Heat asymptotics for Lévy processes.

Rodrigo Bañuelos

Department of Mathematics
Purdue University
West Lafayette, IN 47907

May 29, 2009
Laplacian in regions $D \subset \mathbb{R}^d$. Always $|D| < \infty$ and $|\partial D| < \infty$.

$D$ open connected finite volume, $\Delta_D$ Dirichlet Laplacian.

$$Z_D(t) = \text{trace}(e^{t\Delta_D}) = \sum_{j=0}^{\infty} e^{-t\lambda_j} = \int_D p_t^D(x, x) dx = \frac{1}{(4\pi t)^{d/2}} \int_D P_x\{\tau_D > t|X_t = x\} dx,$$

$\tau_D$ exit time from $D$ of Brownian motion. In fact,

$$p_t^D(x, y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}} P_x\{\tau_D > t|B_t = y\} = p_t(x-y) - \mathbb{E}^x(\tau_D < t, p_{t-\tau_D}(X(\tau_D), y)) = p_t(x-y) - r_t^D(x, y).$$

The function $r_t^D(x, y)$ is called a killing measure.
Theorem (M. Kac '51 (?) )

For any $D \subset \mathbb{R}^d$ of finite volume

$$\lim_{t \to 0} t^{d/2} Z_D(t) = \frac{|D|}{(4\pi)^{d/2}} = p_1(0)|D|$$

Corollary

Then (Karamata tauberian theorem)

$$\lim_{t \to 0} t^\gamma \int_0^\infty e^{-t\lambda} d\mu(\lambda) = A \Rightarrow \lim_{a \to \infty} a^{-\gamma} \mu[0, a] = \frac{A}{\Gamma(\gamma + 1)}$$

gives Weyl’s asymptotics:

$$\lim_{\lambda \to \infty} \lambda^{-d/2} N(\lambda) = \frac{p_1(0)|D|}{\Gamma(d/2 + 1)}$$

$N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ which not exceeding $\lambda$
Theorem (Minakshisundaram ’53–heat invariance)

$D \subset \mathbb{R}^d$ bounded “smooth”. Then

$$Z_D(t) - \frac{1}{(4\pi t)^{d/2}} \sum_{j=0}^{m} c_j t^{j/2} = O(t^{(m-d+1)/2}), \quad t \downarrow 0$$

$$c_1 = |D|, \quad c_2 = -\frac{\sqrt{\pi}}{2} |\partial D|.$$

Theorem (McKean ’67)

$D \subset \mathbb{R}^2$ with $r$ holes. Then

$$\lim_{t \downarrow 0} \left\{ Z_D(t) - \frac{|D|}{4\pi t} + \frac{|\partial D|}{4(4\pi t)^{1/2}} \right\} = \frac{(1 - r)}{6}$$

Theorem ($C^1$-domains: Brossard-Carmona ’86. Lipschitz domains: R. Brown ’93.)

$$Z_D(t) = (4\pi t)^{-d/2} \left( |D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o(t^{1/2}) \right), \quad t \downarrow 0$$
Uniform bounds. There are many

For all Smooth Bounded Convex domains:

\[
\frac{|D|}{(4\pi t)^{d/2}} - \frac{e^{d/2}|\partial D|}{(4\pi t)^{(d-1)/2}} \leq Z_D(t) \leq \frac{|D|}{(4\pi t)^{d/2}}, \quad t > 0
\]

For Smooth Bounded Convex with mean curvature bounded by \( \frac{1}{R} \)

\[
\left| Z_D(t) - \frac{|D|}{(4\pi t)^{d/2}} + \frac{|\partial D|}{4(4\pi t)^{(d-1)/2}} \right| \leq \frac{|\partial D|}{t^{(d-2)/2}} \left\{ C_1 + C_2 \log \left( 1 + \frac{R^2}{t} \right) \right\}
\]

Theorem (van den Berg ’87–sharp in \( t \) and “degree of smoothness”)

