Shot noise processes with randomly delayed cluster arrivals and dependent noises in the large intensity regime

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Abstract. We study shot noise processes with cluster arrivals, in which entities in each cluster may experience random delays (possibly correlated), and noises within each cluster may be correlated. We prove functional limit theorems for the process in the large intensity asymptotic regime, where the arrival rate gets large while the shot shape function, cluster sizes, delays and noises are unscaled. In the FCLT, the limit process is a continuous Gaussian process (assuming the arrival process satisfies an FCLT with a Brownian motion limit). We discuss the impact of the dependence among the random delays and among the noises within each cluster using several examples of dependent structures. We also study infinite-server queues with cluster/batch arrivals where customers in each batch may experience random delays before receiving service, with similar dependence structures.

1. introduction

We consider the shot noise process $X = \{X(t) : t \geq 0\}$ defined by

$$X(t) := \sum_{i=1}^{A(t)} \sum_{j=1}^{K_i} H(t - \tau_i - \xi_{ij}) Z_{ij}, \quad t \geq 0. \tag{1.1}$$

Here $A = \{A(t) : t \geq 0\}$ is a simple point process of clusters with event times $\{\tau_i : i \geq 1\}$, that is, $A(t) = \max\{k \geq 1 : \tau_k \leq t\}$ with $\tau_0 \equiv 0$. K_i represents the number of arrivals in cluster i. Entities of cluster i may arrive at a subsequent time of the cluster time τ_i, that is, at times $\tau_i + \xi_{ij}$, $j = 1, \dots, K_i$ with $\xi_{ij} \geq 0$. For each cluster i, the random delays $\{\xi_{ij} : j \in \mathbb{N}\}$ can be correlated and are allowed to be zero with a positive probability. The real-valued variables Z_{ij} represent the noises, and for each cluster i, $\{Z_{ij}, j \in \mathbb{N}\}$ may be correlated. For each cluster i, the variables K_i, $\{\xi_{ij}, j \in \mathbb{N}\}$ and $\{Z_{ij}, j \in \mathbb{N}\}$ are mutually independent, and they are also independent for different clusters. In addition, the cluster variables $(K_i, \{\xi_{ij}\}_j, \{Z_{ij}\}_j)$ are independent of the arrival process $A(t)$ (and the event times $\{\tau_i\}$). The function $H : \mathbb{R}_+ \to \mathbb{R}$ is the shot shape (response) function.

This model may be used to model financial markets with clustering events, insurance claims with cluster arrivals, and noise processes in some electronic components. For example, insurance claims may arrive in clusters possibly due to natural disasters or accidents, and the claims in each of the clusters may arrive after random delays. The claim sizes may be also dependent due to the clustering effect. Shot noise processes with cluster arrivals and various forms of the shot shape functions have been studied in [7, 31, 32, 1]. In these papers, people have investigated the general formula for the characteristic functional [7, 31], long range dependence under certain structural conditions on the response function [31], CLT results with normal limit distribution [32] and with stable limit distribution [1]. Similar results for shot noise processes without cluster arrivals were established in [33, 20, 21, 9, 34].

In this paper we establish a functional law of large numbers (FLLN) and a functional central limit theorem (FCLT) for the process $X(t)$ in (1.1) in the large intensity asymptotic regime, in which the arrival rate is large while the response function, the delay and noise variables are fixed (unscaled) and there is no scaling in time. Shot noise processes have been studied in this asymptotic regime

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in [3, 8, 10, 30, 27, 29]. In this regime, the limit processes in the FCLT are Gaussian processes of particular structure (assuming the arrival process satisfies an FCLT with a Brownian limit). This asymptotic regime is different from another commonly used scaling regime, in which both the time and space are scaled (noticing the scaling in time involves both $A(t)$ and $H(t-\cdot)$), and results in self-similar Gaussian processes and fractional Brownian motion limits [17, 19, 24] and stable motion limits [18, 11, 12, 13, 14, 15]. Among these scaling results, the models with renewal arrivals, referred to as random process with immigration at epochs of a renewal process, are studied in [12, 14, 15, 22, 23], and the models with an arbitrary point process are also recently studied in [6, 16].

In this asymptotic regime, we assume that the arrival process satisfies an FCLT as the arrival rate gets large (possibly time inhomogeneous), having a stochastic limit process with continuous paths (including Brownian motion, and other Gaussian processes). In the FLLN, we show that the limit is a deterministic function, which is not affected by the dependences between the random delay times, or by those between the noises, in each cluster. In the FCLT, we show that the limit process is composed of four mutually independent processes, one being an integral functional of the arrival limit process, and the other three being continuous Gaussian processes, capturing the variabilities of random delays, noises and clusters. We give a few examples to illustrate how the covariance functions of these Gaussian processes depend on the correlations. In the examples, we consider the random delays in the scenarios: (a) i.i.d., (b) as a sequence of the event times of a renewal counting process, (c) symmetrically correlated among the arrivals in each cluster, and (d) a discrete autoregressive process with order one (DAR(1)), and the noises in the scenarios of (a), (c) and (d) as the random delays.

We also study an infinite server queueing model with cluster (batch) arrivals, where customers in each batch may experience some random delay before receiving service. Using the same notations above ($Z_{ij} \geq 0$ representing service times), we may express the number of customers in service at time t as

$$X(t) := \sum_{i=1}^{A(t)} \sum_{j=1}^{K_i} \mathbf{1}(0 \leq t - \tau_i - \xi_{ij} < Z_{ij}), \quad t \geq 0. \tag{1.2}$$

In [25], heavy-traffic limits (FLLN and FCLT) were established for infinite-server queues with batch arrivals where service times within each batch may be correlated, as a consequence of infinite-server queues with weakly dependent service times in [26] (see also [28]). That approach cannot include (dependent) random delays for customers in each batch. In this model, we tackle the problem with random delays and allow dependence among random delays as well as among service times. We illustrate impact of the dependence among random delays as well as that among the service times in the scenarios discussed above. When the arrival process is stationary, we discuss the effect of the correlations upon the steady-state mean and variances of the limit process. These results have implications in comparing batch delays and customer delays in service systems (see, e.g., [38]).

1.1. **Organization of the paper.** We describe the model in detail and state the assumptions and main results in Section 2. We then give some examples in Section 3, and discuss the impact of dependence among random delays and among noises within each cluster. In Section 4, we state the results and examples for infinite-server queues with batch arrivals. The proofs are given in Section 5 and 6.

1.2. **Notation.** All random variables and processes are defined in a common complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}(\mathbb{R}_+)$ denotes the space of real (nonnegative) number. For $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Also $a^+ = a \vee 0$. Let $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ denote $\mathbb{R}$-valued function space of all càdlàg functions on $\mathbb{R}_+$. $(\mathbb{D}, J_1)$ denotes space $\mathbb{D}$ equipped with Skorohod J_1 topology, see [2, 36], which is complete and separable. Let $\mathbb{C}$ be subset of $\mathbb{D}$ for continuous functions. Notations $\to$ and $\Rightarrow$ mean convergence of real numbers and convergence in distribution, respectively. Let $m \in \mathbb{D}$ be a function of locally bounded variation. The Stieltjes integral with respect to dm is denoted by,

$$\int_a^b f(z)m(dz) = \int_{(a,b]} f(z)m(dz),$$

for every Borel measurable function f, which is the integral of f on $(a,b]$ and that $\int_a^b m(dz) = m(a,b] = m(b) - m(a)$ for any $a < b$. If the integral is on $[a,b]$, we write $\int_{a-}^b f(z)m(dz)$. For every $g \in \mathbb{D}$, the integral $\displaystyle\int_{(a,b]} m(b-z)g(dz)$ is defined by formal integration by parts

$$\begin{aligned}\int_{(a,b]} m(b-z)g(dz) :&= g(b)m(0-) - g(a)m\big((b-a)-\big) - \int_{(a,b]} g(z)m(b-dz),\\ &= g(b)m(0) - g(a)m(b-a) - \int_{[a,b)} g(z)m(b-dz) \\ &= g(b)m(0) - g(a)m(b-a) + \int_{(0,b-a]} g(b-z)m(dz).\end{aligned} \tag{1.3}$$

See, e.g., [35, pp. 206].

2. Model and Results

We consider a sequence of shot noise processes X^n indexed by n in the high intensity asymptotic regime, where the arrival rate of clusters gets large, in the order $O(n)$, while the distributions of the cluster sizes, random delays and noises as well as the shot shape function are fixed. Define the fluid-scaled process $\bar{X}^n := n^{-1}X^n$ and the diffusion-scaled process $\hat{X}^n := \sqrt{n}(\bar{X}^n - \bar{X})$ where $\bar{X}$ is the limit of $\bar{X}^n$.

We make the following assumptions on the input data.

Assumption 2.1. *Assume that $A^n(0) = 0$. There exist a deterministic, continuous and increasing function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous stochastic process $\hat{A}$ such that $\Lambda(0) = 0$,*

$$\hat{A}^n := n^{1/2}(\bar{A}^n - \Lambda) \Rightarrow \hat{A} \quad in \quad (\mathbb{D}, J_1) \quad as \quad n \to \infty,$$

where $\bar{A}^n = n^{-1}A^n$. This implies that $\bar{A}^n \Rightarrow \Lambda$ in $(\mathbb{D}, J_1)$ as $n \to \infty$.

Assumption 2.2. *Assume that the cluster variables $(K_i, \{\xi_{ij}\}_j, \{Z_{ij}\}_j)_i$ are independent of the arrival process A^n (and the associate event times $\{\tau_i^n\}$), and the variables K_i, $\{\xi_{ij}, j \in \mathbb{N}\}$ and $\{Z_{ij}, j \in \mathbb{N}\}$ are mutually independent for each cluster i, and also mutually independent across i. The cluster sizes K_i are i.i.d. with finite mean m_K and variance σ_K^2, and let $p_k = P(K_i = k)$ for $k \in \mathbb{N}$ such that $\sum_k p_k = 1$. For each cluster i, the random delays $\{\xi_{ij}, j \in \mathbb{N}\}$ may be dependent with a marginal distribution G_j and $\xi_{ij} \geq 0$ (we allow $\xi_{ij} = 0$ with a positive probability), and the noises $\{Z_{ij}, j \in \mathbb{N}\}$ are real-valued and may be also correlated with a common marginal distribution F. Assume that Z_{ij} have finite mean m_Z and variance σ_Z^2. In addition, assume that $\mathbb{E}[Z_{ij}^4] < \infty, \mathbb{E}[K_i^4] < \infty$.*

For notational brevity, we occasionally drop index i for the variables K_i, $\{\xi_{ij}, j \in \mathbb{N}\}$ and $\{Z_{ij}, j \in \mathbb{N}\}$, since they have a common joint law for each i.

Assumption 2.3. *Let $H:\mathbb{R}_+\to\mathbb{R}_+$ be a monotone function and $H(u)=0$ for $u<0$. For every fixed $T>0$, there exists $\gamma>\frac{1}{4}$ such that*

$$\sup_{0\le s<t\le T}\frac{|H(t)-H(s)|}{(t-s)^\gamma}<\infty,$$
$$\sup_{\substack{0\le s<t\le T\\(s,t]\cap\mathcal{L}_1=\emptyset}}\sup_j\frac{\mathbb{P}\big(\xi_j\in(s,t]\big)}{(t-s)^{2\gamma}}<\infty\quad\text{and}\quad\sup_{\substack{0\le s<r<t\le T\\(s,t]\cap\mathcal{L}_2=\emptyset}}\sup_{j,j'}\frac{\mathbb{P}\big(\xi_j\in(s,r],\xi_{j'}\in(r,t]\big)}{(t-s)^{4\gamma}}<\infty,\tag{2.1}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two sets of no accumulation point on $[0,\infty)$, with $\mathcal{L}_1$ being the collection of discontinuous points of the distribution function of ξ_j's.

Remark 2.1. *The fourth moments in Assumption 2.2 are needed in the proof of tightnesses where the increments moments are estimated, while the second moments are used in the convergence of the finite dimensional distribution. In Assumption 2.4, the marginal distributions of ξ_j's are assumed to be piecewise Hölder continuous on $\mathbb{R}_+$, but only a finite number of discontinuities in every finite interval is allowed. For the joint distributions of $(\xi_j,\xi_{j'})$, we impose the regularity condition concerning $\mathbb{P}\big(\xi_j\in(s,r],\xi_{j'}\in(r,t]\big)$ over $(s,r]$ and $(r,t]$ for $s<r<t$ in the third display, which is applied in (5.2) for the tightness proof in Lemmas 5.1 and 5.8. The set $\mathcal{L}_2$ is chosen such that the last inequality in (2.2) holds. We give an example to illustrate the sets $\mathcal{L}_1$ and $\mathcal{L}_2$. If $\big(\xi_j,\xi_{j'}\big)$ is discretely distributed on $\mathbb{R}_+^2$ with support $\{(x_{kjj'},y_{kjj'}),k\ge1\}_{j,j'}$ for $j,j'\in\mathbb{N}$, then $\mathcal{L}_1=\{x_{kjj'}|k,j,j'\ge1\}$ and $\mathcal{L}_2=\{x_{kjj'}|x_{kjj'}=y_{kjj'},k,j,j'\ge1\}$.*

In addition, we have assumed Hölder continuity for the function H; however, our main results, Theorem 2.1 and 2.2, will also hold if H has finitely many jumps on every $[0,T]$. That will require much heavier notations in the proofs, which we omit for brevity.

Assumption 2.4. *Let $H:\mathbb{R}_+\to\mathbb{R}_+$ be a monotone function and $H(u)=0$ for $u<0$. Assume that $\mathcal{L}\subset[0,\infty)$ is a set of no accumulation points, and for every fixed $T>0$, there exists $\gamma>\frac{1}{4}$ such that*

$$\sup_{0\le s<t\le T}\frac{|H(t)-H(s)|}{(t-s)^\gamma}<\infty,$$
$$\sup_{\substack{0\le s<t\le T\\(s,t]\cap\mathcal{L}=\emptyset}}\sup_j\frac{\mathbb{P}\big(\xi_j\in(s,t]\big)}{(t-s)^{2\gamma}}<\infty\quad\text{and}\quad\sup_{\substack{0\le s<r<t\le T\\(s,t]\cap\mathcal{L}=\emptyset}}\sup_{j,j'}\frac{\mathbb{P}\big(\xi_j\in(s,r],\xi_{j'}\in(r,t]\big)}{(t-s)^{4\gamma}}<\infty.\tag{2.2}$$

We remark that the fourth moments in Assumption 2.2 are needed in the proof of tightnesses where the increments moments are estimated, while the second moments are used in the convergence of the finite dimensional distribution. In Assumption 2.4, the functions are allowed to be piecewise Hölder continuous, but only a finite number of discontinuities in every finite time interval is allowed.

