quad + \frac{M_1 n^\delta}{\pi} \exp(\tilde{\Delta} n t (r_2 - 1)) + \frac{1}{\pi} \int_{M_2 n^\delta}^{+\infty} \exp(-\varrho tu^2/2) du \\ & \quad + \frac{1}{2\pi} \int_{|u| \geq M_1 n^\delta} \left| \exp(-\varrho tu^2/2) e^{iux} \hat{p}_\theta(0) \right| du. \end{aligned} \quad (3.10)$$

Applying Theorem 1.1, we obtain $\psi_t^n(-u) \rightarrow \exp(-\varrho tu^2/2)$ as $n \rightarrow \infty$. Since $x_n/n^\delta \rightarrow x$ as $n \rightarrow \infty$, by the dominated convergence theorem, we have that the first term converges to 0 as $n \rightarrow \infty$. The second term is arbitrarily small, when c is arbitrarily large. By (3.6), the third term also converges to 0. It is evident that the last two terms also converge to 0. Applying inversion formula and Theorem 3.3.5 in [12], we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\varrho tu^2/2) e^{iux} \hat{p}_\theta(0) du = \psi_t(x) \hat{p}_\theta(0) = \psi_t(x) \int_{-\infty}^{\infty} p_\theta(y) dy. \quad (3.11)$$

Therefore, we have shown (3.9). This completes the proof. \square

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