Solving Large Scale Lyapunov Equations

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Thanks:

- Mark Embree
- John Sabino
- Kai Sun
- NLA Seminar (CAAM 651)

IWASEP *VI*, Penn State U., 22 May 2006

The Lyapunov Equation

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$, $p << n$

Assumptions: $\mathbf{A}$ is stable and $\mathbf{A} + \mathbf{A}^T \prec \mathbf{0}$

$$\mathcal{P} = \mathcal{P}^T \succeq \mathbf{0}$$

Computation in Dense Case (Schur Decomposition)

- Bartels and Stewart (72),
- Hammarling (82) – Factored Form Soln. $\mathcal{P} = \mathbf{L}\mathbf{L}^T$

Outline

- Model Reduction for Dynamical Systems
- Balanced Truncation and Lyapunov Equations
- APM and ADI for Large Scale Lyapunov Equations
- A Parameter Free Algorithm
- A Special Sylvester Equation Solver
- Implementation Issues
- Computational Results

LTI Systems and Model Reduction

Time Domain

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

$$\mathbf{A} \in \mathbb{R}^{n\times n}, \quad \mathbf{B} \in \mathbb{R}^{n\times m}, \quad \mathbf{C} \in \mathbb{R}^{p\times n}, \quad n >> m, p$$

Frequency Domain

$$\begin{aligned} s\mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

Transfer Function

$$\mathbf{H}(s) \equiv \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \qquad \mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s)$$

Model Reduction

Construct a new system $\{\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}\}$ with LOW dimension $k << n$

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\ \hat{\mathbf{y}} &= \hat{\mathbf{C}}\hat{\mathbf{x}} \end{aligned}$$

Goal: Preserve system response

$$\underline{\hat{\mathbf{y}} \ \text{ should approximate } \ \mathbf{y}}$$

Projection: $\mathbf{x}(t) = \mathbf{V}\hat{\mathbf{x}}(t)$ and $\mathbf{V}\dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{V}\hat{x} + \mathbf{B}\mathbf{u}$

Model Reduction by Projection

Krylov Projection Often Used
Approximate $\mathbf{x} \in \mathcal{S}_V = \mathit{Range}(\mathbf{V})$ k-diml. subspace
i.e. Put $\mathbf{x} = \mathbf{V}\hat{\mathbf{x}}$, and then force

$$\begin{aligned} \mathbf{W}^T[\mathbf{V}\dot{\hat{\mathbf{x}}} &- (\mathbf{A}\mathbf{V}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u})] = 0 \\ \hat{\mathbf{y}} &= \mathbf{C}\mathbf{V}\hat{\mathbf{x}} \end{aligned}$$

If $\mathbf{W}^T\mathbf{V} = \mathbf{I}_k$, then the k dimensional reduced model is

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\ \hat{\mathbf{y}} &= \hat{\mathbf{C}}\hat{\mathbf{x}} \end{aligned}$$

where $\hat{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\hat{\mathbf{B}} = \mathbf{W}^T\mathbf{B}$, $\hat{\mathbf{C}} = \mathbf{C}\mathbf{V}$.

Balanced Reduction (Moore 81)

Lyapunov Equations for system Gramians

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0$$

With $\mathcal{P} = \mathcal{Q} = \mathbf{S}$: Want Gramians Diagonal and Equal

States Difficult to Reach are also Difficult to Observe

Reduced Model $\mathbf{A}_k = \mathbf{W}_k^T\mathbf{A}\mathbf{V}_k$, $\mathbf{B}_k = \mathbf{W}_k^T\mathbf{B}$, $\mathbf{C}_k = \mathbf{C}_k\mathbf{V}_k$

- $\mathcal{P}\mathbf{V}_k = \mathbf{W}_k\mathbf{S}_k$ $\qquad$ $\mathcal{Q}\mathbf{W}_k = \mathbf{V}_k\mathbf{S}_k$
- Reduced Model Gramians $\mathcal{P}_k = \mathbf{S}_k$ and $\mathcal{Q}_k = \mathbf{S}_k$.

Hankel Norm Error estimate (Glover 84)

Why Balanced Truncation?