For all $u\in\mathbb{R}_+$, let

$$h_j(u):=\mathbb{E}\big[H(u-\xi_j)\big]\quad\text{and}\quad h(u):=\mathbb{E}\Big[\sum_{j=1}^K H(u-\xi_j)\Big]=\sum_{j\ge1}h_j(u)\mathbb{P}(K\ge j).\tag{2.3}$$

The second equality in $h(u)$ is due to the independence between K_i and $\{\xi_{ij},j\in\mathbb{N}\}$.

Remark 2.2. *By definition, $h_j(z)=h(z)=0$ for $z<0$ and $h_j(0)=H(0)\mathbb{P}(\xi_j=0)$, $h(0)=\sum_{j\ge1}h_j(0)\mathbb{P}(K\ge j)$, h_j,h may fail to be continuous at 0. Recall that we allow the random delays to take zero values with a positive probability. Note that h_j is independent of cluster i and $h\in\mathbb{D}$ is monotone on $[0,\infty)$ under Assumption 2.4, which will be proved in Lemma 5.3.*

Theorem 2.1. *Under Assumptions 2.2 and 2.4, and assuming that $\bar{A}^n\Rightarrow\bar{A}$ in $\mathbb{D}$ as $n\to\infty$,*

$$\bar{X}^n\Rightarrow\bar{X}\quad\text{in}\quad\mathbb{D}\quad\text{as}\quad n\to\infty,\tag{2.4}$$

where the limit $\bar{X}$ is a deterministic function given by

$$\bar{X}(t) = m_Z \int_0^t h(t-s)\Lambda(ds), \quad t \geq 0. \tag{2.5}$$

We remark that the dependence among the noises $\{Z_{ij}\}_j$ and that among the random delays $\{\xi_{ij}\}_j$ do not affect the fluid limit, which only depend on the marginal distribution of ξ_{ij} and the mean of Z_{ij}.

We next state the FCLT for the diffusion-scaled process $\hat{X}^n$. We first introduce some notations. Let

$$\varrho_{ij} := Z_{ij} - \mathbb{E}[Z_{ij}], \quad \varsigma_{ij}(u) := H(u-\xi_{ij}) - h_j(u), \quad \vartheta_i(u) := \sum_{j=1}^{K_i} \big(h_j(u) - h(u)\big), \quad u \in \mathbb{R}_+.$$

Again for notational convenience, we sometimes drop the index i in ϱ_{ij} and ς_{ij}. Define the following quantities:

$$\begin{aligned}
r_2(t,s) &= \mathbb{E}\Big[\sum_{j,j'}^{K} \varsigma_j(t)\varsigma_{j'}(s)\Big] \quad \text{and} \quad R_2(t,s) = \int_0^{t\wedge s} r_2(t-u, s-u)\Lambda(du), \\
r_3(t,s) &= \mathbb{E}\Big[\sum_{j,j'}^{K} h_j(t)h_{j'}(s)\Big] - h(t)h(s) \quad \text{and} \quad R_3(t,s) = \int_0^{t\wedge s} r_3(t-u, s-u)\Lambda(du), \\
r_4(t,s) &= \mathbb{E}\Big[\sum_{j,j'}^{K} \varrho_j\varrho_{j'} H(t-\xi_j)H(s-\xi_{j'})\Big] \quad \text{and} \quad R_4(t,s) = \int_0^{t\wedge s} r_4(t-u, s-u)\Lambda(du).
\end{aligned} \tag{2.6}$$

Theorem 2.2. *Under Assumptions 2.1, 2.2 and 2.4,*

$$\hat{X}^n \Rightarrow \hat{X} \quad \text{in} \quad (\mathbb{D}, J_1) \quad \text{as} \quad n \to \infty, \tag{2.7}$$

where the limit $\hat{X} = m_Z\big(\hat{X}_1(t) + \hat{X}_2(t) + \hat{X}_3(t)\big) + \hat{X}_4(t)$, a sum of four mutually independent processes defined as follows:

$$\hat{X}_1(t) = \int_{(0,t]} h(t-s)\hat{A}(ds) = \hat{A}(t)h(0) - \int_{[0,t)} \hat{A}(s)h(t-ds), \tag{2.8}$$

and $\hat{X}_\ell, \ell = 2,3,4$, are continuous Gaussian processes with covariance functions $R_\ell, \ell = 2,3,4$, defined in (2.6).

Remark 2.3. *If the limit of the diffusion-scaled process is a Brownian motion (BM), that is, $\hat{A} = \sqrt{\lambda c_a^2} B_a$ where λ is the arrival rate (i.e., $\Lambda(t) = \lambda t$ for $t \geq 0$), c_a^2 is a variability parameter and B_a is a standard BM, then*

$$\hat{X}_1(t) = \sqrt{\lambda} c_a \int_0^t h(t-s)dB_a(s), \quad t \geq 0.$$

In particular, if the arrival process $A(t)$ is renewal with interarrival times of mean λ^{-1} and variance σ_a^2, then $c_a^2 = \sigma_a^2/(\lambda^{-1})^2 = \lambda^2\sigma_a^2$ represents the squared coefficient of variation (SCV) of the interarrival times. In general, c_a^2 indicates the variabilities in the arrival process. With non-stationary arrival rates, the limit can be $\hat{A}(t) = c_a B_a(\Lambda(t))$, which gives

$$\hat{X}_1(t) = c_a \int_0^t h(t-s)dB_a(\Lambda(s)), \quad t \geq 0.$$

3. Examples

In this section, we discuss some special cases of the model, including several dependence structures among random delays as well as among the noises.

Assumption 3.1. *The random delay times of each cluster $\{\xi_{ij}\}_j$ satisfy either of the four conditions:*

(a) *ξ_{ij}'s are i.i.d. with c.d.f. G;*

(b) *for each cluster i, $\xi_{ij} = \sum_{\ell=1}^{j} \zeta_{i\ell}$ where $\{\zeta_{i\ell} : \ell \in \mathbb{N}\}$ are i.i.d. with c.d.f. G_ζ. Let $G_\zeta^{(l)}$ be the l-fold convolution of G_ζ;*

(c) *for each cluster i, the sequence $\{\xi_{ij} : j = 1, \dots, K_i\}$ is symmetrically correlated, that is, each pair has a common joint distribution Ψ whose correlation is equal to ρ_ξ, and each ξ_{ij} has the marginal c.d.f. G;*

(d) *for each cluster i, the sequence $\{\xi_{ij} : j = 1, \dots, K_i\}$ is generated by a first-order discrete autoregressive process, referred to as DAR(1) process. Specifically, let $\xi_{ij} = \delta_{i,j-1}\xi_{i,j-1} + (1-\delta_{i,j-1})\eta_{ij}$. Here $\{\delta_{ij} : j \in \mathbb{N}\}$ is a sequence of i.i.d. Bernoulli random variables with $\mathbb{P}(\delta_{ij} = 1) = \alpha \in (0, 1)$. $\{\eta_{ij} : j \in \mathbb{N}\}$, independent of $\{\delta_{ij}\}$ and $\xi_{i1} \sim G$, is a sequence of i.i.d. random variables with c.d.f. G.*

We also consider the following two conditions which are variations of (a) and (b) above:

(a') *for each cluster i, $\xi_{i1} = 0$ and ξ_{ij} for $j \geq 2$ are i.i.d. with c.d.f. G;*

(b') *for each cluster i, $\xi_{i1} = 0$ and $\xi_{ij} = \sum_{\ell=2}^{j} \zeta_{i\ell}$ for $j \geq 2$, where $\{\zeta_{i\ell} : \ell \in \mathbb{N}\}$ are i.i.d. with c.d.f. G_ζ. Let $G_\zeta^{(l)}$ be the l-fold convolution of G_ζ.*

Noticing that, a special case of Assumption 3.1(c) is given by

$$\Psi(u, v) = \rho_\xi G(u \wedge v) + (1-\rho_\xi) G(u) G(v), \quad \text{for all } u, v \geq 0 \tag{3.1}$$

for the marginal G and correlation $\rho_\xi \in [0, 1]$ of $\Psi(u, v)$, see [37], under which

$$\mathbb{P}\big(\xi_j \in du, \xi_{j'} \in dv\big) = \rho_\xi G(du)\delta_{\{u\}}(dv) + (1-\rho_\xi) G(du) G(dv),$$

and $\mathrm{Corr}(\xi_j, \xi_{j'}) = \rho_\xi$, where $\delta_{\{u\}}$ is the Dirac measure at $\{u\}$. Moreover, under Assumption 3.1(d)

$$\mathbb{P}\big(\xi_j \leq u, \xi_{j'} \leq v\big) = \alpha^{|j-j'|} G(u \wedge v) + \big(1-\alpha^{|j-j'|}\big) G(u) G(v), \quad \text{for all } u, v \geq 0$$

thus, $\mathrm{Corr}(\xi_j, \xi_{j'}) = \alpha^{|j-j'|}$ for all $j, j' \geq 1$, namely, the correlation between ξ_j and $\xi_{j'}$ decreases geometrically in the distance $|j-j'|$. Note that Assumption 3.1(d), the sequence $\{\xi_{ij} : j = 1, \dots, K_i\}$ satisfies the ρ-mixing and ϕ-mixing conditions.

We first discuss the limit $\bar{X}$ under Assumption 3.1.

Under Assumptions 3.1(a),

$$\bar{X}(t) = m_K m_Z \int_0^t \int_{0-}^{t-s} H(t-s-u) G(du) \Lambda(ds).$$

Under Assumptions 3.1(a'),

$$\bar{X}(t) = m_Z \int_0^t \Big(H(t-s) + (m_K - 1)\int_{0-}^{t-s} H(t-s-u) G(du)\Big) \Lambda(ds).$$

Under Assumptions 3.1(b),

$$\bar{X}(t) = m_Z \sum_{k=1}^{\infty} p_k \sum_{l=1}^{k} \int_0^t \int_{0-}^{t-s} H(t-s-u) G_\zeta^{(l)}(du) \Lambda(ds).$$

Under Assumptions 3.1(b'),

$$\bar{X}(t) = m_Z \int_0^t \Big(H(t-s) + \sum_{k=2}^{\infty} p_k \sum_{l=1}^{k-1} \int_{0-}^{t-s} H(t-s-u) G_\zeta^{(l)}(du)\Big) \Lambda(ds).$$

We remark that in the cases (a') and (b'), it can have an arbitrary number of arrivals (less than the cluster size) at the event time of the cluster τ_i^n. For example, if there are ℓ entities of the cluster without delay, assuming $\ell \le K_i$ almost surely, then similar to the case under Assumptions 3.1(a'), we have

$$\bar{X}(t) = m_Z \int_0^t \Big(\ell H(t-s) + (m_K - \ell)\int_{0-}^{t-s} H(t-s-u)G(du)\Big)\Lambda(ds).$$

In the extreme case where all entities of the cluster arriving at the same time as the cluster arrival time (that is, without delay), we have

$$\bar{X}(t) = m_K m_Z \int_0^t H(t-s)\Lambda(ds).$$

Now under Assumptions 3.1(c) and (d), we have the same formula as in the case (a); the correlation does not affect the fluid limit, but it does affect the covariance function as we show below.

We next give the examples on the covariance functions in the various cases.

3.1. **i.i.d. noises.** Under Assumption 3.1(a), we have

$$\begin{aligned}
r_2(t,s) &= m_K\bigg(\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du) - h_1(t)h_1(s)\bigg),\\
r_3(t,s) &= \sigma_K^2 h_1(t)h_1(s),\\
r_4(t,s) &= m_K\sigma_Z^2\mathbb{E}[H(t-\xi_1)H(s-\xi_1)] = m_K\sigma_Z^2\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du),
\end{aligned}$$

where $h_1(t) = \int_{0-}^t H(t-u)G(du)$. Under Assumptions 3.1(a'), we have

$$\begin{aligned}
r_2(t,s) &= (m_K-1)\bigg(\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du) - h_1(t)h_1(s)\bigg),\\
r_3(t,s) &= \sigma_K^2 h_1(t)h_1(s),\\
r_4(t,s) &= \sigma_Z^2 H(t)H(s) + (m_K-1)\sigma_Z^2\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du).
\end{aligned}$$

As mentioned above in the fluid limit, we can also allow an arbitrary number of entities in the cluster to arrive without delay. In the extreme case of all entities without delay, we have

$$r_2(t,s) = 0, \quad r_3(t,s) = \sigma_Z^2 H(t)H(s) \quad \text{and} \quad r_4(t,s) = m_K\sigma_Z^2 H(t)H(s).$$

Under Assumption 3.1(b), we have

$$\begin{aligned}
r_2(t,s) = &\sum_{j\ge1}\sum_{k\ge j} p_k\Big(\int_{0-}^{t\wedge s} H(s-u)H(t-u)G_\zeta^{(j)}(du) - h_j(t)h_j(s)\Big)\\
&+\sum_{j\ge1}\sum_{j'\ge1}\sum_{k\ge j+j'} p_k\Big(\int_{0-}^{t\wedge s}\int_{0-}^{t-u} H(s-u)H(t-u-v)G_\zeta^{(j)}(du)G_\zeta^{(j')}(dv) - h_j(s)h_{j+j'}(t)\Big)\\
&+\sum_{j\ge1j}\sum_{j'\ge1}\sum_{k\ge j+j'} p_k\Big(\int_{0-}^{t\wedge s}\int_{0-}^{s-u} H(t-u)H(s-u-v)G_\zeta^{(j)}(du)G_\zeta^{(j')}(dv) - h_j(t)h_{j+j'}(s)\Big),\\
r_3(t,s) = &\sum_{j,j'\ge1} h_j(t)h_{j'}(s)\mathbb{P}\big(K\ge j\vee j'\big) - h(t)h(s)\\
r_4(t,s) = &\,\sigma_Z^2\sum_{j\ge1}\sum_{k\ge j} p_k\int_0^{t\wedge s} H(t-u)H(s-u)G_\zeta^{(j)}(du),
\end{aligned}$$

where $h_j(t) = \int_{0-}^t H(t-u)G_\zeta^{(j)}(du)$.

