18.085 Quiz 3 practice Solns

1 a) $$\frac{dy}{dt} = -y^2$$

Forward Euler: $$\frac{y_{n+1} - y_n}{\Delta t} = -y_n^2$$

$$y_{n+1} = y_n - \Delta t\, y_n^2$$

b) Backward Euler:

$$\frac{y_{n+1} - y_n}{\Delta t} = -y_{n+1}^2$$

$$\Delta t\, y_{n+1}^2 + y_{n+1} - y_n = 0$$

Solve quadratic

$$y_{n+1} = \frac{-1 \pm \sqrt{1 + 4\Delta t\, y_n}}{2\Delta t}$$

Select the "+" to get a positive value

$$y_{n+1} = \frac{-1 + \sqrt{1 + 4\Delta t\, y_n}}{2\Delta t}$$

c) Trapezoidal Rule

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{-y_{n+1}^2 - y_n^2}{2}$$

Write as quadratic in y_{n+1}:

$$\Delta t \, y_{n+1}^2 + 2 y_{n+1} + (y_n^2 \Delta t - 2y_n) = 0$$

Solve

$$y_{n+1} = \frac{-2 \pm \sqrt{4 - 4\Delta t (y_n^2 \Delta t - 2y_n)}}{2\Delta t}$$

Choose "+" branch to get positive value.

2) $$U(x) = U(0) + U'(0)\,x + U''(0)\frac{x^2}{2} + U'''(0)\frac{x^3}{6} + \cdots$$

So

$$U(0) = U(0)$$

$$U(\Delta x) = U(0) + U'(0)\Delta x + U''(0)\frac{\Delta x^2}{2} + U'''(0)\frac{\Delta x^3}{6}$$

$$U(2\Delta x) = U(0) + U'(0)\,2\Delta x + U''(0)\frac{4\Delta x^2}{2} + U'''(0)\frac{8\Delta x^3}{6}$$

We want to find c_0, c_1, c_2 st

$$c_0 U(0) + c_1 U(\Delta x) + c_2(2\Delta x) = 0 + 0 + U''(0)$$

$$c_0 + c_1 + c_2 = 0$$

$$c_1 \cdot \Delta x + 2 c_2 \Delta x = 0$$

$$c_1 \frac{\Delta x^2}{2} + c_2 \frac{4\Delta x^2}{2} = 1$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\Delta x^2} \end{pmatrix}$$

Solving

$$c_0 = \frac{1}{\Delta x^2}$$

$$c_1 = \frac{-2}{\Delta x^2}$$

$$c_2 = \frac{1}{\Delta x^2}$$

This is a first order approximation of $U''(0)$ because $c_0 U(0) + c_1 U(\Delta x) + c_2 U(\Delta x \cdot 2)$ has a nonzero Δx^3 term

3)

$$-\frac{d}{dx}\left(\rho(x)\frac{du}{dx}\right) = f(x)$$
$$u(0) = u(2\pi)$$
$$u'(0) = u'(2\pi)$$

Where $\rho(0) = \rho(2\pi)$

Multiply by $\phi(x)$ which is 2π periodic : $\phi(0) = \phi(2\pi)$

$$\int_0^{2\pi} -\frac{d}{dx}\left(\rho(x)\frac{du}{dx}\right)\phi(x)\,dx = \int_0^{2\pi} f(x)\phi(x)\,dx$$

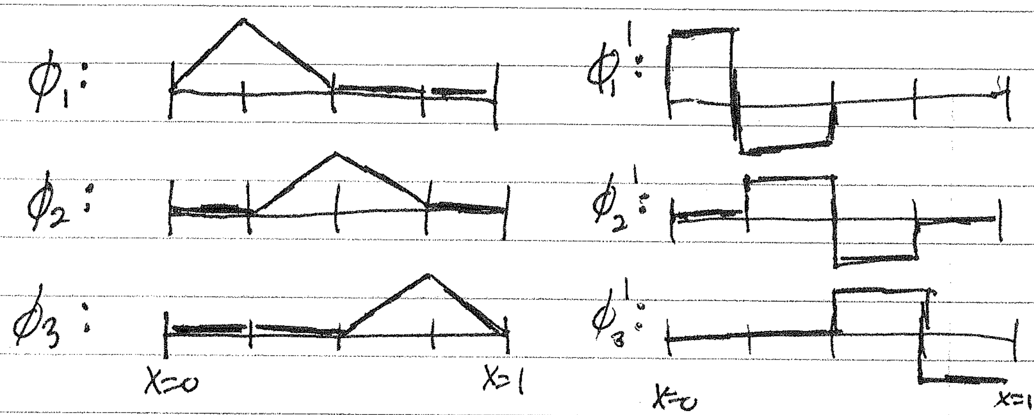
Integrate by parts

$$\int_0^{2\pi} \rho(x)\frac{du}{dx}\frac{d\phi}{dx}\,dx \underbrace{\cancelto{0}{-\rho(x)\frac{du}{dx}\phi(x)\Big|_0^{2\pi}}}_{\rho(2\pi)\frac{du}{dx}(2\pi)\phi(2\pi) - \rho(0)\frac{du}{dx}(0)\phi(0) = 0} = \int_0^{2\pi} f(x)\phi(x)\,dx$$

$$\int_0^{2\pi} \rho(x)\frac{du}{dx}\frac{d\phi}{dx}\,dx = \int_0^{2\pi} f(x)\phi(x)\,dx \quad \text{for all } 2\pi \text{ periodic } \phi$$