- Hankel singular values $= \sqrt{\lambda(\mathcal{P}\mathcal{Q})}$
- Model reduction $\mathcal{H}_\infty$ error (Glover)

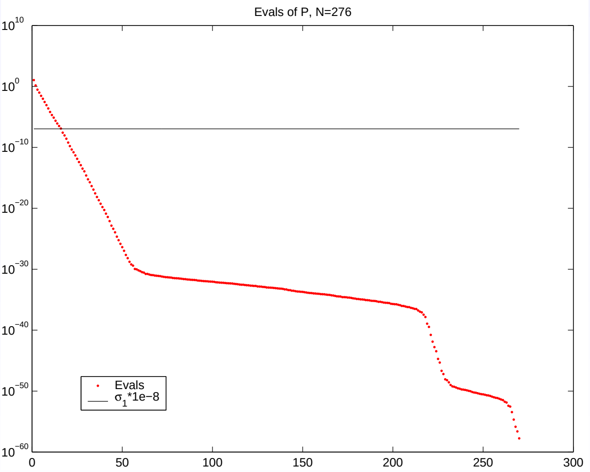
$$\|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq 2 \times (\text{sum neglected singular values})\|u\|_2$$

- Extends to MIMO
- Preserves Stability

Key Challenge

- Approximately solve large scale Lyapunov Equations in Low Rank Factored Form

When are Low Rank Solutions Expected = Eigenvalue Decay



Eigenvalues of $\mathcal{P}$ Often Decay Rapidly

Estimates from Analysis:

- Penzl (00) , Symmetric **A**, must know $\kappa(\mathbf{A})$
- Zhou, Antoulas, S (02) , Nonsymmetric **A**, must know $\sigma(\mathbf{A})$
- Beattie
- Sabino and Embree (in progress)

Approximate Balancing

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0$$

- Sparse Case: Iteratively Solve in Low Rank Factored Form,

$$\mathcal{P} \approx \mathbf{U}_k\mathbf{U}_k^T, \quad \mathcal{Q} \approx \mathbf{L}_k\mathbf{L}_k^T$$

$$[\mathbf{X}, \mathbf{S}, \mathbf{Y}] = \mathrm{svd}(\mathbf{U}_k^T\mathbf{L}_k)$$

$$\mathbf{W}_k = \mathbf{L}\mathbf{Y}_k\mathbf{S}_k^{-1/2} \text{ and } \mathbf{V}_k = \mathbf{U}\mathbf{X}_k\mathbf{S}_k^{-1/2}.$$

Now: $\mathcal{P}\mathbf{W}_k \approx \mathbf{V}_k\mathbf{S}_k$ and $\mathcal{Q}\mathbf{V}_k \approx \mathbf{W}_k\mathbf{S}_k$

Balanced Reduction via Projection

Reduced model of order k:

$$\mathbf{A}_k = \mathbf{W}_k^T \mathbf{A} \mathbf{V}_k, \quad \mathbf{B}_k = \mathbf{W}_k^T \mathbf{B}, \quad \mathbf{C}_k = \mathbf{C} \mathbf{V}_k.$$

$$\begin{aligned} 0 = \mathbf{W}_k^T(\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)\mathbf{W}_k &= \mathbf{A}_k\mathbf{S}_k + \mathbf{S}_k\mathbf{A}_k^T + \mathbf{B}_k\mathbf{B}_k^T \\ 0 = \mathbf{V}_k^T(\mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C})\mathbf{V}_k &= \mathbf{A}_k^T\mathbf{S}_k + \mathbf{S}_k\mathbf{A}_k + \mathbf{C}_k^T\mathbf{C}_k \end{aligned}$$

Reduced model is balanced and asymptotically stable for every k.

Large Scale Lyapunov Equations

- Iterative - Wachpress
- Krylov - Saad; Hu and Reichel
- Quadrature - Gudmundsson and Laub
- Subspace Iteration - Hodel and Tennison, VanDooren, Y. Zhou and S.

Balanced Reduction – Large Scale:

- Lanczos - Golub and Boley
- Restarting – Kasenally et. al.
- Low Rank Lyapunov – Penzl,Li, White , Benner

Low Rank Smith Method

R.A. Smith (68) Variant of ADI

- Wachpress(88)
- Calvetti, Levenberg, Reichel (97)
- Penzl (99) (00)
- Li, White, et. al. (00)
- Gugercin, Antoulas, S (03)

ADI for Lyapunov Equations

From original equation

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0$$

Apply shift $\mu > 0$ from left

$$\mathcal{P} = -(\mathbf{A} - \mu\mathbf{I})^{-1}\left[\mathcal{P}(\mathbf{A} + \mu\mathbf{I})^T + \mathbf{B}\mathbf{B}^T\right]$$

Apply shift μ from right (Alternate Direction)

$$\mathcal{P} = -\left[(\mathbf{A} + \mu\mathbf{I})\mathcal{P} + \mathbf{B}\mathbf{B}^T\right](\mathbf{A} - \mu\mathbf{I})^{-T}$$

Combine these into one step: Get a Stein Equation

Low Rank Smith = ADI

Convert to Stein Equation:

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \Longleftrightarrow \quad \mathcal{P} = \mathbf{A}_\mu \mathcal{P} \mathbf{A}_\mu^T + \mathbf{B}_\mu \mathbf{B}_\mu^T,$$

where

$$\mathbf{A}_\mu = (\mathbf{A} + \mu \mathbf{I})(\mathbf{A} - \mu \mathbf{I})^{-1}, \quad \mathbf{B}_\mu = \sqrt{2|\mu|}(\mathbf{A} - \mu \mathbf{I})^{-1}\mathbf{B}.$$