Under Assumption 3.1(b'), we have

$$
\begin{aligned}
r_2(t,s) &= \sum_{j\geq 1}\sum_{k\geq j} p_{k+1}\Big(\int_{0-}^{t\wedge s} H(s-u)H(t-u)G_\zeta^{(j)}(du) - h_j(s)h_j(t)\Big)\\
&\quad + \sum_{j\geq 1}\sum_{j'\geq 1}\sum_{k\geq j+j'} p_{k+1}\Big(\int_{0-}^{t\wedge s}\int_{0-}^{t-u} H(s-u)H(t-u-v)G_\zeta^{(j)}(du)G_\zeta^{(j')}(dv) - h_j(s)h_{j+j'}(t)\Big)\\
&\quad + \sum_{j\geq 1}\sum_{j'\geq 1}\sum_{k\geq j+j'} p_{k+1}\Big(\int_{0-}^{t\wedge s}\int_{0-}^{s-u} H(t-u)H(s-u-v)G_\zeta^{(j)}(du)G_\zeta^{(j')}(dv) - h_j(t)h_{j+j'}(s)\Big)\\
r_3(t,s) &= \sum_{j,j'\geq 1} h_j(t)h_{j'}(s)\Big(\mathbb{P}\big(K\geq (j+1)\vee(j'+1)\big) - \mathbb{P}(K\geq j+1)\mathbb{P}(K\geq j'+1)\Big),\\
r_4(t,s) &= \sigma_Z^2 H(t)H(s) + \sigma_Z^2 \sum_{j\geq 1}\sum_{k\geq j} p_{k+1}\int_0^{t\wedge s} H(t-u)H(s-u)G_\zeta^{(j)}(du).
\end{aligned}
$$

We next consider the cases where the random delays are correlated in 3.1(c) and (d). The dependence only affect the function $r_2(t,s)$ while the functions $r_3(t,s)$ and $r_4(t,s)$ are the same as in the case of 3.1(a), so we only present the formula for $r_2(t,s)$.

Under Assumption 3.1(c), we have

$$
\begin{aligned}
r_2(t,s) &= m_K\Big(\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du) - h_1(t)h_1(s)\Big)\\
&\quad + \mathbb{E}[K(K-1)]\Big(\int_{0-}^{t}\int_{0-}^{s} H(t-u)H(s-v)\big(\Psi(du,dv) - G(du)G(dv)\big)\Big).
\end{aligned}
$$

If $\Psi(u,v)$ is approximated by $\tilde{\Psi}(u,v)$ defined in (3.1), we can approximate $r_2(t,s)$ by

$$
\tilde{r}_2(t,s) = \big(m_K(1-\rho_\xi) + \rho_\xi\mathbb{E}[K^2]\big)\Big(\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du) - h_1(t)h_1(s)\Big). \tag{3.2}
$$

This approximate function is linear in the correlation parameter ρ_ξ.

Under Assumption 3.1(d), we have

$$
\mathbb{E}[\varsigma_j(t)\varsigma_{j'}(s)] = \big(\mathbb{E}\big[H(t-\xi_1)H(s-\xi_1)\big] - h_1(t)h_1(s)\big)\alpha^{|j-j'|} \quad \text{for all } j,j'.
$$

Thus,

$$
\begin{aligned}
r_2(t,s) &= \Big(\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du) - h_1(t)h_1(s)\Big)\mathbb{E}\Big[\sum_{j,j'}^{K}\alpha^{|j-j'|}\Big]\\
&= \Big(\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du) - h_1(t)h_1(s)\Big)\Big(\frac{m_K(1+\alpha)}{1-\alpha} + \frac{2\alpha\big(\mathbb{E}[\alpha^K]-1\big)}{(1-\alpha)^2}\Big).
\end{aligned}
$$

It is clear that the function $r_2(t,s)$ is increasing (decreasing) nonlinearly in α if the quantity in the first parenthesis is positive (negative).

3.2. Correlated noises.

Assumption 3.2. *For each cluster i, $\{Z_{ij}, j\in\mathbb{N}\}$ satisfies either of the following conditions:*

- (a) *$\{Z_{ij}, j\in\mathbb{N}\}$ are symmetrically correlated with a common c.d.f. F and a bivariate joint distribution for each pair Φ.*
- (b) *$\{Z_{ij}, j\in\mathbb{N}\}$ is a DAR(1) sequence as in Assumption 3.1, with Bernoulli parameter $\beta\in(0,1)$ and marginal c.d.f. F.*

Note that the correlations in noises only affect the function $r_4(t,s)$. We present its formula in the following cases.

Under Assumptions 3.1(a) and 3.2(a), we have

$$r_4(t,s) = m_K\sigma_Z^2\int_{0-}^{t} H(t-u)H(s-u)G(du) + \mathbb{E}\big[K^2-K\big]\mathrm{Cov}(Z_1,Z_2)h_1(t)h_1(s). \tag{3.3}$$

Under Assumptions 3.1(a) and 3.2(b), we have

$$r_4(t,s) = m_K\sigma_Z^2\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du) + h_1(t)h_1(s)\sigma_Z^2\Big(\frac{m_K 2\beta}{1-\beta} + \frac{2\beta\big(\mathbb{E}[\beta^K]-1\big)}{(1-\beta)^2}\Big). \tag{3.4}$$

Under Assumptions 3.1(c) and 3.2(a), we have

$$\begin{aligned} r_4(t,s) = {} & m_K\sigma_Z^2\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du) \\ & + \mathbb{E}\big[K^2-K\big]\mathrm{Cov}(Z_1,Z_2)\int_{0-}^{t}\int_{0-}^{s} H(t-u)H(s-v)\Psi(du,dv). \end{aligned}$$

Similar to (3.2), under Assumption 3.1(c), let ρ_Z be the correlation between Z_j and $Z_{j'}$, and Φ be approximated by the following $\tilde{\Phi}$:

$$\tilde{\Phi}(z_1,z_2) = \rho_Z F(z_1\wedge z_2) - (1-\rho_Z)F(z_1)F(z_2).$$

We can approximate r_4 by $\tilde{r}_4$ given by

$$\begin{aligned} \tilde{r}_4(t,s) = {} & \Big(m_K + \rho_\xi\rho_Z\mathbb{E}[K^2-K]\Big)\sigma_Z^2\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du) \\ & + (1-\rho_\xi)\rho_Z\mathbb{E}[K^2-K]\sigma_Z^2 h_1(t)h_1(s). \end{aligned} \tag{3.5}$$

It is clear that when $\rho_\xi = 0$, this formula reduces to (3.3) under Assumptions 3.1(a) and 3.2(a).

Under Assumptions 3.1(d) and 3.2(b), we have

$$\begin{aligned} r_4(t,s) = {} & \sigma_Z^2\int_{0-}^{t\wedge s} H(t-u)H(s-u)G(du)\Big(\frac{m_K(1+\alpha\beta)}{1-\alpha\beta} + \frac{2\alpha\beta\big(\mathbb{E}[(\alpha\beta)^K-1]\big)}{(1-\alpha\beta)^2}\Big) \\ & + \sigma_Z^2 h_1(t)h_1(s)\Big(\frac{2m_K\beta(1-\alpha)}{(1-\beta)(1-\alpha\beta)} + \frac{2\beta\big(\mathbb{E}[\beta^K]-1\big)}{(1-\beta)^2} - \frac{2\alpha\beta\big(\mathbb{E}[(\alpha\beta)^K]-1\big)}{(1-\alpha\beta)^2}\Big). \end{aligned}$$

It is also clear that when $\alpha = 0$, this formula reduces to (3.4) under Assumptions 3.1(a) and 3.2(b).

Remark 3.1. *The process X in (1.1) can be used to model the total claims in insurance, where $A(t)$ is the arrival process of cluster claims, K_i's are the cluster sizes, the variables $\{Z_{ij}\}_j$ are the claim sizes for each cluster, and $\{\xi_{ij}\}_j$ are the delays for the claims to arrive in each cluster. See, for example, the book by Daley and Vere-Jones [4] and the recent work in [1] and references therein. When the arrival process A is Poisson, under the i.i.d. conditions on the claim sizes and delays, the distribution of X can be characterized, using the probability generating or characteristic functionals [4, 31]. In [1, 32], CLTs with Gaussian and infinite-variance stable limits are proved and used to approximate the total claim distributions as $t\to\infty$. Our results provide distributional approximations for the total claim size at each time t when the arrival rate of cluster claims is large, which are valid for any general non-stationary arrival processes, as well as for various scenarios of correlated claims and delays discussed above. For instance, given that the arrival process results in a BM limit as in Remark 2.3, the total claim $X(t)$ at each time t can be approximated by a Gaussian process with mean $\bar{X}(t)$ in (2.5), and covariance functions $\sum_{i=1}^4 r_i(t,s)$, where $r_1(t,s) = c_a^2\int_0^{t\wedge s} h(t-u)h(s-u)d\Lambda(u)$ and r_2,r_3,r_4 are given as above in the various scenarios. Then, one can approximate the corresponding ruin probability (the first-passage-time or hitting time* *of the total claim), by exploiting the computation of the hitting times for Gaussian processes (see, e.g., [5]).*

4. Infinite-server queues with cluster arrivals and random delays

We consider infinite-server queues with batch/cluster arrivals where the arrivals in each cluster may experience random delays. Let $A(t)$ be the arrival process of batches/clusters, and K_i be the batch/cluster size of cluster i. For each cluster i, ξ_{ij}, $j = 1, \dots, K_i$, are the random delays and Z_{ij}, $j = 1, \dots, K_i$, are the corresponding service times. Note that in Assumption 2.2, the noises can take any real values, but in the queueing setting, the service times must be positive. Let $X(t)$ be the number of customers in service at time t. Then it has the representation in (1.2), that is,

$$X^n(t) = \sum_{i=1}^{A^n(t)} \sum_{j=1}^{K_i} \mathbf{1}(\tau_i^n + \xi_{ij} \le t < \tau_i^n + \xi_{ij} + Z_{ij}).$$

We make the following regularity conditions instead of Assumption 2.4, and impose the same conditions in Assumptions 2.1 and 2.2.

Assumption 4.1. *For every fixed $T > 0$, there exists $\gamma > \frac{1}{4}$ such that*

$$\sup_{\substack{0 \le s < t < T \\ (s,t] \cap \mathcal{L}_1 = \emptyset}} \sup_j \frac{\mathbb{P}\big(\xi_j \in (s,t]\big)}{(t-s)^{2\gamma}} < \infty \quad \text{and} \quad \sup_{\substack{0 \le s < r < t < T \\ (s,t] \cap \mathcal{L}_2 = \emptyset}} \sup_{j,j'} \frac{\mathbb{P}\big(\xi_j \in (s,r], \xi_{j'} \in (r,t]\big)}{(t-s)^{4\gamma}} < \infty,$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the sets of no accumulation point on $\mathbb{R}_+$ as in Assumption 2.4. In addition,

$$\sup_{0 \le s < t \le T} \sup_j \frac{\mathbb{P}(Z_j \in (s,t])}{(t-s)^{2\gamma}} < \infty,$$

and for some $\mathcal{L}_3 \subset \mathbb{R}_+$ of no accumulation point,

$$\sup_{\substack{0 \le s < r < t \le T \\ (s,t] \cap \mathcal{L}_3 = \emptyset}} \sup_{j,j'} \frac{\mathbb{P}\big(Z_j + \xi_j \in (s,r], Z_{j'} + \xi_{j'} \in (r,t]\big)}{(t-s)^{4\gamma}} < \infty. \tag{4.1}$$

Remark 4.1. *In addition to the conditions on ξ_j's in Assumption 2.4, we further assume that the marginal distributions of Z_j's are locally Hölder continuous (and thus, the joint distribution of $(Z_j, Z_{j'})$ is continuous). The condition in (4.1) is the regularity condition imposed upon the joint distributions of $(Z_j + \xi_j, Z_{j'} + \xi_{j'})$ concerning $\mathbb{P}\big(Z_j + \xi_j \in (s,r], Z_{j'} + \xi_{j'} \in (r,t]\big)$ over the intervals $(s,r]$ and $(r,t]$ for $s < r < t$, which is applied in (6.1) for the proof of tightness. If the joint distribution $(Z_j, Z_{j'})$ is itself locally Hölder continuous, that is,*

$$\sup_{\substack{0 \le s < t \le T \\ 0 \le v < u \le T}} \sup_{j \ne j'} \frac{\mathbb{P}\big(Z_j \in (s,t], Z_{j'} \in (v,u]\big)}{(t-s)^{2\gamma}(u-v)^{2\gamma}} < \infty,$$

then $\mathcal{L}_3 = \emptyset$. Note that the joint distributions of $(Z_j, Z_{j'})$ and $(Z_j + \xi_j, Z_{j'} + \xi_{j'})$ are continuous on $\mathbb{R}_+^2$. However, the second and third conditions do not imply the condition in (4.1). (It is believable that the results in this section also hold for Z_j having a discontinuous distribution function, as for H in the cluster model, which would require introducing additional notations in the proof; hence we omit that for brevity.)

We define for $u \ge 0$,

$$\tilde{H}_j(u) = \mathbb{P}(Z_j > u), \quad \tilde{h}_j(u) = \mathbb{P}\big(0 \le u - \xi_j < Z_j\big), \quad \tilde{h}(u) = \mathbb{E}\Big[\sum_{j=1}^{K} \tilde{h}_j(u)\Big], \tag{4.2}$$

and $\tilde{H}_j(u) = \tilde{h}_j(u) = \tilde{h}(u) = 0$ for $u < 0$.

Theorem 4.1. *Under Assumptions 2.2 and 4.1, and assuming that $\bar{A}^n \Rightarrow \bar{A}$ in $\mathbb{D}$ as $n \to \infty$, then (2.4) holds with the limit $\bar{X}(t)$ using $\tilde{h}(u)$ in (4.2).*

We next state the FCLT for the diffusion-scaled process $\hat{X}^n$. We first introduce some notations. For $u \in \mathbb{R}$, let

$$\tilde{\varsigma}_{ij}(u) := \tilde{H}_j(u - \xi_{ij}) - \tilde{h}_j(u), \quad \tilde{\vartheta}_i(u) := \sum_{j=1}^{K_i} \tilde{h}_j(u) - \tilde{h}(u),$$
$$\tilde{\varrho}_{ij}(u) := \mathbf{1}\big(0 \le u - \xi_{ij} < Z_{ij}\big) - \tilde{H}_j(u - \xi_{ij}).$$

Again for notational convenience, we occasionally drop the index i in $\tilde{\varrho}_{ij}$, $\tilde{\varsigma}_{ij}$ and $\tilde{\vartheta}_i$. Define the following quantities:

$$\begin{aligned} r_2(t,s) &= \mathbb{E}\Big[\sum_{j,j'}^{K} \tilde{\varsigma}_j(t)\tilde{\varsigma}_{j'}(s)\Big] \quad \text{and} \quad R_2(t,s) = \int_0^{t\wedge s} r_2(t-u, s-u)\Lambda(du), \\ r_3(t,s) &= \mathbb{E}\Big[\sum_{j,j'}^{K} \tilde{h}_j(t)\tilde{h}_{j'}(s)\Big] - \tilde{h}(t)\tilde{h}(s) \quad \text{and} \quad R_3(t,s) = \int_0^{t\wedge s} r_3(t-u, s-u)\Lambda(du), \\ r_4(t,s) &= \mathbb{E}\Big[\sum_{j,j'}^{K} \tilde{\varrho}_j(t)\tilde{\varrho}_{j'}(s)\Big] \quad \text{and} \quad R_4(t,s) = \int_0^{t\wedge s} r_4(t-u, s-u)\Lambda(du). \end{aligned} \tag{4.3}$$

Theorem 4.2. *Under Assumptions 2.1, 2.2 and 4.1, the convergence in (2.7) holds where the limit $\hat{X} = \sum_{\ell=1}^4 \hat{X}_\ell$, a sum of mutually independent processes, $\hat{X}_1(t)$ is the same as that in Theorem 2.2, and $\hat{X}_\ell, \ell = 2,3,4$, are continuous Gaussian processes with covariance functions $R_\ell, \ell = 2,3,4$, defined in (4.3).*

4.1. **Examples.**

In this section we give explicit expressions for the fluid limit and the functions r_ℓ in the covariance functions, under Assumptions 3.1 and 3.2. Note that except the renewal random delays in Assumption 3.1(b), all the combinations of the cases of random delays and service times, can be regarded as a tandem infinite-server queue with two service stations, where the random delays $\{\xi_{ij}\}$ are the service times in the first station and the service times $\{Z_{ij}\}$ are those for the second station. These are interesting examples themselves, since tandem $G/G/\infty - G/\infty$ queues with correlated service times in each service station have not been studied in the literature.

