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Transient Behaviors of Single-Server Queues with Diffusive Rates

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1 Introduction

Queues in random environment are important models in applied probability. In [10], a family of $M/M/1$ queues in an interactive random environment is introduced, where not only the arrival and service rates depend on the state of a Markov random environment, but also the transitions of the Markov random environment depend on the state of the queueing process. The joint queueing and environment processes are constructed in a certain way such that the stationary distribution of the joint processes takes an explicit simple form (weighted geometric, or viewed as some product form). An explicit estimate for the exponential rate of convergence to the stationary distribution of the joint Markov process is also established via coupling. This family of single-server queues with interactive random environments includes many interesting specific models. In particular, the following special cases are introduced: (a) $M/M/1$ queue with an arrival rate being a reflected Brownian motion (RBM), (b) $M/M/1$ queue with a RBM service rate, and (c) $M/M/1$ queue with both arrival and service rates being a two-dimensional (2-d) RBM in a wedge of the positive orthant. This note poses the transient behavior problems for these models.

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2 Problem Statement

We describe the $M/M/1$ queue with both arrival and service rates being diffusive and the other two can be stated similarly.

Let Z be a two-dimensional RBM with drift in the wedge $\mathscr{D} = \{(z_1, z_2) \in \mathbb{R}^2_+ : z_2 \leq z_1\}$ with normal (or oblique) reflections at the boundaries. Let $\mathscr{A}$ be its generator. For each $z = (z_1, z_2) \in \mathscr{D}$, let $\lambda(z) = z_1$ and $\mu(z) = z_2$, which are used for the arrival and service rates for the single-server queue. We consider the Markov process (N, Z) with values in $\mathbb{N} \times \mathscr{D}$, with the following generator:

$$\mathscr{L} f(n,z) = \mathscr{M}_z f(n,z) + \beta_n \rho(z)^{-n} \mathscr{A} f(n,z),$$

for f in the domain of $\mathscr{L}$, where $\mathscr{M}_z$ is the generator of the queue given the environment state z, that is, for g in the domain of $\mathscr{M}_z$, $\mathscr{M}_z g(n) = \lambda(z)(g(n+1) - g(n)) + \mathbf{1}_{n\neq 0}\mu(z)(g(n-1) - g(n))$, and $\rho(z) = \lambda(z)/\mu(z)$ is the traffic intensity. We assume that $\rho(z) \in [0,1]$ for all $z \in \mathscr{D}$. The constant β_n is the variability coefficient for the diffusive environment, depending on the queueing state. $\rho(z)^{-n}$ can be regarded as the queueing impact factor, capturing the impact from the traffic intensity (congestion) on the queueing process.

The RBM with generator $\mathscr{A}$ is positive recurrent and has a unique stationary/invariant measure ν. (For general diffusions this is assumed to exist in [10].) We assume that $\Xi := \int \frac{1}{1-\rho(z)} \nu(dz) < \infty$. It is shown in [10, Theorem 3.1] that there is a unique invariant measure for (N, Z): $\pi(\{n\}, z) = \Xi^{-1} \rho^n(z) \nu(dz)$.

The transient behaviors for the model remain to be understood. Of particular interests are the associated moment functions. Let $m_{N,k}(t, n_0) = E[N(t)^k | N(0) = n_0]$ and $m_{Z,i,k}(t, z_{i,0}) = E[Z_i(t)^k | Z_i(0) = z_{i,0}]$, $i = 1, 2$, for $k \geq 1$ and $t \geq 0$, and for any given initial conditions $N(0) = n_0$ and $Z_0 = (Z_1(0), Z_2(0)) = (z_{1,0}, z_{2,0})$. Also, let $\{W(t), t \geq 0\}$ be the workload process of the queue and $m_{W,k}(t, w_0) := E[W(t)^k | W(0) = w_0]$ for $k \geq 1$, given the initial condition w_0. These quantities show how the joint distributions of the queue length and the RBM converge to the steady states.

Open Problems: We conjecture that the moment functions $m_{N,k}(t, 0)$, $m_{W,k}(t, 0)$ and $m_{Z,i,k}(t, 0)$ of both the queue and the RBM starting at the origin are nondecreasing in time. The moment functions $m_{N,k}(t, n_0)$, $m_{W,k}(t, w_0)$ and $m_{Z,i,k}(t, z_{i,0})$ with general initial conditions can be decomposed into two nondecreasing components. In addition, asymptotic approximations of the moment functions as $t \to \infty$ can be derived. The $(k+1)^{\text{st}}$ moments converge to equilibria slower than the k^{th} moment as $t \to \infty$.

For the $M/M/1$ queue with only a diffusive arrival rate, we set $\mu \equiv 1$ and $\lambda(z) = z \in [0, 1]$, and the RBM is on $[0, 1]$. For the $M/M/1$ queue with only a diffusive service rate, we set $\lambda \equiv 1$ and $\mu(z) = z \in [1, \infty)$ and the RBM with a negative drift is on $[1, \infty)$.

3 Discussion

In the literature, for standard $M/M/1$ queues, the behavior of the moments of the queue length $N(t)$ at time t, $m_{N,k}(t, n_0)$ for $k \geq 1$ was analyzed in [3,4]. In [4], the transient behaviors of the busy periods and first passage times were also studied using Laplace transforms. For 1-d RBM $Z(t)$ on $\mathbb{R}_+$, the behavior of the moments $m_{Z,k}(t,z_0)$ for $k \geq 1$ was studied in [1,2]. Useful approximations for the moments can be possibly derived using mixtures of exponentials. In [5], the transient behavior for the moments of the workload process in standard $M/G/1$ queues was studied.

One can start with the two special models with arrival or service rates being a RBM as described above. The results on the RBM in [1,2] can be directly used for the service rate process. The transient behaviors of RBMs (with or without drift) in a bounded interval are not a difficult extension of [1,2]. The model with both arrival and service rates being a 2-d RBM in a wedge seems to be much more difficult. Transient behaviors of 2-d RBMs in a wedge are much harder than those in the 1-d case. Existing results for such RBMs are mostly on stationary distributions, see, e.g., [9,7,8]. Under certain conditions on the drift and reflections, the RBM is positive recurrent and has a unique product-of-exponential distributions [11]. It would be interesting to study similar transient properties in [1,2] for 2-d RBMs with drifts in a wedge. Such results will be in turn used to study the $M/M/1$ queues with 2-d RBM rates.

One classical approach is to use double transforms (using generating function and Laplace transform) of probability transition functions $P_t((n,z),(n',z'))$. The connection to the first-passage times and the relationships between the moment functions and probability transition functions as the building blocks for the analysis can be exploited as in [4]. Moreover, multidimensional Laplace transforms and their numerical inversions in [6] and other relevant numerical algorithms may be useful.

Finally, it would be also interesting to consider more general models, including reflected (jump) diffusions for the rates other than RBMs and queues with general arrival or service processes such as $M/G/1$ or $G/M/1$ queues.

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