If \( \partial D \) satisfies uniform inner and outer ball condition with radius \( R \)

\[
\left| Z_D(t) - (4\pi t)^{-d/2} \left( |D| - \frac{\sqrt{\pi t}}{2} |\partial D| \right) \right| \leq \frac{d^4}{\pi^{d/2}} \frac{|D|t}{t^{d/2} R^2}, \quad t > 0.
\]
Problem: Investigate similar properties for “other” Lévy processes, and especially those subordinate to Brownian motion whose generators are simple transformations of the Laplacian.
Problem: Investigate similar properties for “other” Lévy processes, and especially those subordinate to Brownian motion whose generators are simple transformations of the Laplacian.

Definition

A Lévy Process is a stochastic process \( X = (X_t), t \geq 0 \) with

- \( X \) has independent and stationary increments
- \( X_0 = 0 \) (with probability 1)
- \( X \) is stochastically continuous: For all \( \varepsilon > 0 \),
  \[
  \lim_{t \to s} P\{|X_t - X_s| > \varepsilon\} = 0
  \]

Note: Not the same as a.s. continuous paths. However, it gives “cadlag” paths: Right continuous with left limits.
• **Stationary increments:** \(0 < s < t < \infty, A \in \mathbb{R}^d\) Borel

\[
P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}
\]

• **Independent increments:** For any given sequence of ordered times

\[
0 < t_1 < t_2 < \cdots < t_m < \infty,
\]

the random variables

\[
X_{t_1} - X_0, X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}
\]

are independent.

The characteristic function of \(X_t\) is

\[
\varphi_t(\xi) = E\left(e^{i\xi \cdot X_t}\right) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} p_t(dx) = (2\pi)^{d/2} \hat{p}_t(\xi)
\]

where \(p_t\) is the distribution of \(X_t\). Notation (same with measures)

\[
\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(x)dx, \quad f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(\xi)d\xi
\]
The characteristic function has the form
\[ \varphi_t(\xi) = e^{t\rho(\xi)}, \]
where
\[ \rho(\xi) = ib \cdot \xi - \frac{1}{2} \xi \cdot A \xi + \int_{\mathbb{R}^d} \left( e^{i\xi \cdot x} - 1 - i\xi \cdot x 1_{\{|x| < 1\}}(x) \right) \nu(dx) \]
for some \( b \in \mathbb{R}^d \), a non–negative definite symmetric \( n \times n \) matrix \( A \) and a Borel measure \( \nu \) on \( \mathbb{R}^d \) with \( \nu\{0\} = 0 \) and
\[ \int_{\mathbb{R}^d} \min(|x|^2, 1) \nu(dx) < \infty \]
\( \rho(\xi) \) is called the **symbol** of the process or the **characteristic exponent**. The triple \((b, A, \nu)\) is called the **characteristics of the process**.

Converse also true. Given such a triple we can construct a Lévy process.
Example (The rotationally invariant stable processes:)

These are self–similar processes, denoted by $X_t^\alpha$, in $\mathbb{R}^d$ with symbol

$$\rho(\xi) = -|\xi|^\alpha, \quad 0 < \alpha \leq 2.$$ 

$\alpha = 2$ is Brownian motion. $\alpha = 1$ is the Cauchy processes.

Example (Relativistic Brownian motion)

According to quantum mechanics, a particle of mass $m$ moving with momentum $p$ has kinetic energy

$$E(p) = \sqrt{m^2c^4 + c^2|p|^2} - mc^2$$

where $c$ is speed of light. Then $\rho(p) = -E(p)$ is the symbol of a Lévy process, called “relativistic Brownian motion.”