Let $F^c = 1 - F$. With i.i.d. random delays, under Assumption 3.1(a),

$$\bar{X}(t) = m_K \int_0^t \Big(\int_{0-}^{t-s} F^c(t-s-u)G(du) \Big) \Lambda(ds),$$

and under Assumption 3.1(a'),

$$\bar{X}(t) = \int_0^t \Big(F^c(t-s) + (m_K - 1) \int_{0-}^{t-s} F^c(t-s-u)G(du) \Big) \Lambda(ds).$$

With renewal random delays, under Assumption 3.1(b),

$$\bar{X}(t) = \int_0^t \sum_{k=1}^{\infty} p_k \sum_{l=1}^{k} \int_{0-}^{t-s} F^c(t-s-u)G^{(l)}(du)\Lambda(ds),$$

and under Assumptions 3.1(b'),

$$\bar{X}(t) = \int_0^t \Big(F^c(t-s) + \sum_{k=2}^{\infty} p_k \sum_{l=1}^{k-1} \int_{0-}^{t-s} F^c(t-s-u)G^{(l)}(du) \Big) \Lambda(ds).$$

Again, under Assumptions 3.1(c) and (d), the fluid limit $\bar{X}$ is the same as the case (a). The dependence among random delays does not affect the fluid limit.

We next give some examples of the covariance functions under Assumptions 3.1 and 3.2.

4.1.1. *i.i.d. service times.*

Under Assumption 3.1(a), we have

$$\begin{aligned} r_2(t,s) &= m_K\Big(\int_{0-}^{t\wedge s} F^c(t-u)F^c(s-u)G(du) - \tilde{h}_1(t)\tilde{h}_1(s)\Big) \\ r_3(t,s) &= \sigma_K^2\tilde{h}_1(t)\tilde{h}_1(s) \\ r_4(t,s) &= m_K\int_{0-}^{t\wedge s}\big(F^c(t\vee s-u) - F^c(t-u)F^c(s-u)\big)G(du), \end{aligned}$$

where

$$\tilde{h}_1(u) = \int_{0-}^{u} F^c(u-v)G(dv). \tag{4.4}$$

Under Assumption 3.1(a'), we have the same $r_3(t,s)$ above, and

$$\begin{aligned} r_2(t,s) &= (m_K-1)\Big(\int_{0-}^{t\wedge s} F^c(t-u)F^c(s-u)G(du) - \tilde{h}_1(t)\tilde{h}_1(s)\Big) \\ r_4(t,s) &= \big(F^c(t\vee s) - F^c(t)F^c(s)\big) + (m_K-1)\int_{0-}^{t\wedge s}\big(F^c(t\vee s-u) - F^c(t-u)F^c(s-u)\big)G(du). \end{aligned}$$

Under Assumption 3.1(b), we have

$$\begin{aligned} r_2(t,s) &= \sum_{l\geq 1}\sum_{k\geq l} p_k\Big(\int_{0-}^{t\wedge s} F^c(t-u)F^c(s-u)G^{(l)}(du) - \tilde{h}_l(t)\tilde{h}_l(s)\Big) \\ &\quad + \sum_{l,l'\geq 1}\sum_{k\geq l+l'} p_k\Big(\int_{0-}^{t\wedge s}\int_{0-}^{s} F^c(t-u)F^c(s-u-v)G^{(l)}(du)G^{(l')}(dv) - \tilde{h}_l(t)\tilde{h}_{l+l'}(s)\Big) \\ &\quad + \sum_{l,l'\geq 1}\sum_{k\geq l+l'} p_k\Big(\int_{0-}^{t\wedge s}\int_{0-}^{t} F^c(s-u)F^c(t-u-v)G^{(l)}(du)G^{(l')}(dv) - \tilde{h}_l(s)\tilde{h}_{l+l'}(t)\Big) \\ r_3(t,s) &= \sum_{l,l'\geq 1}\tilde{h}_l(t)\tilde{h}_{l'}(s)\big(\mathbb{P}(K\geq l\vee l') - \mathbb{P}(K\geq l)\mathbb{P}(K\geq l')\big) \\ r_4(t,s) &= \sum_{l\geq 1}\sum_{k\geq l} p_k\Big(\int_{0-}^{t\wedge s}\big(F^c(t\vee s-u) - F^c(t-u)F^c(s-u)\big)G^{(l)}(du)\Big), \end{aligned}$$

where $\tilde{h}_l(u) = \int_{0-}^{u} F^c(u-v)G^{(l)}(dv)$.

Under Assumption 3.1(c) and (d), the correlations in the random delays only affect the function $r_2(t,s)$, while the functions $r_3(t,s)$ and $r_4(t,s)$ remain the same as those in the i.i.d. case in Assumption 3.1(a). So we state the function $r_2(t,s)$ in these two scenarios. Under Assumption 3.1(c), we have

$$\begin{aligned} r_2(t,s) &= m_K\Big(\int_{0-}^{t\wedge s} F^c(t-u)F^c(s-u)G(du) - \tilde{h}_1(t)\tilde{h}_1(s)\Big) \\ &\qquad + \mathbb{E}[K^2-K]\Big(\int_{0-}^{t}\int_{0-}^{s} F^c(t-u)F^c(s-v)\big(\Psi(du,dv) - G(du)G(dv)\big)\Big), \end{aligned}$$

where $\tilde{h}_1(u)$ is defined in (4.4). If $\Psi(u,v) = \rho_\xi G(u \wedge v) + (1-\rho_\xi)G(u)G(v)$, then

$$r_2(t,s) = \big(\rho_\xi \mathbb{E}[K^2] + \mathbb{E}[K](1-\rho_\xi)\big)\Big(\int_{0-}^{t\wedge s} F^c(t-u)F^c(s-u)G(du) - \tilde{h}_1(t)\tilde{h}_1(s)\Big).$$

Observe that the function $r_2(t,s)$ is approximately linear in the correlation parameter ρ_ξ. Under Assumption 3.1(d), we have

$$r_2(t,s) = \Big(\int_{0-}^{t\wedge s} F^c(t-u)F^c(s-u)G(du) - \tilde{h}_1(t)\tilde{h}_1(s)\Big)\Big(\frac{m_K(1+\alpha)}{1-\alpha} + \frac{2\alpha\big(\mathbb{E}[\alpha^K]-1\big)}{(1-\alpha)^2}\Big)$$

where $\tilde{h}_1(u)$ is defined in (4.4). Note that the function $r_2(t,s)$ is increasing (decreasing) nonlinearly in the correlation parameter α if the quantity in the first parenthesis is positive (negative).

4.1.2. *Correlated service times.*

We consider the scenarios in Assumptions 3.1(c) and 3.2(a), and Assumptions 3.1(d) and 3.2(b). (The formulas in the scenarios in Assumptions 3.1(a) and 3.2(a), and Assumptions 3.1(a) and 3.2(b) can be obtained from them, respectively, as seen in Section 3.2.) Note that in both scenarios, we have $r_3(t,s) = \sigma_K^2 \tilde{h}_1(t)\tilde{h}_1(s)$, which is not affected by the correlations in random delays and in service times. So we focus on the functions $r_2(t,s)$ and $r_4(t,s)$.

Under Assumptions 3.1(c) and 3.2(a), we have

$$\begin{aligned}
r_2(t,s) &= m_K\Big(\int_{0-}^{t\wedge s} F^c(t-u)F^c(s-u)G(du) - \tilde{h}_1(t)\tilde{h}_1(s)\Big)\\
&\qquad + \mathbb{E}[K^2-K]\Big(\int_{0-}^{t}\int_{0-}^{s} F^c(t-u)F^c(s-v)\big(\Psi(du,dv) - G(du)G(dv)\big)\Big)\\
r_4(t,s) &= m_K\Big(\int_{0-}^{t\wedge s} \big(F^c(t\vee s-u) - F^c(t-u)F^c(s-u)\big)G(du)\Big)\\
&\qquad + \mathbb{E}[K^2-K]\Big(\int_{0-}^{t}\int_{0-}^{s} \big(\Phi^c(t-u,s-v) - F^c(t-u)F^c(s-u)\big)\Psi(du,dv)\Big),
\end{aligned}$$

where $\tilde{h}_1(u)$ is defined in (4.4) and

$$\Phi^c(u,v) := \mathbb{P}\big(Z_1 > u, Z_2 > v\big), \quad \text{for } u,v \geq 0. \tag{4.5}$$

If we further assume the relations similar to (3.5), and let ρ_Z be the correlation between Z_j and $Z_{j'}$, then we can approximate r_2 and r_4 by $\tilde{r}_2$ and $\tilde{r}_4$, respectively:

$$\begin{aligned}
\tilde{r}_2(t,s) &= \big(\rho_\xi\mathbb{E}[K^2] + \mathbb{E}[K](1-\rho_\xi)\big)\Big(\int_{0-}^{t\wedge s} F^c(t-u)F^c(s-u)G(du) - \tilde{h}_1(t)\tilde{h}_1(s)\Big)\\
\tilde{r}_4(t,s) &= \big(\rho_\xi\rho_Z\mathbb{E}[K^2] + \mathbb{E}[K](1-\rho_\xi\rho_Z)\big)\Big(\int_{0-}^{t\wedge s} \big(F^c(t\vee s-u) - F^c(t-u)F^c(s-u)\big)G(du)\Big)\\
&\qquad + \rho_\xi(1-\rho_Z)\mathbb{E}[K^2-K]\Big(\int_{0-}^{t}\int_{0-}^{s} \big(F^c(t\vee s-u) - F^c(t-u)F^c(s-u)\big)G(du)G(dv)\Big).
\end{aligned}$$

Note that the function $r_2(t,s)$ is linear in ρ_ξ, only affected by the correlations in the random delays, and $r_4(t,s)$ is linear in both ρ_ξ and ρ_Z. The formulas for r_2 and r_4 under Assumptions 3.1(a) and 3.2(a) are obtain from $\tilde{r}_2$ and $\tilde{r}_4$, respectively, by setting $\rho_\xi = 0$.

Under Assumptions 3.1(d) and 3.2(b), we have

$$\begin{aligned}
r_2(t,s) &= \Big(\int_{0-}^{t\wedge s} F^c(t-u)F^c(s-u)G(du) - \tilde{h}_1(t)\tilde{h}_1(s)\Big)\Big(\frac{m_K(1+\alpha)}{1-\alpha} + \frac{2\alpha\big(\mathbb{E}[\alpha^K]-1\big)}{(1-\alpha)^2}\Big)\\
r_4(t,s) &= \Big(\int_{0-}^{t\wedge s} \big(F^c(t\vee s-u) - F^c(t-u)F^c(s-u)\big)G(du)\Big)\Big(\frac{m_K(1+\alpha\beta)}{1-\alpha\beta} + \frac{2\alpha\beta\big(\mathbb{E}[(\alpha\beta)^K]-1\big)}{(1-\alpha\beta)^2}\Big)
\end{aligned}$$

$$+\Big(\int_{0-}^t\int_{0-}^s \big(F^c(t\vee s-u)-F^c(t-u)F^c(s-u)\big)G(du)G(dv)\Big)$$
$$\times\Big(\frac{2m_K\beta(1-\alpha)}{(1-\beta)(1-\alpha\beta)}+\frac{2\beta\big(\mathbb{E}[\beta^K]-1\big)}{(1-\beta)^2}-\frac{2\alpha\beta\big(\mathbb{E}[(\alpha\beta)^K]-1\big)}{(1-\alpha\beta)^2}\Big)$$

where $\tilde{h}_1(u)$ is defined in (4.4). Observe that $r_2(t,s)$ is increasing (decreasing) nonlinearly in α if the quantity in the first parenthesis is positive (negative), and is only affected by the correlations in the random delays. On the other hand, the function $r_4(t,s)$ does not necessarily have a monotone property in either α or β as indicated in the second term. The formulas under Assumptions 3.1(a) and 3.2(b) can be obtained from them by setting $\alpha=0$.

4.1.3. *Steady state in the stationary case.*

We consider the stationary case with $\Lambda(t)=\lambda t$ for $t\ge 0$ and the arrival limit $\hat{A}(t)=\sqrt{\lambda c_a^2}B_a(t)$ for $c_a>0$ and a standard BM B_a. In this case we obtain the equilibrium point of the fluid limit $\bar{X}(t)$ and the steady-state distribution of the stochastic limit $\hat{X}(t)$, which is a Gaussian process. We state the steady state limit $\bar{X}(\infty)=\lim_{t\to\infty}\bar{X}(t)$ and the variance $\mathrm{Var}(\hat{X}(\infty))$ of the limiting Gaussian random variable $\hat{X}(\infty)$ of $\hat{X}(t)$ as $t\to\infty$.

Recall that for infinite-server queues with batch arrivals and i.i.d. service times, it is shown in [25] that

$$\bar{X}(\infty)=\lambda m_K\int_0^\infty F^c(s)ds=\lambda m_K m_Z,$$

and

$$\mathrm{Var}(\hat{X}(\infty))=\lambda m_K m_Z+\lambda m_K\big(m_K(c_a^2+c_K^2)-1\big)\int_0^\infty \big(F^c(u)\big)^2du, \tag{4.6}$$

where $c_K^2=\sigma_K^2/m_K^2$ is the SCV of K.

