Stokes' Theorem: Crib Sheet

First some review:
Definition 1: Suppose $U \subset \mathbf{R}^n$ is open, and $f : U \to \mathbf{R}$ is a function. Then the *support* of $f =$ supp $f =$ closure of $\{x \in U : f(x) \neq 0\}$.

Definition 2: Z a normed vector space, then $C_0^\infty(U, Z) = \{f \in C^\infty(U, Z) :$ supp f is compact$\}$

Definition 3: $\Omega_0^k(U) = C_0^\infty(U, \Lambda^k(\mathbf{R}^n))$, $0 \leq k \leq n$.

Definition 4: Suppose $\omega \in \Omega_0^n(U)$, $\omega = g dx_1 \wedge ... \wedge dx_n$, $g \in C_0^\infty(U, \mathbf{R})$. Then

$$\int_U \omega = \int_U g.$$

Remark: To make sense out of this, you must convince yourself that there is a bounded Jordan-measurable set C with supp $g \subset C \subset U$, and *define* the integral on the right to *mean* $\int_C g$. Then you must see that you get the same result no matter which Jordan measurable set C you pick that satisfies these conditions.

Recall that for any differentiable map $\phi : U \to \mathbf{R}^n$, open V with $\phi(U) \subset V$, and $\omega \in \Omega^n(V)$,

$$\phi^* \omega = (\omega \circ \phi) \det D\phi$$

Theorem 1: If ϕ is C^∞-invertible, $\det D\phi > 0$, and $\omega \in \Omega_0^n(V)$,

$$\int_U \phi^* \omega = \int_V \omega$$

Proof: In view of all the definitions above, this is merely a restatement of the Change of Variables Theorem.

Definition 5: $S \subset U$ is a *smooth surface* iff there exists a C^∞-invertible $\phi : U \to \mathbf{R}^n$ so that $S = \{x \in U : \phi_n(x) = 0\}$.

Example 1: Take $U = \mathbf{R}^n$, $S = S_n \equiv \{x : x_n = 0\}$, and $\phi =$ the identity map. This is the prototype smooth surface!

Nomenclature: ϕ as in the definition of smooth surface is a *chart* for S in U.

Definition 6: Suppose that ϕ is a chart for S in U, $x \in S$. Then

$$T_x^\phi S = \{v \in \mathbf{R}^n : (D\phi(x)v)_n = 0\}.$$

For $w \in \mathbf{R}^n$, write $w = (w', w_n)$, $w' \in \mathbf{R}^{n-1}$.

Lemma 1: $T_x^\phi S = \{D\phi^{-1}(\phi(x))(w', 0) : w' \in \mathbf{R}^{n-1}\}$ *Proof:* $D\phi(x)D\phi^{-1}(\phi(x))(w', 0) = (w', 0)$, so $\{D\phi^{-1}(\phi(x))(w', 0) : w' \in \mathbf{R}^{n-1}\} \subset T_x^\phi S$. On the other hand, if $v \in T_x^\phi S$, then $D\phi(x)v = (w', 0)$ for some $w' \in \mathbf{R}^{n-1}$, so $v = D\phi^{-1}(\phi(x))D\phi(x)v = D\phi^{-1}(\phi(x))(w', 0)$.

Proposition 1: Suppose that ϕ and ψ are charts for S in U. Then for every $x \in S$,

$$T_x^\phi S = T_x^\psi S$$

Definition 7: The *tangent space* $T_x S$ to S at x is the subset $T_x^\phi S$ for any chart ϕ for S in U.

Remark: You can identify the tangent space at x with the tangent vectors of smooth curves in S passing through x.

Definition 8: a couple of useful linear maps: $\Pi \in L(\mathbf{R}^n, \mathbf{R}^{n-1})$: $\Pi(w', w_n) = w'$. $\Pi^T \in L(\mathbf{R}^{n-1}, \mathbf{R}^n)$: $\Pi^T w' = (w', 0)$.

Lemma 2: If ϕ is a chart for S in U, then $\Pi(\phi(S)) = \Pi(S_n \cap \phi(U)) \subset \mathbf{R}^{n-1}$ is open.