In fact, these are Lévy processes of the form $X_t = B_{T_t}$ where $B_t$ is Brownian motion and $T_t$ is a “subordinator” independent of $B_t$. 
Example (Subordinators)

A subordinator is a one-dimensional Lévy process \( \{T_t\} \) such that

(i) \( T_t \geq 0 \) a.s. for each \( t > 0 \)

(ii) \( T_{t_1} \leq T_{t_2} \) a.s. whenever \( t_1 \leq t_2 \)

Theorem (Laplace transforms)

\[
E(e^{-\lambda T_t}) = e^{-t \psi(\lambda)}, \quad \lambda > 0,
\]

\[
\psi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda s}) \nu(ds)
\]

\( b \geq 0 \) and the Lévy measure satisfies \( \nu(-\infty, 0) = 0 \) and \( \int_0^\infty \min(s, 1) \nu(ds) < \infty \). \( \psi \) is called the Laplace exponent of the subordinator.

Example (\( \alpha/2 \)-Stable subordinator)

\[\psi(\lambda) = \lambda^{\alpha/2}, \quad 0 < \alpha < 2\] gives the stable with \( b = 0 \) and

\[
\nu(ds) = \frac{\alpha/2}{\Gamma(1 - \alpha/2)} s^{-1-\alpha/2} ds
\]
Example (Relativistic stable subordinator:)

$0 < \alpha < 2$ and $m > 0$, $\Psi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$.

$$\nu(ds) = \frac{\alpha/2}{\Gamma(1 - \alpha/2)} e^{-m^{2/\alpha}s} s^{-1-\alpha/2} ds$$

Many others: “Gamma subordinators, Geometric stable subordinators, iterated geometric stable subordinators, Bessel subordinators,…" 

Theorem

If $X$ is an arbitrary Lévy process and $T$ is a subordinator independent of $X$, then $Z_t = X_{T_t}$ is a Lévy process. For any Borel $A \subset \mathbb{R}^d$,

$$p_{Z_t}(A) = \int_0^{\infty} p_{X_s}(A) p_{T_t}(ds)$$
\[ P^x (X_t^\alpha \in A) = \int_A p_t^\alpha (x - y) dy, \quad p_t^\alpha (x) = t^{-d/\alpha} p_1^\alpha \left( \frac{x}{t^{1/\alpha}} \right). \]

Heat Semigroup in \( D \) is the self-adjoint operator

\[ T_t^D f(x) = E_x \left[ f(X_t^\alpha); \tau_D > t \right] = \int_D p_t^{D,\alpha} (x, y) f(y) dy, \]

\[ p_t^{D,\alpha} (x, y) \leq p_t^\alpha (x - y) \leq p_1^\alpha (0) t^{-d/\alpha} \]

\[ = \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\xi|\alpha} d\xi \right) t^{-d/\alpha} = t^{-d/\alpha} \frac{\omega_d (d/\alpha)}{(2\pi)^d} \int_0^\infty e^{-s} s^{(n/\alpha - 1)} ds \]

\[ = t^{-d/\alpha} \omega_d \Gamma (d/\alpha) \left( \frac{1}{2\pi} \right)^d \alpha, \quad \omega_d = \sigma (S_d) \]

As before,

\[ p_t^{D,\alpha} (x, y) = p_t^\alpha (x - y) - E^x \left( \tau_D < t, p_t^\alpha (X(\tau_D), y) \right) \]

\[ = p_t^\alpha (x - y) - r_t^{D,\alpha} (x, y). \]
Symmetric Stable, $0 < \alpha < 2$

Two expressions for the free heat kernel: $g_{\alpha/2}(t, s) = \text{density of } T_t$.

$$p_t^\alpha(x) = \int_0^\infty p_s^{(2)}(x) g_{\alpha/2}(t, s) ds = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-|x|^2/4s} g_{\alpha/2}(t, s) ds$$

and

$$p_t^\alpha(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t|\xi|^\alpha} d\xi$$

This leads to:

$$p_t^\alpha(x - y) \leq c \left( \frac{t}{|x - y|^{d+\alpha}} \wedge \frac{1}{t^{d/\alpha}} \right), \quad x, y \in \mathbb{R}^d, \ t > 0$$

and

$$r_t^{D,\alpha}(x, x) \leq c \left( \frac{t}{\delta_D^d(x) + \alpha} \wedge \frac{1}{t^{d/\alpha}} \right), \quad x \in D, \ t > 0$$
Relativistic Symmetric Stable, $0 < \alpha < 2$, $m > 0$