Solution:

$$\mathcal{P} = \sum_{j=0}^{\infty} \mathbf{A}_\mu^j \mathbf{B}_\mu \mathbf{B}_\mu^T (\mathbf{A}_\mu^j)^T = \mathbf{L}\mathbf{L}^T,$$

where $\mathbf{L} = [\mathbf{B}_\mu, \ \mathbf{A}_\mu \mathbf{B}_\mu, \ \mathbf{A}_\mu^2 \mathbf{B}_\mu, \ \ldots]$ Factored Form

Convergence - Low Rank Smith

$$\mathcal{P}_m = \sum_{j=0}^{m} \mathbf{A}_\mu^j \mathbf{B}_\mu \mathbf{B}_\mu^T (\mathbf{A}_\mu^j)^T$$

$$\mathcal{P}_{m+1} = \mathbf{A}_\mu \mathcal{P}_m \mathbf{A}_\mu^T + \mathbf{B}_\mu \mathbf{B}_\mu^T$$

Easy:

$$\mathcal{E}_{m+1} = \mathbf{A}_\mu \mathcal{E}_m \mathbf{A}_\mu^T = \mathbf{A}_\mu^{m+2} \mathcal{P} (\mathbf{A}_\mu^{m+2})^T \rightarrow 0,$$

$$\text{where} \;\; \mathcal{E}_\mathrm{m} = \mathcal{P} - \mathcal{P}_\mathrm{m}.$$

Note: $\rho(\mathbf{A}_\mu) < 1.$

Modified Low Rank Smith

LR - Smith: Update Factored Form $\mathcal{P}_m = L_m L_m^T$: (*Penzl*)

$$\begin{aligned} \mathbf{L}_{m+1} &= [\mathbf{A}_\mu \mathbf{L}_m, \mathbf{B}_\mu] \\ &= [\mathbf{A}_\mu^{m+1} \mathbf{B}_\mu, \mathbf{L}_m] \end{aligned}$$

Modified LR - Smith:

Update and Truncate SVD

Re-Order and Aggregate Shift Applications

Much Faster and Far Less Storage

$$\begin{aligned} \mathbf{B} &\leftarrow \mathbf{A}_\mu \mathbf{B}; \\ [\mathbf{V}, \mathbf{S}, \mathbf{Q}] &= \mathrm{svd}([\mathbf{A}_\mu \mathbf{B}, \mathbf{L}_\mathrm{m}]); \\ \mathbf{L}_{m+1} &\leftarrow \mathbf{V}_k \mathbf{S}_k; \quad (\sigma_{k+1} < tol \cdot \sigma_1) \end{aligned}$$

Modified Low Rank Smith - Multishift

```
Bm = B;   L = [];
for j = 1:kshifts,
   mu = -u(j);
   rho = sqrt(2*mu);
   Bm = ((A - mu*eye(n))\Bm);
   L = [L rho*Bm];
   for j = 1:ksteps,
      Bm = (A + mu*eye(n))*((A - mu*eye(n))\Bm);
      L = [L rho*Bm];
   end
   Bm = (A + mu*eye(n))*Bm;
end
[V,S,Q] = svd(L,0);
L = V(:,1:k)*S(1:k,1:k);
```

Performance of Modified Low Rank Smith

$$\frac{\|\mathcal{P}_S - \mathcal{P}_{MS}\|}{\|\mathcal{P}_S\|} < tol, \quad \frac{\|\mathcal{Q}_S - \mathcal{Q}_{MS}\|}{\|\mathcal{Q}_S\|} < tol.$$

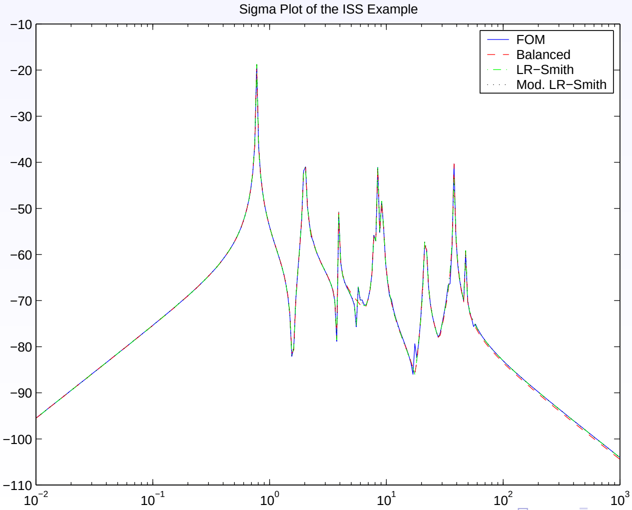
tol = SVD drop tol, S = LR Smith, MS = Modified LR Smith