For the steady state $\bar{X}(\infty)$ of our model, we still have

$$\bar{X}(\infty)=\lambda\int_0^\infty\sum_{j=1}^\infty \mathbb{P}(K\ge j)\mathbb{P}(\xi_j\le s<\xi_j+Z_j)\,ds=\lambda m_K m_Z,$$

if $\mathbb{E}[Z_j]\equiv m_Z$ for all $j\in\mathbb{N}$, where the second equality follows by Fubini's theorem. We next provide the steady-state variance formulas in various cases, which are new to the literature.

i.i.d. service times. For notational convenience, let

$$\chi_1=\int_0^\infty\big(F^c(u)\big)^2du\quad\text{and}\quad\chi_2=\int_0^\infty\Big(\int_{0-}^u F^c(u-v)G(dv)\Big)^2du.$$

Under Assumptions 3.1(a), we obtain

$$\mathrm{Var}(\hat{X}(\infty))=\lambda m_K m_Z+\lambda m_K\big(m_K(c_a^2+c_K^2)-1\big)\chi_2. \tag{4.7}$$

This result is a direct generalization of the case of i.i.d. service times without random delays, comparing (4.7) with (4.6). As we have alluded earlier, this case can be regarded as a tandem $G/G/\infty-G/\infty$ queue with i.i.d. service times at both service stations. When counting the number of customers in the second station, we count those that have completed service at the first station and are still in service at the second station. Under Assumption 3.1(a'), we obtain

$$\mathrm{Var}(\hat{X}(\infty))=\lambda m_K m_Z+\lambda(c_a^2-1)\chi_1+\lambda(m_K-1)\big((m_K-1)(c_a^2+c_{K-1}^2)-1\big)\chi_2$$
$$+2\lambda c_a^2(m_K-1)\int_0^\infty F^c(s)\int_{0-}^s F^c(s-u)G(du)ds.$$

Note that $c_{K-1}^2=\mathrm{Var}(K-1)/(m_K-1)^2=\sigma_K^2/(m_K-1)^2$.

Under Assumption 3.1(b), we obtain

$$\begin{aligned}\mathrm{Var}(\hat{X}(\infty)) &= \lambda m_K m_Z + 2\lambda \sum_{j,j'\geq 1} \mathbb{P}(K \geq j+j') \int_{0-}^{\infty} G^{(j')}(dv) \int_0^{\infty} F^c(s)F^c(s+v)ds \\ &\quad + \lambda(c_a^2 - 1) \int_0^{\infty} \Big(\sum_{j\geq 1} \mathbb{P}(K \geq j) \int_{0-}^{s} G^{(j)}(du)F^c(s-u)\Big)^2 ds.\end{aligned}$$

Under Assumption 3.1(c), we obtain

$$\begin{aligned}\mathrm{Var}(\hat{X}(\infty)) &= \lambda m_K m_Z + \lambda m_K \big(m_K(c_a^2 + c_K^2) - 1\big)\chi_2 \\ &\quad + \lambda m_K \big(m_K(1+c_K^2) - 1\big) \int_0^{\infty} \int_{0-}^{s} \int_{0-}^{s} F^c(s-u)F^c(s-v)\big(\Psi(du,dv) - G(du)G(dv)\big)ds.\end{aligned}$$

Note that in the special case of i.i.d. random delays, $\Psi(du, dv) = G(du)G(dv)$, thus the identity above is consistent with (4.7). Also, if $\Psi(u, v) = \rho_\xi G(u \wedge v) + (1-\rho_\xi)G(u)G(v)$, then

$$\mathrm{Var}(\hat{X}(\infty)) = \lambda m_K m_Z + \lambda m_K \big(m_K(c_a^2 + c_K^2) - 1\big)\chi_2 + \lambda \rho_\xi m_K \big(m_K(1+c_K^2) - 1\big)\big(\chi_1 - \chi_2\big).$$

It is clear that when $\rho_\xi = 0$, this reduces to the formula in (4.7).

Under Assumption 3.1(d), we obtain

$$\mathrm{Var}(\hat{X}(\infty)) = \lambda m_K m_Z + \lambda\big(m_K(c_a^2 + c_K^2) - 1\big)\chi_2 + \lambda\bigg(\frac{2m_K\alpha}{1-\alpha} + \frac{2\alpha\big(\mathbb{E}[\alpha^K] - 1\big)}{(1-\alpha)^2}\bigg)\big(\chi_1 - \chi_2\big).$$

Observe that if $\chi_1 - \chi_2 > 0$, then $\mathrm{Var}(\hat{X}(\infty))$ is increasing in α nonlinearly.

Correlated service times. Let

$$\chi_3 := \int_{0-}^{\infty} G(du) \int_{0-}^{\infty} G(dv) \int_{u\vee v}^{\infty} F^c(s - u \wedge v)ds.$$

Under Assumptions 3.1(c) and 3.2(a), let Φ^c be defined in (4.5), we obtain

$$\begin{aligned}\mathrm{Var}(\hat{X}(\infty)) &= \lambda m_K m_Z + \lambda m_K \big(m_K(c_a^2 + c_K^2) - 1\big)\chi_2 \\ &\quad + \lambda m_K \big(m_K(1+c_K^2) - 1\big) \int_0^{\infty} \int_{0-}^{s} \int_{0-}^{s} \big(\Phi^c(s-u, s-v) - F^c(s-u)F^c(s-v)\big)\Psi(du,dv)ds \\ &\quad + \lambda m_K \big(m_K(1+c_K^2) - 1\big) \int_0^{\infty} \int_{0-}^{s} \int_{0-}^{s} F^c(s-u)F^c(s-v)\big(\Psi(du,dv) - G(du)G(dv)\big)ds.\end{aligned} \tag{4.8}$$

If $\Psi(u, v) = \rho_\xi G(u \wedge v) + (1-\rho_\xi)G(u)G(v)$ and $\Phi(u, v) = \rho_Z F(u \wedge v) + (1-\rho_Z)F(u)F(v)$, then

$$\begin{aligned}\mathrm{Var}(\hat{X}(\infty)) &= \lambda m_K m_Z + \lambda m_K \big(m_K(c_a^2 + c_K^2) - 1\big)\chi_2 \\ &\quad + \lambda \rho_\xi \rho_Z m_K \big(m_K(1+c_K^2) - 1\big)(m_Z - \chi_2) \\ &\quad + \lambda \rho_\xi (1-\rho_Z) m_K \big(m_K(1+c_K^2) - 1\big)(\chi_1 - \chi_2) \\ &\quad + \lambda (1-\rho_\xi) \rho_Z m_K \big(m_K(1+c_K^2) - 1\big)(\chi_3 - \chi_2).\end{aligned} \tag{4.9}$$

Under Assumptions 3.1(d) and 3.2(b), we obtain

$$\begin{aligned}
\mathrm{Var}(\hat{X}(\infty)) &= \lambda m_K m_Z + \lambda m_K \big(m_K(c_a^2 + c_K^2) - 1\big)\chi_2 \\
&\quad + \lambda\Big(\frac{m_K 2\alpha\beta}{1-\alpha\beta} + \frac{2\alpha\beta(\mathbb{E}[(\alpha\beta)^K - 1])}{(1-\alpha\beta)^2}\Big)\big(m_Z - \chi_2\big) \\
&\quad + \lambda\Big(\frac{m_K 2\alpha(1-\beta)}{(1-\alpha)(1-\alpha\beta)} + \frac{2\alpha(\mathbb{E}[\alpha^K] - 1)}{(1-\alpha)^2} - \frac{2\alpha\beta(\mathbb{E}[(\alpha\beta)^K - 1])}{(1-\alpha\beta)^2}\Big)\big(\chi_1 - \chi_2\big) \\
&\quad + \lambda\Big(\frac{m_K 2\beta(1-\alpha)}{(1-\beta)(1-\alpha\beta)} + \frac{2\beta(\mathbb{E}[\beta^K] - 1)}{(1-\beta)^2} - \frac{2\alpha\beta(\mathbb{E}[(\alpha\beta)^K - 1])}{(1-\alpha\beta)^2}\Big)\big(\chi_3 - \chi_2\big).
\end{aligned} \tag{4.10}$$

Observe that the first two terms in these two scenarios in (4.8)–(4.10) are the same as the steady-state variance in (4.7), and the other terms capture the effect of correlations among the random delays, as well as among service times.

5. Proof of Theorem 2.2

This section is dedicated to the proof of Theorem 2.2. Since Theorem 2.1 follows directly from the consequence Theorem 2.2, we omit its proof for brevity.

We first provide a decomposition of the process $\hat{X}^n$. Recall $h_j(u)$ and $h(u)$ defined in (2.3). We have

$$\hat{X}^n(t) = \mathbb{E}[Z_1]\big(\hat{X}_1^n(t) + \hat{X}_2^n(t) + \hat{X}_3^n(t)\big) + \hat{X}_4^n(t), \tag{5.1}$$

where for every $t > 0$, the subprocesses are given by

$$\begin{aligned}
\hat{X}_1^n(t) &:= \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{A^n(t)} h(t-\tau_i^n) - \int_0^t h(t-s)\Lambda(ds)\Big) = \int_{(0,t]} h(t-s)\hat{A}^n(ds) \\
&= \hat{A}^n(t)h(0) - \int_{[0,t)} \hat{A}^n(s)h(t-ds) = \hat{A}^n(t)h(0) + \int_{(0,t]} \hat{A}^n(t-s)h(ds)
\end{aligned}$$

where the integration by parts in (1.3) and the fact that $\hat{A}^n(0) = 0$ are applied, and

$$\begin{aligned}
\hat{X}_2^n(t) &:= \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\sum_{j=1}^{K_i}\Big(H(t-\tau_i^n-\xi_{ij}) - h_j(t-\tau_i^n)\Big) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\sum_{j=1}^{K_i}\varsigma_{ij}(t-\tau_i^n), \\
\hat{X}_3^n(t) &:= \frac{1}{\sqrt{n}}\sum_{j=1}^{A^n(t)}\Big(\sum_{j=1}^{K_i} h_j(t-\tau_i^n) - h(t-\tau_i^n)\Big) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\vartheta_i(t-\tau_i^n), \\
\hat{X}_4^n(t) &:= \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\sum_{j=1}^{K_i} H(t-\tau_i^n-\xi_{ij})\big(Z_{ij} - \mathbb{E}[Z]\big) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\sum_{j=1}^{K_i} H(t-\tau_i^n-\xi_{ij})\varrho_{ij}.
\end{aligned}$$

In the proofs below, we further assume that $\mathbb{E}[Z] = 1$ w.l.o.g. to simplify our notations. Moreover, by Assumption 2.4, for fixed $T > 0$, there are $c_0 > 1$ such that $\forall s < t \le T$ with small $t - s$,

$$\begin{aligned}
\big(H(t) - H(s)\big)^2 &\le c_0\Big((t-s)^{2\gamma} + \mathbf{1}\big(0 \in (s,t]\big)\Big), \\
\mathbb{P}\big(\xi_j \in (s,t]\big) &\le c_0\Big((t-s)^{2\gamma} + \sum_{k\ge 0}\mathbf{1}\big(q_k \in (s,t]\big)\Big), \\
\mathbb{P}\big(\xi_j \in (s,r], \xi_{j'} \in (r,t]\big) &\le c_0\Big((t-s)^{4\gamma} + (t-s)^{2\gamma}\sum_{k\ge 0}\mathbf{1}\big(q_k \in (s,t]\big)\Big),
\end{aligned} \tag{5.2}$$

where $\{q_k, k \ge 0\} = \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{0\}$ with $q_0 = 0$, and in the last inequality the indicators are equal to 0 except for at most one term. For the last inequality above, we only consider the nontrivial case where $j \neq j'$, $r \geq 0$ and $s < r < t \leq T$ with $t-s$ small enough. Then there is at most one possible point, say, $q \in (s,t] \cap \mathcal{L}$ and $q = q_0 = 0$ if $s < 0$. We thus have case-by-case

$$
\begin{aligned}
&\mathbb{P}\big(\xi_j \in (s,r], \xi_{j'} \in (r,t]\big)\\
&= \mathbf{1}\big(s<0\big)\mathbf{1}\big((0,t]\cap\mathcal{L}=\emptyset\big)\Big(\mathbb{P}\big(\xi_j \in (0,r], \xi_{j'} \in (r,t]\big) + \mathbb{P}\big(\xi_j = 0, \xi_{j'} \in (r,t]\big)\Big)\\
&\quad + \mathbf{1}\big(s \geq 0\big)\mathbb{P}\big(\xi_j \in (s,r], \xi_{j'} \in (r,t]\big)\\
&\qquad \times \Big(\mathbf{1}\big((s,t]\cap\mathcal{L}=\emptyset\big) + \mathbf{1}\big(q \in (s,r]\big)\mathbf{1}\big((r,t]\cap\mathcal{L}=\emptyset\big) + \mathbf{1}\big((s,r]\cap\mathcal{L}=\emptyset\big)\mathbf{1}\big(q\in(r,t]\big)\Big)\\
&\leq c_0(t-s)^{4\gamma} + c_0(t-r)^{2\gamma}\mathbf{1}\big(q\in(s,r]\big) + c_0(r-s)^{2\gamma}\mathbf{1}\big(q\in(r,t]\big)\\
&\leq c_0(t-s)^{4\gamma} + c_0(t-s)^{2\gamma}\mathbf{1}\big(q\in(s,t]\big).
\end{aligned}
$$

In the following proofs, we fix the constant c_0 and $\{q_k\}$ in (5.2).

Lemma 5.1. *Under Assumption 2.4, for all $v < w < u \leq T$ with small enough $u-v$, we have*

$$
\mathbb{E}\big[H(u-\xi_j) - H(v-\xi_j)\big]^2 \leq 2c_0^2\Big((u-v)^{2\gamma} + \sum_{k\geq 0}\mathbf{1}\big(q_k \in (v,u]\big)\Big), \tag{5.3}
$$

$$
\begin{aligned}
&\mathbb{E}\big[(H(u-\xi_j) - H(w-\xi_j))^2(H(w-\xi_{j'}) - H(v-\xi_{j'}))^2\big]\\
&\qquad \leq 4c_0^3\Big((u-v)^{4\gamma} + (u-v)^{2\gamma}\sum_{k\geq 0}\mathbf{1}\big(q_k\in(v,u]\big)\Big),
\end{aligned} \tag{5.4}
$$

where c_0 is the constant in (5.2).

Proof. Applying (5.2), we have

$$
\mathbb{E}\big[H(u-\xi_j) - H(v-\xi_j)\big]^2 \leq c_0\Big((u-v)^{2\gamma} + \mathbb{P}\big(\xi_j \in (v,u]\big)\Big)
$$

which gives (5.3) by further applying (5.2).