Two expressions for the “free density"

$$p_t^{\alpha, m}(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|x|^2}{4s}} e^{(-m^{1/\beta} s)} g_{\alpha/2}(t, s) ds,$$

$$p_t^{\alpha, m}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-t\left\{\left(m^{2/\alpha} + |\xi|^2\right)^{\alpha/2} - m\right\}} d\xi$$

$$p_t^{\alpha, m}(x - y) \leq c(\alpha, d) \left\{ \frac{m^{d/\alpha - d/2}}{t^{d/2}} + \frac{1}{t^{d/\alpha}} \right\}, \quad x, y \in \mathbb{R}^d, \ t > 0$$
Relativistic Symmetric Stable, \(0 < \alpha < 2, \ m > 0\)

Two expressions for the “free density”

\[
p_t^{\alpha,m}(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-|x|^2/4s} e^{-(m^{1/\beta}s) \frac{g_{\alpha/2}(t, s)}{2}} ds,
\]

\[
p_t^{\alpha,m}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-t\left\{\left(\frac{m^2}{\alpha} + |\xi|^2\right)^{\alpha/2} - m\right\}} d\xi
\]

\[
p_t^{\alpha,m}(x - y) \leq c(\alpha, d) \left\{ \frac{m^{d/\alpha - d/2}}{t^{d/2}} + \frac{1}{t^{d/\alpha}} \right\}, \quad x, y \in \mathbb{R}^d, \ t > 0
\]

\[
p_t^{\alpha,m}(x - y) \leq c_1 e^{mt} \left\{ \frac{t e^{-c_2|x-y|}}{|x-y|^{d+\alpha}} \wedge \frac{1}{t^{d/\alpha}} \right\}, \quad x, y \in \mathbb{R}^d, \ t > 0
\]

\[
r_t^{D,\alpha,m}(x, x) \leq c_1 e^{mt} \left\{ \frac{t e^{-c_2\delta_D(x)}}{\delta_D(x)^{d+\alpha}} \wedge \frac{1}{t^{d/\alpha}} \right\}, \quad x \in D, \ t > 0
\]

\[
\lim_{t \to 0} p_t^{\alpha,m}(0) e^{-mt} t^{d/\alpha} = C_1(\alpha, d) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d/\alpha}}, \quad \omega_d = \sigma(S_d)
\]
Trace, stable and relativistic stable (we drop the $\alpha$, and $m$)

\[
Z_D(t) = \int_D p_D(t, x, x)\,dx = \int_D p_t(x - x)\,dx - \int_D r_t^D(x, x)\,dx
\]

\[
= p_t(0)|D| - \int_D r_t^D(x, x)\,dx
\]

Lemma (Both Stable and Relativistic Stable)

\[
\lim_{t \to 0} t^{d/\alpha} \int_D r_t^D(x, x)\,dx = 0
\]

Proof.

Recall $t^{d/\alpha} r_t^D(x, x) \leq C\left(\frac{t^{d/\alpha + 1}}{\delta_D^d(x)} \wedge 1\right)$. Set $D_t = \{x \in D : d_d(x) > t^{1/2\alpha}\}$. Then

\[
t^{d/\alpha} \int_{D \setminus D_t} r_t^D(x, x)\,dx \leq C|D \setminus D_t|,
\]

\[
t^{d/\alpha} \int_{D_t} r_t^D(x, x)\,dx \leq Ct^{d/2\alpha + 1/2}|D|, \quad t \ll 1
\]
Corollary (For any set of finite volume $D$)

$$\lim_{t \to 0} t^{d/\alpha} Z_D(t) = C_1(\alpha, d)|D| = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha} |D|, \quad \text{Stable}$$

$$\lim_{t \to 0} t^{d/\alpha} e^{-mt} Z_D(t) = C_1(\alpha, d)|D| = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha} |D|, \quad \text{Relativistic Stable}$$

Stable proved under assumption $vol_d(\partial D) = 0$ by Blumenthal-Getoor 1959.