Storage - Cols. $\mathbf{U}_k$, $\mathcal{P} \approx \mathbf{U}_k \mathbf{U}_k^T$

Prob.	n	LR Smith	Modified
CD120	120	70	25
ISS	270	210	106
Penzl	1006	300	19

ISS module comparison

k = 26 , n = 270



Approximate Power Method (Hodel)

$$\mathbf{A}\mathcal{P}\mathbf{U} + \mathcal{P}\mathbf{A}^T\mathbf{U} + \mathbf{B}\mathbf{B}^T\mathbf{U} = \mathbf{0}$$

$$\mathbf{A}\mathcal{P}\mathbf{U} + \mathcal{P}\mathbf{U}\mathbf{U}^T\mathbf{A}^T\mathbf{U} + \mathbf{B}\mathbf{B}^T\mathbf{U} + \mathcal{P}(\mathbf{I} - \mathbf{U}\mathbf{U}^T)\mathbf{A}^T\mathbf{U} = \mathbf{0}$$

Thus

$$\mathbf{A}\mathcal{P}\mathbf{U} + \mathcal{P}\mathbf{U}\mathbf{H}^T + \mathbf{B}\mathbf{B}^T\mathbf{U} \approx \mathbf{0} \quad \text{where} \quad \mathbf{H} = \mathbf{U}^T\mathbf{A}\mathbf{U}$$

Solving

$$\mathbf{A}\mathbf{Z} + \mathbf{Z}\mathbf{H}^T + \mathbf{B}\mathbf{B}^T\mathbf{U} = \mathbf{0}$$

gives approximation to

$$\mathbf{Z} \approx \mathcal{P}\mathbf{U}$$

Iterate $\Rightarrow$ *Approximate Power Method* $\mathbf{Z}_j \rightarrow \mathbf{U}\mathbf{S}$ with $\mathcal{P}\mathbf{U} = \mathbf{U}\mathbf{S}$

A Parameter Free Synthesis ($\mathcal{P} \approx \mathbf{US}^2\mathbf{U}^T$)

Step 1: (APM step) Solve a projected Sylvester equation

$$\mathbf{AZ} + \mathbf{ZH}^T + \mathbf{B}\hat{\mathbf{B}}^T = \mathbf{0}, \quad \text{with} \quad \mathbf{H} = \mathbf{U}_k{}^T\mathbf{AU}_k, \quad \hat{\mathbf{B}} = \mathbf{U}_k{}^T\mathbf{B}.$$

Step 2: Solve the reduced order Lyapunov equation

$$\text{Solve} \quad \mathbf{H}\hat{\mathcal{P}} + \hat{\mathcal{P}}\mathbf{H}^T + \hat{\mathbf{B}}\hat{\mathbf{B}}^T = \mathbf{0}.$$

Step 3: Modify $\mathbf{B}$

$$\text{Update} \quad \mathbf{B} \leftarrow (\mathbf{I} - \mathbf{Z}\hat{\mathcal{P}}^{-1}\mathbf{U}^T)\mathbf{B}.$$

Step 4: (ADI step) Update factorization and basis $\mathbf{U}_k$

$$\text{Re-scale} \quad \mathbf{Z} \leftarrow \mathbf{Z}\hat{\mathcal{P}}^{-1/2}.$$

Update (and truncate) $[\mathbf{U}, \mathbf{S}] \leftarrow svd[\mathbf{US}, \mathbf{Z}]$.

$\mathbf{U}_k \leftarrow \mathbf{U}(:, 1:k)$, basis for dominant subspace.

Monotone Convergence of $\mathcal{P}_j$

Recall $\mathcal{P}_{j+1} = \mathcal{P}_j + \mathbf{Z}_j \hat{\mathcal{P}}_j^{-1} \mathbf{Z}_j^T \succeq \mathcal{P}_j$.

Key Lemma

$$\mathbf{A}\mathcal{P}_j + \mathcal{P}_j\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{B}_j\mathbf{B}_j^T \quad \text{for} \quad j = 1, 2, \ldots$$

where $\mathbf{B}_{j+1} = (\mathbf{I} - \mathbf{Z}_j \hat{\mathcal{P}}_j^{-1} \mathbf{U}_j^T)\mathbf{B}_j$ with $\mathbf{B}_1 = \mathbf{B}$, $\mathcal{P}_1 = \mathbf{0}$

Montone Convergence!

$$\mathcal{P}_j \preceq \mathcal{P}_{j+1} \preceq \mathcal{P} \quad \text{for} \quad j = 1, 2, \ldots$$

This implies

$$\lim_{j \to \infty} \mathcal{P}_j = \mathcal{P}_o \preceq \mathcal{P}, \quad \text{monotonically}$$

Implementation Details

Note: $\mathbf{A}\mathcal{P}_j + \mathcal{P}_j\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{B}_j\mathbf{B}_j^T$ gives Residual Norm for Free!