Similarly, we have from (5.2) that

$$
\begin{aligned}
&\mathbb{E}\big[(H(u-\xi_j) - H(w-\xi_j))^2(H(w-\xi_{j'}) - H(v-\xi_{j'}))^2\big]\\
&\leq c_0^2\mathbb{E}\Big[\Big((u-w)^{2\gamma} + \mathbf{1}\big(\xi_j\in(w,u]\big)\Big)\Big((w-v)^{2\gamma} + \mathbf{1}\big(\xi_{j'}\in(v,w]\big)\Big)\Big]\\
&\leq c_0^2\bigg((u-v)^{4\gamma} + c_0(u-w)^{2\gamma}\Big((w-v)^{2\gamma} + \sum_{k\geq 0}\mathbf{1}\big(q_k\in(v,w]\big)\Big)\\
&\qquad + c_0(w-v)^{2\gamma}\Big((u-w)^{2\gamma} + \sum_{k\geq 0}\mathbf{1}\big(q_k\in(w,u]\big)\Big)\\
&\qquad + c_0\Big((u-v)^{4\gamma} + (u-v)^{2\gamma}\sum_{k\geq 0}\mathbf{1}\big(q_k\in(v,u]\big)\Big)\bigg)
\end{aligned}
$$

which gives (5.4). This finishes the proof. $\square$

5.1. **Convergence of $\hat{X}_1^n$.**

For the convergence of $\hat{X}_1^n$, we need the following lemma, which is a simplified version of [27, Lemma 6.1], noticing that the Lebesgue-Stieltjes integral for ψ_g is defined on the interval $[0,t)$. We state it here for completeness and provide a proof in the appendix.

Lemma 5.2. *Let g be a function in $\mathbb{D}$ with locally bounded variation. Define the mapping ψ_g on $\mathbb{D}$:*

$$
\psi_g(z)(t) := \int_{[0,t)} z(s)g(t-ds) = -\int_{(0,t]} z(t-s)g(ds), \tag{5.5}
$$

for $z \in \mathbb{D}$ and $t > 0$. Then the following hold:

(i) for any $z \in \mathbb{D}$, $\psi_g(z) \in \mathbb{D}$ and $\psi_g(z)(0) = 0$;
(ii) if $g \in \mathbb{C}$ or $z \in \mathbb{C}$ and $z(0) = 0$, then $\psi_g(z) \in \mathbb{C}$;
(iii) if $z \in \mathbb{C}$, then ψ_g is continuous at z in $(\mathbb{D}, J_1)$.

Lemma 5.3. *Under Assumptions 2.2 and 2.4, the functions h_j and h defined in (2.3) are monotonic functions in $\mathbb{D}$, which is piecewise Hölder continuous.*

Proof. Monotonicity and càdlàg of h_j and h follow from the corresponding properties of H. For all $s < t < T$ and $j \in \mathbb{N}$, we have from (5.3) that

$$(h_j(t) - h_j(s))^2 \le \mathbb{E}\big[H(t-\xi_j) - H(s-\xi_j)\big]^2 \le 2c_0^2\bigg((t-s)^{2\gamma} + \sum_{k\ge 0}\mathbf{1}\big(q_k \in (s,t]\big)\bigg). \tag{5.6}$$

This proves the result. $\square$

Lemma 5.4. *Under Assumptions 2.1, 2.2 and 2.4, $\hat{X}_1^n \Rightarrow \hat{X}_1$ in $(\mathbb{D}, J_1)$ as $n \to \infty$, where $\hat{X}_1$ is the continuous process as given in Theorem 2.2.*

Proof. Firstly, the continuity of $\hat{X}_1$ is a consequence of (2.8), (ii) in Lemma 5.2, the continuity of $\hat{A}$ and the fact $\hat{A}(0) = 0$. Moreover, by definition

$$\hat{X}_1^n(t) = \hat{A}^n(t)h(0) - \int_{[0,t)} \hat{A}^n(s)h(t-ds),$$

the claim follows from (iii) in Lemma 5.2 and the continuous mapping theorem [2, Theorem 2.7]. $\square$

5.2. Convergence of $\hat{X}_2^n, \hat{X}_3^n$ and $\hat{X}_4^n$ in $\mathbb{D}$.

This proceeds in the following steps:

Step 1: The existence of the limit Gaussian process $\hat{X}_k$ with sample paths in $\mathbb{C}$ (Lemma 5.5).
Step 2: The convergence of finite dimensional distributions of $\hat{X}_k^n$ to those of $\hat{X}_k$ (Lemma 5.7).
Step 3: Verifying the convergence criterion with the modulus of continuity as in [2, Theorem 13.5] and completing the proof (Lemma 5.8).

Lemma 5.5. *Under Assumptions 2.1, 2.2 and 2.4, there exists continuous modifications of the Gaussian processes $\hat{X}_k, k = 2, 3, 4$ with mean zero and covariance functions in (2.6).*

Proof. Recalling the covariance functions (2.6) of the limit distribution $\hat{X}_k$, its existence as a Gaussian process follows from the consistency condition for Gaussian distributional property. To prove $\hat{X}_k \in \mathbb{C}$, it is sufficient to check each $\hat{X}_k$ has continuous quadratic mean.

Let $\eta(t) := \sum_{j=1}^{K} H(t-\xi_j)Z_j$. For every $v < u < T$, by (5.3), we have

$$\begin{aligned}\mathrm{Var}(\eta(u) - \eta(v)) &\le \mathbb{E}\bigg[K\sum_{j=1}^{K}\varrho_j^2(H(u-\xi_j) - H(v-\xi_j))^2\bigg] \\ &\le 2c_0^2\mathbb{E}\big[K^2\big]\mathbb{E}[\varrho^2]\bigg((u-v)^{2\gamma} + \sum_{k\ge 0}\mathbf{1}\big(q_k \in (v,u]\big)\bigg).\end{aligned}$$

And it can be found that

$$\sum_{j=1}^{K}\varrho_j H(t-\xi_j) \in \sigma\{K,\xi,Z\}, \quad \sum_{j=1}^{K}\varsigma_j(t) \in \sigma\{K,\xi\} \quad \text{and} \quad \vartheta(t) \in \sigma\{K\},$$

are centered variables adapted to successively descending filtrations, where we use ξ, Z to denote the sequence of the corresponding variables. Therefore, for $0 \le s < t < T$

$$\mathbb{E}\big[\big(\hat{X}_2(t) - \hat{X}_2(s)\big)^2\big] + \mathbb{E}\big[\big(\hat{X}_3(t) - \hat{X}_3(s)\big)^2\big] + \mathbb{E}\big[\big(\hat{X}_4(t) - \hat{X}_4(s)\big)^2\big]$$

$$= \int_0^t \mathrm{Var}\big(\eta(t-u) - \eta(s-u)\big)\Lambda(du)$$
$$\leq 2c_0^2 \mathbb{E}\big[Z^2\big]\mathbb{E}\big[K^2\big]\bigg((t-s)^{2\gamma}\Lambda(T) + \sum_{k\geq 0}\big(\Lambda(t-q_k) - \Lambda(s-q_k)\big)\bigg).$$

Since Λ is continuous and only finitely many q_k on $[0,T]$, this finishes the proof. $\square$

To prove the convergence of the finite-dimensional distributions of $\hat{X}_k^n$, $k=2,3,4$, noticing that the processes are essentially independent sum of centered random variables, we follow the idea of proving CLT under the Lindeberg condition used in [35, Theorem.III.4.1], where we need the 2^{nd} moments for $\hat{X}_k^n$, $k=2,3,4$. Let $\mathcal{F}_A^n(t) = \sigma\{A^n(s) : t \geq s \geq 0\}$.

Lemma 5.6. *For $0 < t < T$, we have*

$$\mathbb{E}\Big[\big(\hat{X}_2^n(t)\big)^2\Big|\mathcal{F}_A^n(t)\Big] = \int_{(0,t]} r_2(t-u,t-u)\bar{A}^n(du),$$
$$\mathbb{E}\Big[\big(\hat{X}_3^n(t)\big)^2\Big|\mathcal{F}_A^n(t)\Big] = \int_{(0,t]} r_3(t-u,t-u)\bar{A}^n(ds),$$
$$\mathbb{E}\Big[\big(\hat{X}_4^n(t)\big)^2\Big|\mathcal{F}_A^n(t)\Big] = \int_{(0,t]} r_4(t-u,t-u)\bar{A}^n(du).$$

Proof. For fixed $t>0$, notice that conditioning on $\mathcal{F}_A^n(t)$, $\hat{X}_4^n(t)$ is a summand of independent and centralized random variable variables. It is straightforward that

$$\mathbb{E}\big[\big(\hat{X}_4^n(t)\big)^2\big|\mathcal{F}_A^n(t)\big] = \frac{1}{n}\sum_{i=1}^{A^n(t)}\mathbb{E}\bigg[\bigg(\sum_{j=1}^{K} H(u_i - \xi_j)\varrho_j\bigg)^2\bigg]\bigg|_{u_i = t-\tau_i^n}.$$

Thus, the formula for $\hat{X}_4^n(t)$ follows. The conditional second moments of $\hat{X}_2^n$ and $\hat{X}_3^n$ are derived similarly by conditioning. $\square$

A direct application of Lemma 5.6 shows that for $s,t \in [0,T]$

$$\mathbb{E}\Big[\hat{X}_2^n(t)\hat{X}_2^n(s)\Big|\mathcal{F}_A^n(t)\Big] = \int_{(0,t]} r_2(t-u,s-u)\bar{A}^n(du),$$
$$\mathbb{E}\Big[\hat{X}_3^n(t)\hat{X}_3^n(s)\Big|\mathcal{F}_A^n(t)\Big] = \int_{(0,t]} r_3(t-u,s-u)\bar{A}^n(du),$$
$$\mathbb{E}\Big[\hat{X}_4^n(t)\hat{X}_4^n(s)\Big|\mathcal{F}_A^n(t)\Big] = \int_{(0,t]} r_4(t-u,s-u)\bar{A}^n(du).$$

Lemma 5.7. *Under Assumptions 2.1, 2.2 and 2.4, the finite-dimensional distributions of the processes $\big(\hat{X}_2^n, \hat{X}_3^n, \hat{X}_4^n\big)$ converge to those of $(\hat{X}_2, \hat{X}_3, \hat{X}_4)$, in which $\hat{X}_k, k=2,3,4$ are mutually independent.*

Proof. For fixed $t>0$ and $\alpha,\beta,\gamma \in \mathbb{R}$, we consider first the limit distribution of

$$\begin{aligned}\hat{Y}^n &:= \alpha\hat{X}_2^n(t) + \beta\hat{X}_3^n(t) + \gamma\hat{X}_4^n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\breve{\eta}_i(s)\Big|_{s=t-\tau_i^n},\\ \breve{\eta}_i(s) &:= \frac{1}{\sqrt{n}}\bigg(\alpha\sum_{j=1}^{K_i}\varsigma_{ij}(s) + \beta\vartheta_i(s) + \gamma\sum_{j=1}^{K_i}H(s-\xi_{ij})\varrho_{ij}\bigg),\end{aligned} \tag{5.7}$$

where, by assumption, $\breve{\eta}_i(s)$ are independent for i with mark s, and $\breve{\eta}(s) = \alpha(\eta(s) - h(s))$ comparing with η defined in the proof of Lemma 5.5 if $\alpha=\beta=\gamma$. Applying the continuity theorem, it suffices to show that the characteristic function of $\hat{Y}^n$ converges point-wise to that of $\big(\alpha\hat{X}_2(t)+\beta\hat{X}_3(t)+\gamma\hat{X}_4(t)\big)$, where $\hat{X}_2, \hat{X}_3$ and $\hat{X}_4$ are mutually independent.

To which end, we examine the distribution of $\breve{\eta}(s)$ for $s\geq 0$ making use of the following identities,

$$\big|e^{iu}-(1+iu)\big|\leq \frac{u^2}{2},\quad \big|e^{iu}-(1+iu-\frac{u^2}{2})\big|\leq \frac{|u|^3}{6}\quad \text{and}\quad \big|\ln(1+v)-v\big|\leq |v|^2,$$

where ln denotes the principal value of the logarithm, which are valid for all $u\in\mathbb{R}$ and v is a complex number with $|v|\leq\frac{1}{2}$. It follows from the fact $\mathbb{E}[\breve{\eta}(s)]=0$ that for every $a>0$

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(\frac{iz}{\sqrt{n}}\breve{\eta}(s)\Big)\Big] &= \mathbb{E}\Big[\exp\Big(\frac{iz}{\sqrt{n}}\breve{\eta}(s)\Big);|\breve{\eta}(s)|>a\Big]+\mathbb{E}\Big[\exp\Big(\frac{iz}{\sqrt{n}}\breve{\eta}(s)\Big);|\breve{\eta}(s)|\leq a\Big]\\ &=1+\frac{z^2\theta_1}{2n}\mathbb{E}\big[\breve{\eta}^2(s);|\breve{\eta}|>a\big]-\frac{z^2}{2n}\mathbb{E}\big[\breve{\eta}^2(s);|\breve{\eta}|\leq a\big]+\frac{z^3\theta_2}{6n^{3/2}}\mathbb{E}\big[|\breve{\eta}(s)|^3;|\breve{\eta}|\leq a\big]\\ &=1-\frac{z^2}{2n}\mathbb{E}\big[\breve{\eta}^2(s)\big]+\frac{z^2}{n}R_s\end{aligned}$$

where θ_1,θ_2 are complex numbers with $|\theta_1|,|\theta_2|\leq 1$ and

$$|R_s|\leq \mathbb{E}\big[\breve{\eta}^2(s);|\breve{\eta}(s)|>a\big]+\frac{|z|a}{6\sqrt{n}}\mathbb{E}\big[\breve{\eta}^2(s)\big]\quad\text{and}\quad \Big|\mathbb{E}\big[e^{\frac{iz}{\sqrt{n}}\breve{\eta}(s)}\big]-1\Big|\leq\frac{z^2}{2n}\mathbb{E}\big[\breve{\eta}^2(s)\big].$$

Now, taking $z\in\mathbb{R}$ such that $z^2\mathbb{E}[\breve{\eta}^2(s)]\leq n$ then for some $|\theta_3|\leq 1$

$$\ln\mathbb{E}\Big[\exp\Big(\frac{iz}{\sqrt{n}}\breve{\eta}(s)\Big)\Big]=\frac{-z^2}{2n}\mathbb{E}\big[\breve{\eta}^2(s)\big]+\frac{z^2}{n}R_s+\frac{z^4\theta_3}{4n^2}\mathbb{E}^2[\breve{\eta}^2(s)]. \tag{5.8}$$

For the same reasoning as in the proof of Lemma 5.5, we have

$$\begin{aligned} g(s):=\mathbb{E}(\breve{\eta}^2(s)) &=\alpha^2\mathbb{E}\Big[\Big(\sum_{j=1}^K\varsigma_j(s)\Big)^2\Big]+\beta^2\mathbb{E}\big[\vartheta(s)^2\big]+\gamma^2\mathbb{E}\Big[\Big(\sum_{j=1}^K H(s-\xi_j)\varrho_j\Big)^2\Big]\\ &=\gamma^2\Big(\mathbb{E}\Big[\Big(\sum_{j=1}^K H(s-\xi_j)Z_j\Big)^2\Big]-h^2(s)\Big)+\big(\alpha^2-\gamma^2\big)\Big(\mathbb{E}\Big[\big(\sum_{j=1}^K H(s-\xi_j)\big)^2\Big]-h^2(s)\Big)\\ &\quad+\big(\beta^2-\alpha^2-\gamma^2\big)\Big(\mathbb{E}\Big[\Big(\sum_{j=1}^K h_j(s)\Big)^2\Big]-h^2(s)\Big) \end{aligned}\tag{5.9}$$

which shows $g\in\mathbb{D}$ has bounded variation. Applying integration by parts (1.3) and Lemma 5.2,

$$\begin{aligned}\int_{(0,t]}g(t-u)\bar{A}^n(du)&=g(0)\bar{A}^n(t)+\int_{[0,t)}\bar{A}^n(u)g(t-du)\\ &\to g(0)\bar{A}(t)+\int_{[0,t)}\bar{A}(u)g(t-du)=\int_{(0,t]}g(t-u)\bar{A}(du)\end{aligned}$$

in probability as $n\to\infty$.