Corollary

Gives Weyl’s asymptotics:

$$\lim_{\lambda \to \infty} \lambda^{-d/\alpha} N(\lambda) = \frac{C_1(\alpha, d)|D|}{\Gamma(d/2 + 1)}, \quad \text{Stable}$$

and

$$\lim_{\lambda \to \infty} \lambda^{-d/\alpha} e^{m/\lambda} N(\lambda) = \frac{C_1(\alpha, d)|D|}{\Gamma(d/2 + 1)}, \quad \text{Relativistic Stable}$$

$N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ which not exceeding $\lambda$
From now on, only $\alpha$-stable, $0 < \alpha < 2$

**Theorem ($R$-smooth domains: B.–Kulczycki ’08)**

$$\left| Z_D(t) - \frac{C_1(\alpha, d)|D|}{t^{d/\alpha}} + \frac{C_2(\alpha, d)|\partial D|t^{1/\alpha}}{t^{d/\alpha}} \right| \leq \frac{C_3|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}, \ t > 0.$$

**Theorem (Lipschitz domains: B.–Kulczycki–Siudeja (preprint))**

$$t^{d/\alpha} Z_D(t) = C_1(\alpha, d)|D| - C_2(\alpha, d)|\partial D|t^{1/\alpha} + o\left(t^{1/\alpha}\right), \ t \downarrow 0$$

\[ C_1(\alpha, d) = p_1^\alpha(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}, \]
\[ C_2(\alpha, d) = \int_0^\infty r_1^H(q, 0, \ldots, 0), (q, 0, \ldots, 0))dq, \text{ where } H = \{x : x_1 > 0\}. \]
Lemma (A geometric property of $R$–smooth domains)

Let $D \subset \mathbb{R}^d$ be $R$-smooth. Set $D_q = \{x \in D : d_D(x) > q\}$. Then for any $0 < q \leq R/2$

(i) \[ 2^{-d+1} |\partial D| \leq |\partial D_q| \leq 2^{d-1} |\partial D|, \]

(ii) \[ |\partial D| \leq \frac{2^d |D|}{R}, \]

(iii) \[ |\partial D_q| - |\partial D| \leq \frac{2^d d q |\partial D|}{R} \leq \frac{2^{2d} d q |D|}{R^2}. \]
Proposition \((t^{1/\alpha} > R/2)\)

\[
Z_D(t) \leq \frac{C_1 |D|}{t^{d/\alpha}} \leq \frac{C_1 |D| t^{2\alpha}}{R^2 t^{d/\alpha}}
\]

and by (ii),

\[
\frac{C_2 |\partial D| t^{1/\alpha}}{t^{d/\alpha}} \leq \frac{2^d C_2 |D| t^{1/\alpha}}{R t^{d/\alpha}} \leq \frac{2^{d+1} C_2 |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}
\]

This implies Theorem for \(t^{1/\alpha} > R/2\).

\[
Z_D(t) - \frac{C_1 |D|}{t^{d/\alpha}} = - \int_D r_t^D(x, x) dx = - \int_{D_{R/2}} r_t^D(x, x) dx - \int_{D \setminus D_{R/2}} r_t^D(x, x) dx
\]

As before, for \(t^{1/\alpha} \leq R/2\),

\[
\int_{D_{R/2}} r_t^D(x, x) dx \leq \frac{C |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}
\]
Lemma

For $x \in D \setminus D_{R/2}$, let $x_* \in \partial D$ with $d_D(x) = |x - x_*|$. Let $B_1(z_1, R)$ and $B_2(z_2, R)$ be the balls of radius $R$ passing through $x_*$ with $B_1 \subset D$ and $B_2 \subset D^c$. Let $H(x)$ be the half space containing $B_1$ perpendicular to $z_1z_2$. For $t^{1/\alpha} < R$,

$$\left| \int_{D \setminus D_{R/2}} r_t^D(x, x) \, dx - \int_{D \setminus D_{R/2}} r_t^{H(x)}(x, x) \, dx \right| \leq \frac{C |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}$$