Stopping Rules:

1. $\dfrac{\|\mathcal{P}_{j+1} - \mathcal{P}_j\|_2}{\|\mathcal{P}_{j+1}\|_2} = \dfrac{\|\mathbf{Z}_j\hat{\mathcal{P}}_j^{-1/2}\|_2^2}{\|\mathbf{S}_{j+1}\|_2^2} \le tol$
2. $\dfrac{\|\mathbf{B}_j\|_2}{\|\mathbf{B}\|_2} \le \sqrt{tol}$

Must monitor and control cond($\hat{\mathcal{P}}$):

Truncate SVD($\hat{\mathcal{P}}_j$) $\approx \mathbf{W}\hat{\mathbf{S}}^2\mathbf{W}^T$

$$1) \quad \mathbf{Z}_j \leftarrow \mathbf{Z}_j\mathbf{W}\hat{\mathbf{S}}^{-1} \quad \text{and} \quad 2) \quad \mathbf{B}_{j+1} = \mathbf{B}_j - \mathbf{Z}_j\hat{\mathbf{S}}^{-1}\mathbf{U}_j^T\mathbf{B}_j$$

Convergence Theory still valid

Solving the projected Sylvester equation:

Must solve:

$$\mathbf{A}\mathbf{Z} + \mathbf{Z}\mathbf{H}^T + \mathbf{B}\hat{\mathbf{B}}^T = \mathbf{0},$$

W.L.O.G. ... $\mathbf{H}^T = \mathbf{R} = (\rho_{ij})$ is in Schur form.

for j = 1:k,

  Solve $(\mathbf{A} - \rho_{jj}\mathbf{I})\mathbf{z}_j = \mathbf{B}\hat{\mathbf{B}}^T\mathbf{e}_j - \sum_{i=1}^{j-1} \mathbf{z}_i \rho_{i,j};$

end

Main Cost: Sparse Direct Factorization $\mathbf{L}\mathbf{U} = \mathbf{P}(\mathbf{A} - \rho_{jj}\mathbf{I})\mathbf{Q}$

Alternative: Note $\mathbf{Z} = \mathbf{X}\mathbf{Y}^{-1}$ where

$$\begin{bmatrix} \mathbf{A} & \mathbf{M} \\ \mathbf{0} & -\mathbf{H}^T \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} \hat{\mathbf{R}}$$

Low Rank Smith on Sylvester

Convert to Stein Equation:

$$\mathbf{A}\mathbf{Z} + \mathbf{Z}\mathbf{H}^T + \mathbf{B}\hat{\mathbf{B}}^T = 0 \quad \Longleftrightarrow \quad \mathbf{Z} = \mathbf{A}_\mu \mathbf{Z}\mathbf{H}_\mu^T + \mathbf{B}_\mu \hat{\mathbf{B}}_\mu^T,$$

where

$$\mathbf{A}_\mu = (\mathbf{A} + \mu\mathbf{I})(\mathbf{A} - \mu\mathbf{I})^{-1}, \quad \mathbf{B}_\mu = \sqrt{2|\mu|}(\mathbf{A} - \mu\mathbf{I})^{-1}\mathbf{B}.$$

$$\mathbf{H}_\mu = (\mathbf{H} + \mu\mathbf{I})(\mathbf{H} - \mu\mathbf{I})^{-1}, \quad \hat{\mathbf{B}}_\mu = \sqrt{2|\mu|}(\mathbf{H} - \mu\mathbf{I})^{-1}\hat{\mathbf{B}}.$$

Solution:

$$\mathbf{Z} = \sum_{j=0}^{\infty} \mathbf{A}_\mu^j \mathbf{B}_\mu \hat{\mathbf{B}}_\mu^T (\mathbf{H}_\mu^j)^T$$

Avantages L.R. Smith

Use Multi=Shift variant of:

$$\mathbf{Z} = \sum_{j=0}^{\infty} \mathbf{A}_\mu^j \mathbf{B}_\mu \hat{\mathbf{B}}_\mu^T (\mathbf{H}_\mu^j)^T$$

Note: Spectral Radius $\rho(\mathbf{A}_\mu) < 1$ regardless of μ.

- Choose shifts to minimize $\rho(\mathbf{H}_\mu)$
  We use Bagby ordering of eigenvalues of $\mathbf{H}$
  (cf: Levenberg and Reichel,93) .
- Monitor $\|\mathbf{H}_\mu^j \hat{\mathbf{B}}_\mu\|$ during iteration

Big Avantage:

Apply k shifts (roots of $\mathbf{H}$) with few factorizations $\mathbf{A} - \mu\mathbf{I}$.