Definition 9: Suppose that ϕ is a chart for S in U, and $\omega \in \Omega_0^{n-1}(U)$. Then

$$\int_S^\phi \omega = \int_{\Pi(\phi(S))} (\phi^{-1} \circ \Pi^T)^* \omega.$$

Remark: The integral on the right is the $n-1$-dimensional version of the n-form integral defined above, so makes sense. The next thing to do is to show that the definition is actually independent of the choice of chart, as was the definition of tangent space - these two facts are related. To see that every choice of chart yields the same definition of the integral requires comparing the pull-back forms $(\phi^{-1} \circ \Pi^T)^* \omega$ and $(\psi^{-1} \circ \Pi^T)^* \omega$ for two (possibly) different charts ϕ and ψ. It would be natural to compose the inverse of one map with the other, and in fact that's essentially what we will do, but since neither map is strictly speaking

invertible we have to be a bit more subtle. The idea is to examine the composition of the two pull-backs, with more or less the inverse of the one associated with ϕ in the middle:

$$(\psi^{-1} \circ \Pi^T)^*(\Pi \circ \phi)^*(\phi^{-1} \circ \Pi^T)^*\omega$$

You can associate this composition as

$$= (\Pi \circ \phi \circ \psi^{-1} \circ \Pi^T)^*(\phi^{-1} \circ \Pi^T)^*\omega \tag{1}$$

Lemma 3: $\Pi \circ \phi \circ \psi^{-1} \circ \Pi^T$ is a C^∞-invertible map: $\Pi(\psi(S)) \to \Pi(\phi(S))$ (these are open sets in $\mathbf{R}^{n-1}$). Furthermore, if $\det D\phi > 0$ and $\det D\psi > 0$ on U, then $\det D(\Pi \circ \phi \circ \psi^{-1} \circ \Pi^T) > 0$ on $\Pi(\psi(S))$.

This Lemma almost links $(\phi^{-1} \circ \Pi^T)^*\omega$ and $(\psi^{-1} \circ \Pi^T)^*\omega$ by pull-back by a C^∞-invertible map, which because of the C-of-V theorem would mean their integral would be the same - but not quite, because it's not ω that'e being pulled back. However it's just as good, because of the following two observations:

Proposition 2: Suppose that Suppose that ψ is a chart for S in U, and $\omega \in \Omega_0^{n-1}(U)$.Then $(\psi^{-1} \circ \Pi^T)^*\omega = 0$ iff for every $x \in S$, $(v_1, v_2, ...v_{n-1}) \in (T_xS)^{n-1}$, $\omega(x)(v_1, ..., v_{n-1}) = 0$.

Remark: That is, as far as integration over S is concerned, it's only the action of $n-1$-forms on the tangent space of S that matters. This leads naturally to a definition of forms *on* S, but we shall not have need of that abstraction - see Ch. 5 in Spivak if you want to know more about this.

Proposition 3: For $x \in S$, $(\omega - (\Pi \circ \phi)^*(\phi^{-1} \circ \Pi^T)^*\omega)(x)$ vanishes on all $(v_1, v_2, ...v_{n-1}) \in (T_xS)^{n-1}$.

Theorem 2: Suppose that ϕ and ψ are charts for S in U, and that $\det D\phi > 0, \det D\psi > 0$ in U. Then

$$\int_S^\phi \omega = \int_S^\psi \omega$$

for any $\omega \in \Omega_0^n(U)$.
Proof:

$$\int_S^\psi \omega = \int_{\Pi(\psi S)} (\psi^{-1} \circ \Pi^T)^*\omega.$$

$$= \int_{\Pi(\psi(S))} (\psi^{-1} \circ \Pi^T)^*(\omega - (\Pi \circ \phi)^*(\phi^{-1} \circ \Pi^T)^*\omega)$$

$$+\int_{\Pi(\psi(S))}(\Pi\circ\phi\circ\psi^{-1}\circ\Pi^T)^*(\phi^{-1}\circ\Pi^T)^*\omega$$

using equation 1. However the first integrand is the pull-back by $\psi^{-1}\circ\Pi^T$ of a form that vanishes on every T_xS, according to Proposition 3, and by Proposition 2 this pull-back vanishes - here is where we use the invariance of the definition of T_xS with respect to choice of chart (i.e. it didn't matter whether we used ϕ or ψ to define T_xS). That is, the first integral vanishes. The second is the integral of a pull-back by a C^∞-invertible map with positive determinant (Lemma 3), so the C-of-V theorem (Theorem 1) finishes the proof.