4)

a) $$\int_0^1 \rho(x)\frac{du}{dx}\frac{d\phi}{dx}\,dx = \int_0^1 \phi(x)\,dx \quad \text{for all } \phi \text{ with } \phi(0)=0,\ \phi(1)=0$$

b) 

$$K_{ij} = \int_0^1 \rho(x)\,\phi_i'(x)\,\phi_j'(x)\,dx$$

$$K_{12} = \int_0^1 \underbrace{\rho(x)\phi_1'(x)\phi_2'(x)}_{\text{non zero for}}\,dx \qquad \tfrac{1}{4}<x<\tfrac{1}{2}: \quad \phi_1' = \frac{-1}{\Delta x} = -4, \quad \phi_2' = \frac{1}{\Delta x} = 4, \quad \rho(x) = 1$$

$$= 1\cdot\left(\frac{-1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right)\Delta x = -\frac{1}{\Delta x} = -4$$

$$K_{13} = 0$$

$$K_{23} = \int_0^1 \underbrace{\rho(x)\phi_2'(x)\phi_3'(x)}_{\text{non zero for}}\,dx \qquad \tfrac{1}{2}<x<\tfrac{3}{4}: \quad \phi_2' = -\frac{1}{\Delta x}, \quad \phi_3' = \frac{1}{\Delta x}, \quad \rho(x) = 2$$

$$= 2\left(-\frac{1}{\Delta x}\right)\frac{1}{\Delta x}\Delta x = -\frac{2}{\Delta x} = -8$$

$$K_{11} = \int_0^1 \underbrace{\rho(x)\, \phi_1'(x)\phi_1'(x)}_{\text{nonzero for } 0<x<1/2\ ;\ \rho=1}\, dx$$

$$= 1\cdot\left(\frac{1}{\Delta x}\right)^2 2\Delta x = \frac{2}{\Delta x} = 8$$

$$K_{33} = \int_0^1 \rho(x)\, \underbrace{\phi_3'\phi_3'}_{\text{nonzero for } 1/2<x<1\ ;\ \rho=2}\, dx$$

$$= (2)\cdot\left(\frac{1}{\Delta x}\right)^2 2\Delta x = \frac{4}{\Delta x} = 16$$

$$K_{22} = \int_0^1 \rho(x)\, \phi_2'\phi_2'\, dx$$

break into $\frac{1}{4}<x<\frac{1}{2}$ and $\frac{1}{2}<x<\frac{3}{4}$

$$= \int_{1/4}^{1/2} 1\ \phi_2'\cdot\phi_2'\, dx + \int_{1/2}^{3/4} 2\ \phi_2'\phi_2'\, dx$$

$$= 1\cdot\frac{1}{\Delta x}\cdot\frac{1}{\Delta x}\Delta x + 2\cdot\frac{1}{\Delta x}\frac{1}{\Delta x}\Delta x$$

$$= \frac{3}{\Delta x} = 12$$

$$F_i = \int_0^1 \phi_i(x)\, dx = \frac{1}{2}(2\Delta x)\cdot 1 = \Delta x = 1/4$$

$$Ku = F$$

$$\begin{pmatrix} 8 & -4 & 0 \\ -4 & 12 & -8 \\ 0 & -8 & 16 \end{pmatrix} u = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$$

5) $f(x) = 1 + \sin(4x)$

$$= 1e^{i0x} + \frac{1}{2i}e^{4ix} - \frac{1}{2i}e^{-4ix}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

Observe $\hat{f}(0) = 1$

$\hat{f}(4) = 1/2i$

$\hat{f}(-4) = -1/2i$

$\hat{f}(k) = 0$ for all other k.