Now, conditioning on $\mathcal{F}^n_A(t)$, plugging the limit above into the following, we have

$$\begin{aligned}\log\Big(\mathbb{E}\big[\exp\big(iz\hat{Y}^n\big)\big|\mathcal{F}^n_A(t)\big]\Big)&=\int_{(0,t]}\log\mathbb{E}\Big[\exp\Big(\frac{iz}{\sqrt{n}}\breve{\eta}(t-u)\Big)\Big]A^n(du)\\ &=\frac{-z^2}{2}\int_{(0,t]}g(t-u)\bar{A}^n(du)+z^2\int_{(0,t]}R(t-u)\bar{A}^n(du)+\frac{z^4}{n}\int_{(0,t]}\theta_3 g^2(t-u)\bar{A}^n(du)\\ &\to -\frac{z^2}{2}\int_{(0,t]}g(t-u)\Lambda(du)=-\frac{z^2}{2}\Big(\alpha^2R_2(t,t)+\beta^2R_3(t,t)+\gamma^2R_4(t,t)\Big),\end{aligned}$$

in probability for every $z \in \mathbb{R}$ by passing $n \to \infty$ and then $a \to \infty$, where the boundedness in (5.8) and $\sup_{s\in[0,T]} \mathbb{E}\big[\breve{\eta}^2(s)\big] < \infty$ are applied, which shows immediately the desired limit distributions of $\hat{X}^n_k(t)$ at fixed t, and their mutual independency and that with respect to $\hat{X}^n_1$.

The above convergence can be generalised to the joint Laplace of $\big(\hat{X}^n_2, \hat{X}^n_3, \hat{X}^n_4\big)$, that is,

$$\sum_{l=1}^{m}\Big(\sum_{j=1}^{K}\big(\alpha_{l,j}\varsigma(s_l)\big) + \beta_l\vartheta(s_l) + \gamma_{l,j}H(s_l-\xi_j)\varrho_j\Big)$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$ and $0 < s_1 < \cdots < s_m < T$. Applying the same procedure will complete the proof of the convergence of the finite-dimensional distributions of $\hat{X}^n_k, k = 2, 3, 4$, as well as the mutual independency between the limit processes. We only remark that the 2nd-moment of the random variable above, associated g in (5.9), is an m-dimensional function which may fail to be a continuous function on the domain, however we can always take $u \to g(s_1 - u, s_2 - u, \cdots, s_m - u)$ as a càdlàg function with bound variation which induces a sign measure on $[0, T]$. □

For the tightness of $\hat{X}^n_k$, $k = 2, 3, 4$, we obtain the following probability bound for the increments of the prelimit processes $\hat{X}^n_k$, $k = 2, 3, 4$, where the idea for empirical processes is applied.

Lemma 5.8. *Under Assumptions 2.1, 2.2 and 2.4, for* $0 \le s < r < t \le T$*,*

$$\begin{aligned}
\max\Big\{&\mathbb{P}\Big(\big|\hat{X}^n_2(t)-\hat{X}^n_2(r)\big| \wedge \big|\hat{X}^n_2(r)-\hat{X}^n_2(s)\big| \ge \lambda\Big|\mathcal{F}^n_A(t)\Big),\\
&\mathbb{P}\Big(\big|\hat{X}^n_3(t)-\hat{X}^n_3(r)\big| \wedge \big|\hat{X}^n_3(r)-\hat{X}^n_3(s)\big| \ge \lambda\Big|\mathcal{F}^n_A(t)\Big),\\
&\mathbb{P}\Big(\big|\hat{X}^n_4(t)-\hat{X}^n_4(r)\big| \wedge \big|\hat{X}^n_4(r)-\hat{X}^n_4(s)\big| \ge \lambda\Big|\mathcal{F}^n_A(t)\Big)\Big\}\\
&\le \frac{c\big(\bar{A}^n(T)+1\big)^2}{\lambda^4}\bigg((t-s)^{4\gamma} + \Big(\sum_{k\ge0}\bar{A}^n(t-q_k) - \sum_{k\ge0}\bar{A}^n(s-q_k)\Big)^2\bigg).
\end{aligned}$$

for some constant $c > 0$ independent of n and $\sigma\{A^n\}$.

Proof. In the proof, for fixed $0 \le s < r < t \le T$ and every $1 \le i \le A^n(T)$, we always put $(u_i, w_i, v_i) = (t - \tau^n_i, r - \tau^n_i, s - \tau^n_i)$ which are constants less than T conditioning on A^n. And we need the following expansion

$$x^2y^2 = \big(\sum_k x_k\big)^2\big(\sum_k y_k\big)^2 = \sum_{i,j,l,m} x_ix_jy_ly_m = \sum_k x_k^2y_k^2 + \sum_{i\ne j}\big(x_i^2y_j^2 + 2x_ix_jy_iy_j\big) + r(x,y), \quad (5.10)$$

where $r(x, y)$ collects those with at least one single subscript.

We first consider the increments of $\hat{X}^n_4$. For every $v < w < u \le T$, we take

$$x = \sum_{j=1}^{K}\varrho_j\big(H(u-\xi_j) - H(w-\xi_j)\big) \quad \text{and} \quad y = \sum_{j=1}^{K}\varrho_j\big(H(w-\xi_j) - H(v-\xi_j)\big).$$

Applying the Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}\big[x^2y^2\big] &\le \mathbb{E}\Big[K^2\sum_{j,j'}^{K}\varrho_j^2\varrho_{j'}^2\big(H(u-\xi_j)-H(w-\xi_j)\big)^2\big(H(w-\xi_{j'})-H(v-\xi_{j'})\big)^2\Big]\\
&\le 4c_0^3\mathbb{E}[K^4]\mathbb{E}[\varrho^4]\bigg((u-v)^{4\gamma} + (u-v)^{2\gamma}\sum_{k\ge0}\mathbf{1}\big(q_k\in(v,u]\big)\bigg), \quad (5.11)
\end{aligned}$$

where (5.4) is applied in the last inequality above. Similarly, one can check that

$$\mathbb{E}\big[x^2\big] \le \mathbb{E}\Big[K\sum_{j=1}^{K}\varrho_j^2\big(H(u-\xi_j)-H(w-\xi_j)\big)^2\Big]$$
$$\le 2c_0^2\mathbb{E}[K^2]\mathbb{E}[\varrho^2]\bigg((u-v)^{2\gamma}+\sum_{k\ge 0}\mathbf{1}\big(q_k\in(v,u]\big)\bigg), \tag{5.12}$$

by (5.3), and the same inequality holds also for $\mathbb{E}\big[y^2\big]$ and $\mathbb{E}[|xy|]$.

Then, we take for $1\le i\le A^n(T)$,

$$x_i=\sum_{j=1}^{K_i}\varrho_{ij}\mathbf{1}\big(\xi_{ij}\in(w_i,u_i]\big) \quad \text{and} \quad y_i=\sum_{j=1}^{K_i}\varrho_{ij}\mathbf{1}\big(\xi_{ij}\in(v_i,w_i]\big).$$

By the conditional independent for i and centralisation, we have from the last identity in (5.10),

$$\mathbb{E}\Big[\big(\hat{X}_4^n(t)-\hat{X}_4^n(r)\big)^2\big(\hat{X}_4^n(r)-\hat{X}_4^n(s)\big)^2\Big|\mathcal{F}_A^n(T)\Big]$$
$$=\frac{1}{n^2}\sum_{i=1}^{A^n(t)}\mathbb{E}\big[x_i^2y_i^2\big|\tau_i^n\big]+\frac{1}{n^2}\sum_{i\neq i'}^{A^n(t)}\Big(\mathbb{E}\big[x_i^2\big|\tau_i^n\big]\mathbb{E}\big[y_{i'}^2\big|\tau_{i'}^n\big]+2\mathbb{E}\big[x_iy_i\big|\tau_i^n\big]\mathbb{E}\big[x_{i'}y_{i'}\big|\tau_{i'}^n\big]\Big)$$
$$\le\frac{1}{n^2}\sum_{i=1}^{A^n(t)}\mathbb{E}\big[x_i^2y_i^2\big|\tau_i^n\big]+\frac{1}{n^2}\Big(\sum_{i=1}^{A^n(t)}\mathbb{E}\big[x_i^2\big|\tau_i^n\big]\Big)\Big(\sum_{i=1}^{A^n(t)}\mathbb{E}\big[y_i^2\big|\tau_i^n\big]\Big)+\frac{2}{n^2}\Big(\sum_{i=1}^{A^n(t)}\mathbb{E}\big[x_iy_i\big|\tau_i^n\big]\Big)^2.$$

Plugging (5.11) and (5.12) into the inequality above, noticing that $u_i-v_i\equiv(t-s)$ for every $i\in\mathbb{N}$ and $\mathbf{1}\big(q_k\in(v_i,u_i]\big)=\mathbf{1}\big(\tau_i^n\in(s-q_k,t-q_k)\big)$ for all $k\ge 0$, we have almost surely

$$\text{RHS}\le 4c_0^3\mathbb{E}[K^4]\mathbb{E}[\varrho^4]\bigg((t-s)^{4\gamma}\bar{A}^n(T)+(t-s)^{2\gamma}\sum_{k\ge 0}\big(\bar{A}^n(t-q_k)-\bar{A}^n(s-q_k)\big)\bigg)$$
$$+12c_0^4\mathbb{E}^2[K^2]\mathbb{E}^2[\varrho^2]\bigg((t-s)^{2\gamma}\bar{A}^n(T)+\sum_{k\ge 0}\big(\bar{A}^n(t-q_k)-\bar{A}^n(s-q_k)\big)\bigg)^2$$
$$\le 28c_0^4\mathbb{E}[K^4]\mathbb{E}[\varrho^4]\big(\bar{A}^n(T)+1\big)^2\bigg((t-s)^{4\gamma}+\bigg(\sum_{k\ge 0}\big(\bar{A}^n(t-q_k)-\bar{A}^n(s-q_k)\big)\bigg)^2\bigg),$$

which shows the inequality for $\hat{X}_4^n$.

For the 2nd moments of increment of $\hat{X}_2^n$, we denote by for $v<w<u<T$,

$$x_j=H(u-\xi_j)-H(w-\xi_j) \quad \text{and} \quad x=\sum_{j=1}^{K}\big(\varsigma_j(u)-\varsigma_j(w)\big)=\sum_{j=1}^{K}\big(x_j-\mathbb{E}(x_j)\big),$$
$$y_j=H(w-\xi_j)-H(v-\xi_j) \quad \text{and} \quad y=\sum_{j=1}^{K}\big(\varsigma_j(w)-\varsigma_j(v)\big)=\sum_{j=1}^{K}\big(y_j-\mathbb{E}(y_j)\big).$$

Applying (5.3), (5.4) and the fact $H(z)=0$ for $z<0$, it can be checked that for all $j,j'\ge 1$,

$$\max\Big\{\mathbb{E}[x_j^2y_{j'}^2],\mathbb{E}[x_j^2]\mathbb{E}[y_{j'}^2],\mathbb{E}^2[x_j]\mathbb{E}[y_{j'}^2],\mathbb{E}[x_j^2]\mathbb{E}^2[y_{j'}],\mathbb{E}^2[x_j]\mathbb{E}^2[y_{j'}],$$
$$\mathbb{E}^2[x_jy_{j'}],\big|\mathbb{E}[x_j]\mathbb{E}[x_jy_{j'}^2]\big|,\big|\mathbb{E}[x_j^2y_{j'}]\mathbb{E}[y_{j'}]\big|\Big\}=\max\big\{\mathbb{E}[x_j^2y_{j'}^2],\mathbb{E}[x_j^2]\mathbb{E}[y_{j'}^2]\big\}$$
$$\le 4c_0^4\bigg((u-v)^{4\gamma}+(u-v)^{2\gamma}\sum_{k\ge 0}\mathbf{1}\big(q_k\in(v,u]\big)\bigg).$$

Therefore, we have by applying the Hölder's inequality,

$$\begin{aligned}\mathbb{E}[x^2y^2] &\le \mathbb{E}\Big[K^2\sum_{j,j'}^{K}\big(x_j-\mathbb{E}[x_j]\big)^2\big(y_{j'}-\mathbb{E}[y_{j'}]\big)^2\Big]\\ &\le 64c_0^4\mathbb{E}[K^4]\bigg((u-v)^{4\gamma}+(u-v)^{2\gamma}\sum_{k\ge0}\mathbf{1}\big(q_k\in(v,u]\big)\bigg),\end{aligned}$$

and the inequality similar to (5.12)

$$\max\{\mathbb{E}[x^2],|\mathbb{E}[xy]|,\mathbb{E}[y^2]\}\le 2c_0^2\mathbb{E}[K^2]\bigg((u-v)^{2\gamma}+\sum_{k\ge0}\mathbf{1}\big(q_k\in(v,u]\big)\bigg).$$

For the last, applying the last identity in (5.10), we will have almost surely

$$\begin{aligned}&\mathbb{E}\Big[\big(\hat X_2^n(t)-\hat X_2^n(r)\big)^2\big(\hat X_2^n(r)-\hat X_2^n(s)\big)^2\Big|\mathcal{F}_A^n(T)\Big]\\ &\le 64c_0^4\mathbb{E}[K^4]\bigg((t-s)^{4\gamma}\bar A^n(T)+(t-s)^{2\gamma}\sum_{k\ge0}\big(\bar A^n(t-q_k)-\bar A^n(s-q_k)\big)\bigg)\\ &\quad+12c_0^4\mathbb{E}^2[K^2]\bigg((t-s)^{2\gamma}\bar A^n(T)+\sum_{k\ge0}\big(\bar A^n(t-q_k)-\bar A^n(s-q_k)\big)\bigg)^2\\ &\le\big(88c_0^4\mathbb{E}[K^4]\big)\mathbb{E}[\varrho^4]\big(\bar A^n(T)+1\big)^2\bigg((t-s)^{4\gamma}+\bigg(\sum_{k\ge0}\big(\bar A^n(t-q_k)-\bar A^n(s-q_k)\big)\bigg)^2\bigg),\end{aligned}$$

which shows the inequality for $\hat X_2^n$.