This Theorem justifies the

Definition 10: For $\omega\in\Omega_0^{n-1}(U)$, define

$$\int_S\omega=(-1)^{n-1}\int_{\Pi(\phi(S))}(\phi^{-1}\circ\Pi^T)^*\omega.$$

where ϕ is any chart for S in U with $\det D\phi>0$. [The sign factor is necessary - you will see why shortly.]

Example 2: Suppose the setup is as in Example 1, and write

$$\omega=\sum_{i=1}^n f_i dx_1\wedge...\widehat{dx_i}\wedge...dx_n$$

in which the hat indicates which factor in the wedge product to omit. Show that $(\phi^{-1}\Pi^T)^*\omega=0$ iff $f_n=0$. Then show that

$$\int_{S_n}\omega=(-1)^{n-1}\int_{\mathbf{R}^{n-1}}f_n(x_1,...,x_{n-1},0)dx_1\wedge....\wedge dx_{n-1} \tag{2}$$

The use of this definition is to supply boundary terms in the multidimensional analogue of the Fundamental Theorem of Calculus. The next definition gives the local analogue of integration over an interval:

Definition 11: Assume as before that $U\subset\mathbf{R}^n$ is open, $\phi\in C^\infty(U,\mathbf{R}^n)$ is C^∞-invertible, and denote $U_-^\phi=\{x\in U:\phi_n(x)\le 0\}$. For $\omega=gdx_1\wedge...\wedge dx_n\in\Omega_0^n(U)$, define

$$\int_{U_-^\phi}\omega=\int_U\chi_{U_-^\phi}g \tag{3}$$

Remark: Note that this definition differs from Definition 4, in that the coefficient is integrated over a subset. It's quite easy to see that the intersection of U_-^ϕ with a Jordan-measureable set, for instance one containing the support of ω, is itself Jordan-measureable,

so this defintion transfers to differential forms part of the concept of integration over Jordan-measureable sets.

In fact, the RHS of equation 3 is

$$= \int_U \chi_{U_-^\phi \cap C} g$$

in which $C \subset U$ is a compact Jordan-measureable set containing the support of ω, and according to the C-of-V theorem this in turn is equal to

$$= \int_{\phi(U)} (\chi_{U_-^\phi \cap C} g) \circ \phi^{-1} |\det D\phi^{-1}|$$

$$= \int_{\phi(U)} \chi_{\phi(U_-^\phi \cap C)} g \circ \phi^{-1} |\det D\phi^{-1}|.$$

Note that $\phi(U_-^\phi \cap C$ is Jordan-measureable (proof!) so this is the definition of

$$= \int_{\phi(U_-^\phi \cap C)} g \circ \phi^{-1} |\det D\phi^{-1}|$$

$$= \int_{\phi(U_-^\phi)} (\phi^{-1})^* \omega$$

(could drop the intersection with C, redundant in view of its containing the support of ω)) *provided* that $\det D\phi > 0$, so $\det D\phi^{-1} > 0$. Remembering what U_-^ϕ is, this is the same as

$$= \int_{x_n \leq 0} (\phi^{-1})^* \omega. \tag{4}$$

Definition 12: A subset $B \subset U$ of an open set U is *locally smoothly bounded* if there exists a C^∞-invertible $\phi \in C^\infty(U, \mathbf{R}^n)$ with positive Jacobian determinant, called a chart for B, so that $B = U_-^\phi$. In that case, $\partial B \cap U = S$, where S is the smooth surface defined by $\phi_n = 0$, and ϕ is also a chart for S.