Problem: SUPG discretization

Problem Description

function [A,b]=supg(N, theta, nu, delta)

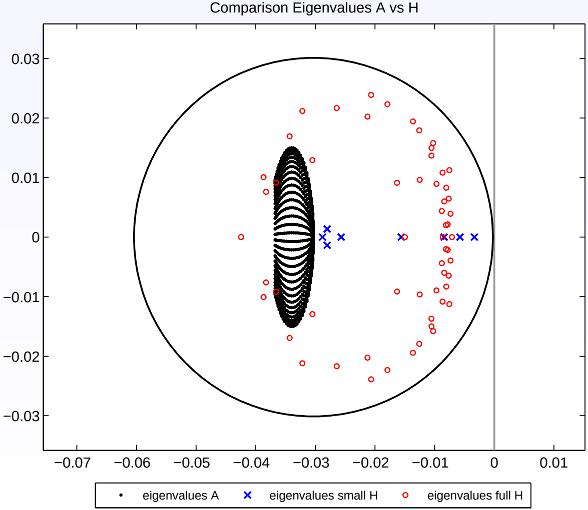
SUPG: matrix and RHS from SUPG discretization of the advection-diffusion operator on square grid of bilinear finite elements.

Mark Embree: Essentially distilled from
the IFISS software of Elman and Silvester.

- Fischer, Ramage, Silvester, Wathen: "Towards Parameter-Free Streamline Upwinding for Advection-Diffusion Problems" Comput. Methods Appl. Mech. Eng., 179:185-202, 1999.

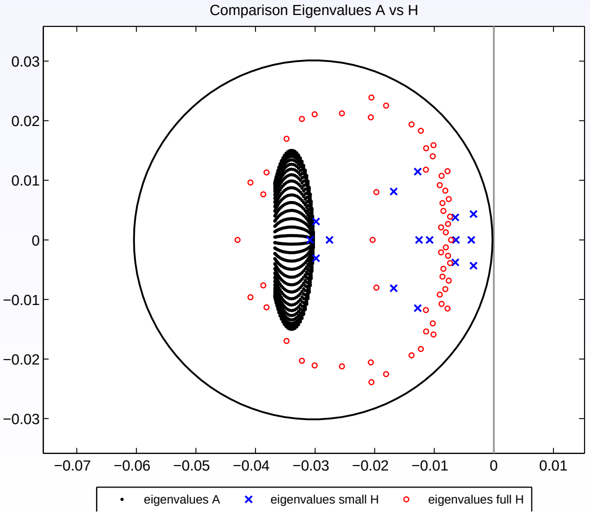
Automatic Shift Selection - Placement?

k = 8, m = 59 , n = 32*32, Thanks Embree



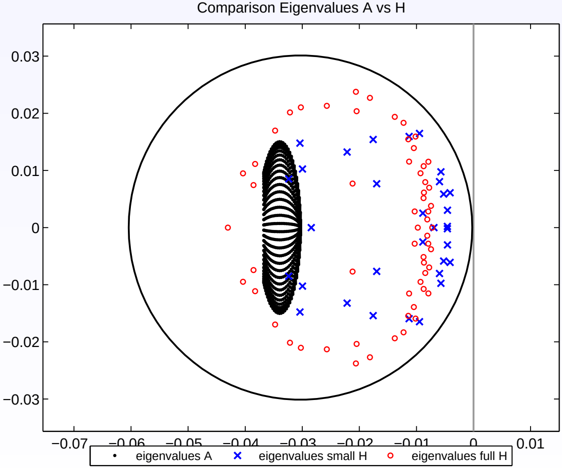
Automatic Shift Selection - Placement?

k = 16, m = 59 , n = 32*32,   Thanks Embree



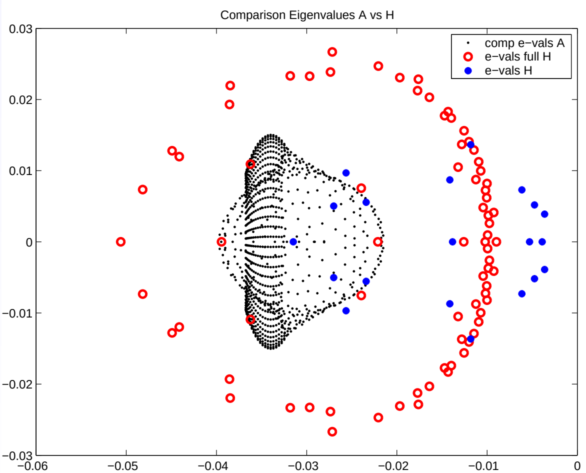
Automatic Shift Selection - Placement?