Recall

$$H = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_1 > 0\}$$

Set

$$f_H(t, q) = r_t^H((q, 0, \ldots, 0), (q, 0, \ldots, 0)), \quad q > 0$$

Then,

$$r_t^{H(x)}(x, x) = f_H(t, d_H(x)(x))$$

and

$$f_H(t, q) = t^{-d/\alpha} f_H(1, qt^{-1/\alpha}), \quad f_H(1, q) \leq c(q^{-d-\alpha} \land 1).$$
\[
\int_{D \setminus D_{R/2}} r_t^{H(x)}(x, x) \, dx = \int_0^{R/2} |\partial D_u| f_H(t, u) \, du \\
= \frac{1}{t^{d/\alpha}} \int_0^{R/2} |\partial D_u| f_H(1, ut^{-1/\alpha}) \, du \\
= \frac{t^{1/\alpha}}{t^{d/\alpha}} \int_0^{R/(2t^{1/\alpha})} |\partial D_{t^{1/\alpha}q}| f_H(1, q) \, dq,
\]

For \( R \)-smooth regions,

\[
\frac{t^{1/\alpha}}{t^{d/\alpha}} \int_0^{R/(2t^{1/\alpha})} \left| \left| \partial D_{t^{1/\alpha}q} \right| - |\partial D| \right| f_H(1, q) \, dq \leq \frac{c|D|t^{2/\alpha}}{R^2t^{d/\alpha}} \int_0^{R/(2t^{1/\alpha})} qf_H(1, q) \, dq \\
\leq \frac{c|D|t^{2/\alpha}}{R^2t^{d/\alpha}} \int_0^\infty q(q^{d-\alpha} \wedge 1) \, dq \\
\leq \frac{c|D|t^{2/\alpha}}{R^2t^{d/\alpha}}.
\]
Remains to show:

\[
\left| \frac{t^{1/\alpha} |\partial D|}{t^{d/\alpha}} \right| \int_0^{R/(2t^{1/\alpha})} f_H(1, q) \, dq - \left| \frac{t^{1/\alpha} |\partial D|}{t^{d/\alpha}} \right| \int_0^\infty f_H(1, q) \, dq \leq \frac{c |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}.
\]

or

\[
\left| \frac{t^{1/\alpha} |\partial D|}{t^{d/\alpha}} \right| \int_0^{R/(2t^{1/\alpha})} f_H(1, q) \, dq \leq \frac{c |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}.
\]
Remains to show:

\[
\left| \frac{t^{1/\alpha} \partial D}{td/\alpha} \int_{0}^{R/(2t^{1/\alpha})} f_H(1, q) \, dq - \frac{t^{1/\alpha} \partial D}{td/\alpha} \int_{0}^{\infty} f_H(1, q) \, dq \right| \leq \frac{c |D| t^{2/\alpha}}{R^2 td/\alpha}.
\]

or

\[
\left| \frac{t^{1/\alpha} \partial D}{td/\alpha} \int_{R/(2t^{1/\alpha})}^{\infty} f_H(1, q) \, dq \right| \leq \frac{c |D| t^{2/\alpha}}{R^2 td/\alpha}.
\]

Recall: \( R/(2t^{1/\alpha}) \geq 1 \). Thus, for \( q \geq R/(2t^{1/\alpha}) \) we have

\[
f_H(1, q) \leq cq^{-d-\alpha} \leq cq^{-2},
\]

\[
\Rightarrow \int_{R/(2t^{1/\alpha})}^{\infty} f_H(1, q) \, dq \leq c \int_{R/(2t^{1/\alpha})}^{\infty} \frac{dq}{q^2} \leq \frac{ct^{1/\alpha}}{R}.
\]

Again, use

\[
|\partial D| \leq \frac{2^d |D|}{R},
\]

to conclude.