For the moment of $\hat X_3^n$, we denote by

$$x=\sum_{j=1}^{K}\big(h_j(u)-h_j(w)\big)\quad\text{and}\quad y=\sum_{j=1}^{K}\big(h_j(w)-h_j(v)\big).$$

then, $\vartheta(u)-\vartheta(w)=x-\mathbb{E}[x]$. Applying (5.6), we have almost surely

$$\begin{aligned}x^2&\le 2c_0^2K^2\bigg((u-w)^{2\gamma}+\sum_{k\ge0}\mathbf{1}\big(q_k\in(w,u]\big)\bigg),\\ y^2&\le 2c_0^2K^2\bigg((w-v)^{2\gamma}+\sum_{k\ge0}\mathbf{1}\big(q_k\in(v,w]\big)\bigg),\end{aligned}$$

which shows for small $u-v$,

$$\begin{aligned}\mathbb{E}\big[(x-\mathbb{E}[x])^2(y-\mathbb{E}[y])^2\big]&\le 64c_0^4\mathbb{E}[K^4]\bigg((u-v)^{4\gamma}+(u-v)^{2\gamma}\sum_{k\ge0}\mathbf{1}\big(q_k\in(v,u]\big)\bigg),\\ \max\big\{\mathbb{E}[x-\mathbb{E}[x]]^2,\mathbb{E}[y-\mathbb{E}[y]]^2\big\}&\le\max\big\{\mathbb{E}[x^2],\mathbb{E}[y^2]\big\}\\ &\le 2c_0^2\mathbb{E}[K^2]\bigg((u-v)^{2\gamma}+\sum_{k\ge0}\mathbf{1}\big(q_k\in(v,u]\big)\bigg),\\ \big|\mathbb{E}\big[(x-\mathbb{E}[x])(y-\mathbb{E}[y])\big]\big|&\le 2c_0^2\mathbb{E}[K^2]\bigg((u-v)^{2\gamma}+\sum_{k\ge0}\mathbf{1}\big(q_k\in(v,u]\big)\bigg).\end{aligned}$$

Therefore, we have by applying (5.10),

$$\mathbb{E}\Big[\big(\hat X_3^n(t)-\hat X_3^n(r)\big)^2\big(\hat X_3^n(r)-\hat X_3^n(s)\big)^2\Big|\mathcal{F}_A^n(T)\Big]$$

$$\leq 64c_0^4\mathbb{E}[K^4]\Big((t-s)^{4\gamma}\bar{A}^n(T)+(t-s)^{2\gamma}\sum_{k\geq 0}\big(\bar{A}^n(t-q_k)-\bar{A}^n(s-q_k)\big)\Big)$$
$$+12c_0^4\mathbb{E}^2[K^2]\Big((t-s)^{2\gamma}\bar{A}^n(T)+\sum_{k\geq 0}\big(\bar{A}^n(t-q_k)-\bar{A}^n(s-q_k)\big)\Big)^2.$$

This proves the last inequality for $\hat{X}_3^n$ and completes the proof. □

6. Proof of Theorem 4.2

Recalling $\tilde{H}_j, \tilde{h}_j$ and $\tilde{h}$ defined in (4.2), we decompose the process $\hat{X}^n(t)$ as follows:

$$\hat{X}^n(t) = \hat{X}_1^n(t) + \hat{X}_2^n(t) + \hat{X}_3^n(t) + \hat{X}_4^n(t)$$

where the subprocesses are defined by

$$\hat{X}_1^n(t) = \sqrt{n}\Big(\sum_{i=1}^{A^n(t)}\frac{\tilde{h}(t-\tau_i^n)}{n} - \int_0^t \tilde{h}(t-s)\Lambda(ds)\Big) = \int_{(0,t]}\tilde{h}(t-s)\hat{A}^n(ds)$$
$$\hat{X}_2^n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\sum_{j=1}^{K_i}\Big(\tilde{H}_j(t-\tau_i^n-\xi_{ij}) - \tilde{h}_j(t-\tau_i^n)\Big) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\sum_{j=1}^{K_i}\tilde{\varsigma}_{ij}(t-\tau_i^n)$$
$$\hat{X}_3^n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\Big(\sum_{j=1}^{K_i}\tilde{h}_j(t-\tau_i^n) - \tilde{h}(t-\tau_i^n)\Big) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\tilde{\vartheta}_i(t-\tau_i^n)$$
$$\hat{X}_4^n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\sum_{j=1}^{K_i}\Big(\mathbf{1}\big(0\leq t-\tau_i^n-\xi_{ij}<Z_{ij}\big) - \tilde{H}_j(t-\tau_i^n-\xi_{ij})\Big) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\sum_{j=1}^{K_i}\tilde{\varrho}_{ij}(t-\tau_i^n).$$

Noticing that $\mathbf{1}(0\leq u<v) = \mathbf{1}(u\geq 0) - \mathbf{1}(u\geq v)$, $\hat{X}_4^n$ can be further written as

$$\hat{X}_4^n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A^n(t)}\sum_{j=1}^{K_i}\Big(F_j(t-\tau_i^n-\xi_{ij}) - \mathbf{1}(\tau_i^n+\xi_{ij}+Z_{ij}\leq t)\Big) \quad \text{for all } t\geq 0.$$

The convergence of the new $\hat{X}_1^n$ in $(\mathbb{D}, J_1)$ is proved by making use of integration by part and the continuous mapping theorem. It is true that under Assumption 4.1, $\tilde{h}\in\mathbb{D}$ has bounded variation and plays the same role as h does for the former $\hat{X}_1^n$. The discussion in subsection 5.1 and Lemma 5.2 can be applied directly, which proves the convergence. For the proof of $\hat{X}_k^n, k=2,3,4$, we follow the same procedures as that stated at the beginning of subsection 5.2.

Step 1). The existence of continuous modifications of the new Gaussian processes $\hat{X}_k$ follows from their continuous quadratic mean as that in Lemma 5.5.

Step 2). Since $\hat{X}_k^n$ are still the random sums of independent variable with marks depending on A^n, the idea of Lemma 5.7 can be applied. The convergence of their finite-dimensional distributions as well as their mutual independency are proved by examining the second moments of joint distributions of the new processes.

Step 3). The tightness properties of the families of the law of $\hat{X}_2^n$ and $\hat{X}_3^n$ are proved by examining the probability bound of their increments. It is true that $\tilde{H}_j\in\mathbb{C}[0,\infty)$ in this model is decreasing on $[0,\infty)$ and vanishes on $(-\infty,0)$, and more importantly, uniformly Hölder continuous of order γ by the assumption 4.1. Therefore, the proof for the former $\hat{X}_2^n$ and $\hat{X}_3^n$ in the proof of Lemma 5.8 can be applied.

Step 4). The proof of tightnesses of $\hat{X}_4^n$ is slightly different, as it is the difference of two indicator functions and related to two variables. Here, we only give a sketch of proof.

To find the bound for the 2nd moments of the increments

$$\mathbb{E}\Big[\big(\hat{X}_4^n(t)-\hat{X}_4^n(r)\big)^2\big(\hat{X}_4^n(r)-\hat{X}_4^n(s)\big)^2\Big],$$

for every $0 \le s < r < t \le T$. We denote by for $v < w < u \le T$

$$\begin{aligned} x_j &= \big(F_j(u-\xi_j)-F_j(w-\xi_j)\big)-\mathbf{1}\big(\xi_j+Z_j\in(w,u]\big),\\ y_j &= \big(F_j(w-\xi_j)-F_j(v-\xi_j)\big)-\mathbf{1}\big(\xi_j+Z_j\in(v,w]\big),\end{aligned}$$

where (v,w,u) will represents $(s-\tau_i^n, r-\tau_i^n, t-\tau_i^n)$, which are constants conditioning on $\mathcal{F}_A^n(t)$ and possibly negative. By the boundedness of the variables, it is not hard to find that

$$\begin{aligned} \mathbb{E}\big[x_j^2\big] \le \mathbb{P}\big(\xi_j+Z_j\in(w,u]\big) = \int_{0-}^{u}\mathbb{P}(\xi_j\in dz)\mathbb{P}\big(Z_j\in(w-z,u-z]\big) \le c_0(u-w)^{2\gamma}\\ \mathbb{E}\big[y_j^2\big] \le \mathbb{P}\big(\xi_j+Z_j\in(v,w]\big) \le c_0(w-v)^{2\gamma},\end{aligned}$$

from the Hölder continuity of Z under Assumption 4.1. Moreover,

$$\begin{aligned} x_j^2 &\le \mathbf{1}(\xi_j+Z_j\in(w,u]) + \big(F_j(u-\xi_j)-F_j(w-\xi_j)\big)^2 \le \mathbf{1}(\xi_j+Z_j\in(w,u]) + c_0(u-w)^{2\gamma}\\ y_j^2 &\le \mathbf{1}(\xi_j+Z_j\in(v,w]) + \big(F_j(w-\xi_j)-F_j(v-\xi_j)\big)^2 \le \mathbf{1}(\xi_j+Z_j\in(w,u]) + c_0(u-w)^{2\gamma}.\end{aligned}$$

Therefore, we have from Assumption 4.1, for small $u-v$,

$$\begin{aligned} \mathbb{E}\big[x_j^2y_{j'}^2\big] &\le 3c_0^2(u-v)^{4\gamma} + \mathbb{P}\big(\xi_j+Z_j\in(w,u], \xi_{j'}+Z_{j'}\in(v,w]\big)\\ &\le 4c_0^2(u-v)^{4\gamma} + c_0(u-v)^{2\gamma}\sum_{k=0}^{\infty}\mathbf{1}\big(q_k\in(v,u]\big)\big). \end{aligned} \tag{6.1}$$

Applying the Hölder inequality and (5.10), similar to previous discussion, we have

$$\begin{aligned} &\mathbb{E}\Big[\big(\hat{X}_4^n(t)-\hat{X}_4^n(r)\big)^2\big(\hat{X}_4^n(r)-\hat{X}_4^n(s)\big)^2\Big|\mathcal{F}_A^n(T)\Big]\\ &\le c\mathbb{E}[K^4]\big(\bar{A}^n(T)+1\big)^2\left((t-s)^{4\gamma}+\left(\sum_{k\ge0}\big(\bar{A}^n(t-q_k)-\bar{A}^n(s-q_k)\big)\right)^2\right),\end{aligned}$$

for some constant $c>0$, which proves the tightness. This completes the proof.

7. Appendix

For completeness, we provide a proof of Lemma 5.2.

Proof of Lemma 5.2. Since every bounded variation function can be expressed as the difference of two increasing functions, it is sufficient to prove the result for the case g being a decreasing function. Moreover, since the integral only depends on the value of g on $(0,\infty)$, we further assume w.l.o.g. that $g(z)=g(0)$ for $z\le 0$. Then $g(t-ds)$ is a positive measure on $\mathbb{R}$ induced by the càglàd increasing function $s\to g(t-s)$, and $\int_{[a,b)} g(t-ds) = g(t-b)-g(t-a)$ for all $b>a$.

For any $\varepsilon>0$, by the separability of $(\mathbb{D},J_1)$, z is approximated by a simple function such that

$$f(t)=\sum_i f(r_i)\mathbf{1}\big(t\in[r_i,r_{i+1})\big) \quad \text{and} \quad ||z-f||_\infty<\varepsilon,$$

where $||\cdot||_\infty$ represents the uniform norm on $[0,T]$. For any $T>t>s\ge 0$, we have

$$\begin{aligned} \big|\psi_g(z)(t)-\psi_g(z)(s)\big| &\le 2||\psi_g(z)-\psi_g(f)||_\infty + \big|\psi_g(f)(t)-\psi_g(f)(s)\big|\\ &\le 2\varepsilon(g(0)-g(t)) + \big|\psi_g(f)(t)-\psi_g(f)(s)\big|.\end{aligned}$$

On the other hand, by the definition of f

$$\begin{aligned}\psi_g(f)(t)-\psi_g(f)(s) &= \int_{[0,s)} f(r)\big(g(t-dr)-g(s-dr)\big)+\int_{[s,t)} f(r)g(t-dr)\\ &= \sum_i f(r_i)\Big(\big(g(t-r_{i+1})-g(s-r_{i+1})\big)-\big(g(t-r_i)-g(s-r_i)\big)\Big)\\ &\quad+\theta\cdot||f||_\infty\big(g(0)-g(t-s)\big),\end{aligned}$$

where θ is a number with $|\theta|\le 1$ and $||f||_\infty=\sup_{u\in[0,T]}|f(u)|$.

For (i) and (ii), by the right-continuity of g, let $t\downarrow s=s_0\ge 0$ and then $\varepsilon\downarrow 0+$ proves the right continuity of $\psi_g(z)$ at s_0. Moreover, let $t,s\uparrow s_0>0$ for some $s_0>0$ with $t-s\to 0+$ and then $\varepsilon\downarrow 0+$, by the left-limit of g this also proves the existence of a left limit $\psi_g(z)$ at s_0. And $\psi_g(z)(0)=0$ by definition. This also proves the continuity of $\psi_g(z)$ if g is continuous at $s>0$. On the other hand, if $z\in\mathbb{C}$ and $z(0)=0$, by the second identity in the definition of ψ_g in (5.5),

$$\psi_g(z)(s)-\psi_g(z)(t)=\int_{(0,s]}\big(z(t-u)-z(s-u)\big)g(du)+\int_{(s,t]} z(t-u)g(du),$$

the continuity follows from the uniformly continuity of z on $[0,T]$ and right-continuity of g.

For (iii), let $z_n\to z$ in $(\mathbb{D},J_1)$, since $z\in\mathbb{C}$ we have $||z_n-z||_\infty\to 0$. It suffices to prove that $||\psi_g(z_n)-\psi_g(z)||_\infty\to 0$ as $n\to\infty$. By the monotonicity of g on $[0,T]$, we have

$$||\psi_g(z_n)-\psi_g(z)||_\infty\le ||z_n-z||_\infty\big(g(0)-g(T)\big),$$

which converge to zero as $n\to\infty$. □

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