k = 32, m = 59 , n = 32*32, Thanks Embree

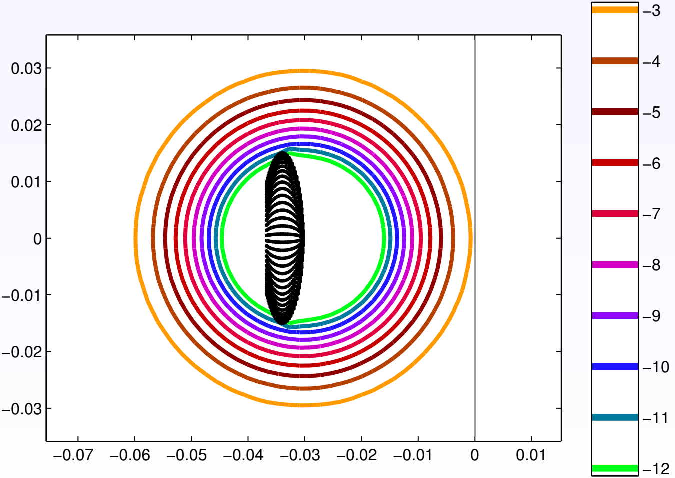


Automatic Shift Selection - Placement?

$k = 20,\ m = 76\ ,\ n = 32*32,$



ϵ-Pseudospectra for **A** from SUPG, n=32*32



Convergence History , Supg, $n = 32$, $N = 1024$

Laptop

Iter	$\frac{\|\mathcal{P}_+ - \mathcal{P}\|}{\|\mathcal{P}_+\|}$	$\|\mathbf{B}_j\|$	$\|\hat{\mathbf{B}}_j\|$
1	2.7e-1	1.6e+0	4.7e+0
2	7.2e-2	1.6e-1	1.5e+0
3	6.6e-3	1.1e-2	1.3e-1
4	3.7e-4	3.5e-7	1.3e-2
5	9.3e-9	1.4e-11	3.4e-7

$\mathcal{P}_f$ is rank $k = 59$ Comptime$(\mathcal{P}_f) = 16.2$ secs

$\frac{\|\mathcal{P} - \mathcal{P}_f\|}{\|\mathcal{P}\|} = 8.8e - 9$ Comptime$(\mathcal{P}) = 810$ secs

Convergence History , Supg, $n = 800$, $N = 640{,}000$

CaamPC

Iter	$\frac{\|\mathcal{P}_+ - \mathcal{P}\|}{\|\mathcal{P}_+\|}$	$\|\mathbf{B}_j\|$	$\|\hat{\mathbf{B}}_j\|$
6	1.3e-01	2.5e+00	2.4e+00
7	7.5e-02	1.1e+00	1.2e+00
8	3.5e-02	6.74e-01	5.0e-01
9	2.0e-02	1.2e-02	6.7e-01
10	2.0e-04	7.1e-07	1.2e-02
11	1.0e-08	2.3e-11	6.4e-07

$\mathcal{P}_f$ is rank $k = 120$ Comptime($\mathcal{P}_f$) $= 157$ mins $= 2.6$ hrs

Power Grid Delivery Network (thanks Kai Sun)

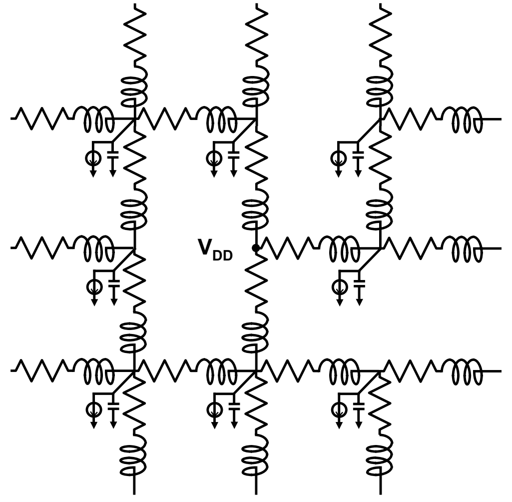
Modified Nodal Analysis (MNA) Description

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$

- $\mathbf{x}$ an $N-$vector of node voltages and inductor currents,
- $\mathbf{A}$ is an $N \times N$ conductance matrix,
- $\mathbf{E}$ (diagonal) represents capacitance and inductance terms,
- $\mathbf{u}(t)$ includes the loads and voltage sources.

The size N can be an order of millions for normal power grids.

RLC Circuit Diagram



Power Grid Results, $N = 1{,}048{,}340$

CaamPC

- $\mathcal{P}_f$ is rank k = 17   No Inductance
- Computational time = 77.52 min = 1.29 hrs.
- Residual Norm ≈ 9.5e-06

Sabino Code: Optimal shifts , much faster !!