Theorem 3 (*Local Stokes Theorem*): Suppose U is open, $B \subset U$ is locally smoothly bounded with boundary $S \subset U$, and $\omega \in \Omega_0^{n-1}(U)$. Then

$$\int_B d\omega = \int_S \omega. \tag{5}$$

Proof: Choose a chart ϕ for B and S. From equation 4 applied to $d\omega$,

$$\int_B d\omega = \int_{x_n \leq 0} (\phi^{-1})^*(d\omega) = \int_{x_n \leq 0} d((\phi^{-1})^* \omega)$$

Write

$$(\phi^{-1})^*\omega = \sum_{i=1}^{n} f_i dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n$$

Then

$$d(\phi^{-1})^*\omega = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} dx_i \wedge dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n$$

$$= \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} (-1)^{i-1} \wedge dx_1 \wedge ... \wedge dx_i \wedge ... \wedge dx_n$$

Note that for any $f \in C_0^\infty(\mathbf{R}^n)$,

$$\int_{x_n \leq 0} \frac{\partial f}{\partial x_i} = \begin{cases} 0, \ i < n, \\ \int_{\mathbf{R}^{n-1}} f(x_1, ..., x_{n-1}, 0) dx_1 ... dx_{n-1} \end{cases}$$

thanks to Fubini's Theorem. Thus

$$\int_B d\omega = (-1)^{n-1} \int_{\mathbf{R}^{n-1}} f_n(x_1, ..., x_{n-1}, 0) dx_1 ... dx_{n-1}$$

$$= (-1)^{n-1} \int_{\Pi(\phi(S))} (\Pi^T)^*(f_n dx_1 \wedge ... \wedge dx_{n-1}). \tag{6}$$

The same logic as was used in Exercise 2 shows that

$$(\Pi^T)^*(f_n dx_1 \wedge ... \wedge dx_{n-1}) = (\Pi^T)^*(\sum_{i=1}^{n} f_i dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n)$$

$$= (\Pi^T)^*(\phi^{-1})^*\omega = (\phi^{-1} \circ \Pi^T)^*\omega$$

Combine this observation with Definition 10 and equation 6 to finish the proof.

This theorem takes a very clean form, easy to remember, but not of the form of the classical integration by parts theorems. In fact, it is equivalent to the Divergence Theorem, which involves the *surface volume (or area) form* dS - not an exact differential, generally, despite the notation. This object in turn depends for its definition on the *outward unit normal* vector field $n_S \in C^\infty(U, \mathbf{R}^n)$, defined by

$$n_S = \frac{\nabla \phi_n}{|\nabla \phi_n|}.$$

for any choice of chart ϕ. Here and for the rest of the discussion $|\cdot|$ is the Euclidean (l^2) norm. Note that $|n_S| = 1$ and for $x \in S$, sufficiently small $t > 0$, $x + tn_S(x) \in U \setminus B$, that is, n_S points *out of* B. It is *normal* to S, that is, $n_S(x)$ is orthogonal to T_xS: this is just a restatement of the definition, since $v \in T_xS \Leftrightarrow$

$$\langle n_S(x), v \rangle = \frac{1}{|\nabla \phi_n(x)|} \langle \nabla \phi(x), v \rangle = \frac{1}{|\nabla \phi_n(x)|} D\phi_n(x) v = 0.$$

Since T_xS is an $(n-1)-$dimensional vector space, it has two orientations that is, two equivalence class of bases as defined in Spivak p. 82. The *usual orientation* consists of the bases $(v_1, ..., v_{n-1})$ of T_xS for which the basis $(v_1, ..., v_{n-1}, n_S(x))$ belongs to the usual orientation of $\mathbf{R}^n$, that is, the determinant of the $n \times n$ basis matrix is positive.

T_xS inherits an inner product from R^n, that is, the restriction of the Euclidean inner product. If $(v_1, ..., v_{n-1})$ is an orthonormal basis of T_xS belonging to the usual orientation, then $(v_1, ..., v_{n-1}, n_S(x))$ is an orthonormal basis of $\mathbf{R}^n$ belonging to the usual orientation.

Definition 12: $dS \in \Omega^{n-1}(U)$ is a volume form on S iff for every $x \in S$, and every orthonormal basis $(v_1, ..., v_{n-1})$ of T_xS belonging to the usual orientation, $dS(x)(v_1, ..., v_{n-1}) = 1$. That is, $dS(x)$ is a volume element on T_xS in the sense defined in Spivak p. 83. Spivak also shows that any subspace, provided with an orientation, has a volume element. If we extend the definition of T_xS away from S, defining it to be the null space of $D\phi_n(x)$ at each $x \in U$, then Spivak's definition provides a function $dS : U \mapsto \Lambda^{n-1}(\mathbf{R}^n)$. It will follow from the calculations below that dS is actually smooth, that is, $dS \in \Omega^{n-1}(U)$.