6) $$-\frac{d^2 v}{dx^2} = e^{4ix} + e^{-3ix}$$

$$v(0) = v(2\pi)$$

$$v'(0) = v'(2\pi)$$

Identify $f(x) = e^{4ix} + e^{-3ix}$

$$\hat{f}(n) = \begin{cases} 1 & \text{if } n=4 \text{ or } n=-3 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

$$v(x) = \sum_{n=-\infty}^{\infty} \hat{v}(n) e^{inx}$$

$$-\frac{d^2 v}{dx^2} = -\sum_{n=-\infty}^{\infty} -n^2 \hat{v}(n) e^{inx}$$

$$+n^2 \hat{v}(n) = \hat{f}(n)$$

$$\Rightarrow \quad \hat{v}(n) = \begin{cases} \frac{1}{4^2} & \text{if } n=4 \\ \frac{1}{3^2} & \text{if } n=-3 \\ \text{undefined} & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$

$$v(x) = \frac{1}{16} e^{4ix} + \frac{1}{9} e^{-3ix} + C$$

7) $$\begin{cases} -\dfrac{d^2U}{dX^2} = \delta(X - L/3) \\ U(0) = 0 \\ U(L) = 0 \end{cases}$$

a) For $0 < X < L/3$

$U(X) = a + bX$ because $\dfrac{d^2U}{dx^2} = 0$

$U(0) = 0 \Rightarrow U(X) = bX$

For $\frac{L}{3} < X < L$

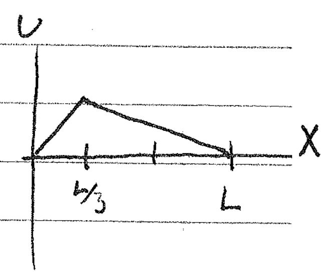
$U(X) = c + d(X - L)$

$U(L) = 0 \Rightarrow c = 0 \qquad U = c(X - L)$

Impose continuity

$U(\frac{L}{3}^-) = U(\frac{L}{3}^+)$

$b\frac{L}{3} = c\left(-\frac{2L}{3}\right) \Rightarrow b = -2c$



Jump in derivative:

$-\left[\dfrac{dU}{dX}\right]_{X = L/3} = 1$

$-c + b = 1$

$\Rightarrow b = 2/3$
$c = -1/3$

$\Rightarrow U(X) = \begin{cases} \frac{2}{3}X & 0 < X < \frac{1}{3} \\ -\frac{1}{3}(X - L) & \frac{1}{3} < X < 1 \end{cases}$

7b)

$$f(x) = \delta(x - L/3) = \sum_{n=1}^{\infty} \hat{f}(n) \sin\frac{n\pi x}{L}$$

$$\hat{f}(n) = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} = \frac{2}{L} \sin\frac{n\pi\, L/3}{L}$$

$$= \frac{2}{L} \sin n\pi/3$$

$$U(x) = \sum_{n=1}^{\infty} \hat{U}(n) \sin\frac{n\pi x}{L}$$

$$-\frac{d^2}{dx^2} U = \sum_{n=1}^{\infty} \hat{U}(n) \frac{n^2\pi^2}{L^2} \sin\frac{n\pi x}{L}$$

So $$\hat{U}(n) = \frac{2}{L} \frac{\sin n\pi/3}{n^2\pi^2/L^2}$$

$$\hat{U}(n) = \frac{2L}{n^2\pi^2} \sin n\pi/3$$

$$U(x) = \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2} \sin n\pi/3 \sin\frac{n\pi x}{L}$$

This is the Fourier Sine Series of the function computed in 7a)

8

$$f(x) = \sum_{n=1}^{\infty} \hat{f}(n)\, \phi_n(x)$$

$$\langle f(x), \phi_m^{(x)} \rangle = \sum_{n=1}^{\infty} \langle \hat{f}(n)\phi_n(x), \phi_m(x) \rangle$$

$$\langle f, \phi_m \rangle = \sum_{n=1}^{\infty} \hat{f}(n) \underbrace{\langle \phi_n(x), \phi_m(x) \rangle}_{= 0 \text{ if } n \neq m}$$

because ϕ_n is orthogonal basis

$$\hat{f}(n) = \frac{\langle f, \phi_m \rangle}{\langle \phi_m, \phi_m \rangle}$$

$$= \frac{\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx}{\int_0^L \sin^2 \frac{n\pi x}{L}\, dx}$$

$f(x) = 1$

$$\hat{f}(n) = \frac{\int_0^L \sin \frac{n\pi x}{L}\, dx}{\int_0^L \sin^2 \frac{n\pi x}{L}\, dx}$$

$$\int_0^L \sin^2 \frac{n\pi x}{L}\, dx = \frac{L}{2} \quad (\sin^2 \text{ averages to } \tfrac{1}{2})$$

$$\int_0^L \sin \frac{n\pi x}{L}\, dx = \frac{-L}{n\pi} \cos \frac{n\pi x}{L} \Big|_0^L = \frac{-L}{n\pi}(\cos n\pi - 1)$$

$$= \begin{cases} \frac{2L}{n\pi} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

$$f(x) = \boxed{1 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{L}\frac{2L}{n\pi} \sin \frac{n\pi x}{L}} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi x}{L}$$