- Symmetric problem $\mathbf{A} = \mathbf{A}^T$
- Modified Smith with Multi-shift strategy
- Optimal shifts

Power Grid Results, $N = 1821$, With Inductance

CaamPC

Problem is Non-Symmetric

In Sylvester

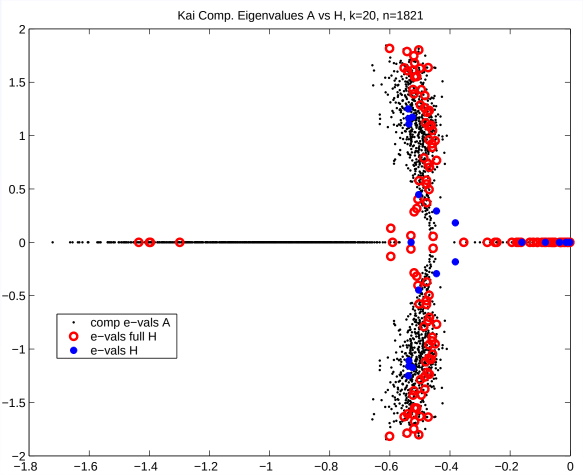
Used 4 real shifts, 30 iters per shift

$\mathcal{P}_f$ is rank $k = 121$ Comptime($\mathcal{P}_f$) $= 11.4$ secs

$\frac{\|\mathcal{P} - \mathcal{P}_f\|}{\|\mathcal{P}\|} = 2.5e - 6$ Comptime($\mathcal{P}$) $= 370$ secs

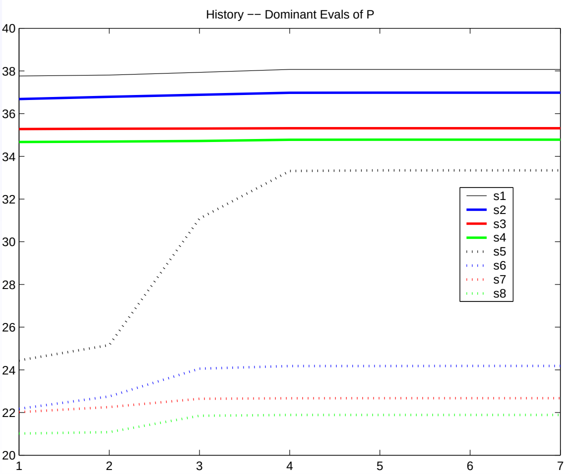
Automatic Shift Selection - Placement?

Power Grid, k = 20, m = 121 , N = 1821,



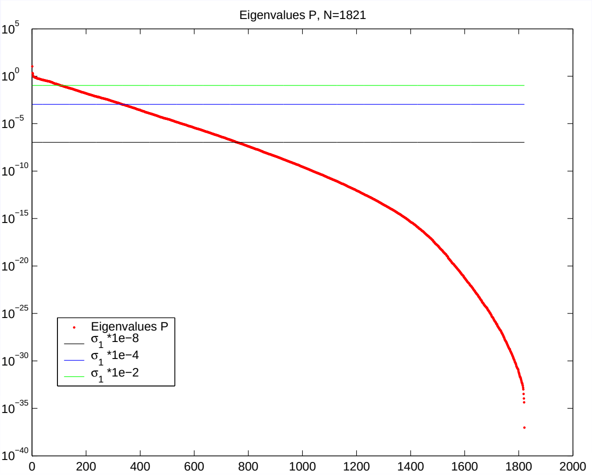
Convergence History– Evals of $\mathcal{P}_j$

Power Grid, N = 1821,



Decay Rate for Evals of $\mathcal{P}$

Power Grid, N $=$ 1821,



Convergence History , Power Grid, N = 48,892

CaamPC

Iter	$\frac{\|\mathcal{P}_+ - \mathcal{P}\|}{\|\mathcal{P}_+\|}$	$\|\mathbf{B}_j\|$	$\|\hat{\mathbf{B}}_j\|$
4	2.3e-01	8.5e+00	8.2e+01
5	5.6e-02	2.2e+00	8.2e+00
6	8.0e-02	1.1e+00	2.2e+00
7	6.4e-02	2.4e-03	1.1e+00
8	7.4e-05	3.0e-06	2.3e-03
9	5.8e-08	1.9e-08	2.4e-06

$\mathcal{P}_f$ is rank $k = 307$ Comptime($\mathcal{P}_f$) = 28 mins

At $N = 196K$ we run out of memory due to rank of $\mathcal{P}$.

Contact Info.

Shift Selection and Decay Rate Estimation
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My web page: www.caam.rice.edu/ ˜ sorensen/

Model Reduction: www.caam.rice.edu/ ˜ modelreduction/

ARPACK: www.caam.rice.edu/software/ARPACK/