Theorem 4 (*Local Divergence Theorem*): Same hypotheses as Theorem 3. Suppose that $F \in C_0^\infty(U, \mathbf{R}^n)$. Then

$$\int_U \text{div } F = \int_S \langle n_S, F \rangle dS.$$

[Note that the RHS is an integral of a smooth function times an (n-1)-form over the surface S, so well-defined.]

Proof: For any such vector field, define (n-diml version of Spivak problem 4-19(a), done in class):

$$\omega_F = \sum_{i=1}^{n} (-1)^{i-1} F_i dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n$$

Then

$$d\omega_F = \text{ div } F dx_1 \wedge ... \wedge dx_n$$

so

$$\int_U \text{div } F = \int_U d\omega_F = \int_S \omega_F.$$

There are two keys to recognizing the RHS as the dot product of F with the outward unit normal, times the surface volume form. First, recall that many (n-1)-forms give the same integral over S: in particular, from Proposition 2, given a chart ϕ, there is a form $\eta \in \Omega^{n-1}(\Pi(\phi(S)))$ so that

$$\int_S \omega_F = \int (\Pi \circ \phi)^* \eta,$$

in fact $\eta = (\phi^{-1} \circ \Pi)^* \omega_F$ - but all we need to know is that there is such an η, which must take the form

$$\eta = g dy_1 \wedge ... \wedge dy_{n-1}.$$

Here $y_1, ..., y_{n-1}$ are coordinates on $\mathbf{R}^{n-1}$, identified as a subset of the coordinates $y_1, .., y^n$ on $\mathbf{R}^n$. From the definition of Π, $\Pi^* dy_i = dy_i$ (this is abuse of notation, hopefully transparent). Thus

$$(\Pi \circ \phi)^* \eta = g \circ (\Pi \circ \phi) d\phi_1 \wedge ... \wedge d\phi_{n-1}.$$

Here is a "doodle on the napkin" step: note that

$$(\Pi \circ \phi)^* \eta \wedge d\phi_n = g \circ (\Pi \circ \phi) d\phi_1 \wedge ... \wedge d\phi_n = g \circ (\Pi \circ \phi) \det D\phi dx_1 \wedge ... \wedge dx_n.$$

On the other hand, a short calculation shows that

$$\omega_F \wedge d\phi_n = (-1)^{n-1} \langle F, \nabla \phi_n \rangle dx_1 \wedge ... \wedge dx_n$$

Choose $x \in S$, an orthonormal basis $(v_1, ..., v_{n-1})$ of $T_x S$. Since $d\phi_n(x) v_i = 0, i = 1, ..., n-1$, the definition of wedge product (Spivak, p. 79) shows that

$$((\Pi \circ \phi)^* \eta \wedge d\phi_n)(x)(v_1, ..., v_{n-1}, n_S(x)) = (\Pi \circ \phi)^* \eta(x)(v_1, ..., v_{n-1}) d\phi_n(x)(n_S(x))$$
$$= (\Pi \circ \phi)^* \eta(x)(v_1, ..., v_{n-1}) |\nabla \phi_n(x)|$$
$$= g \circ (\Pi \circ \phi)(x)(d\phi_1 \wedge ... \wedge d\phi_{n-1} \wedge d\phi_n)(x)(v_1, ..., v_{n-1}, n_S(x)) = g \circ (\Pi \circ \phi)(x) \det D\phi(x)$$

Similarly,

$$(\omega_F \wedge d\phi_n)(x)(v_1, ..., v_{n-1}, n_S(x)) = \langle F, \nabla \phi_n(x) \rangle (dx_1 \wedge ... \wedge dx_n)(v_1, ..., v_{n-1}, n_S(x))$$
$$= \langle F(x), \nabla \phi_n(x) \rangle$$

(since $dx_1 \wedge ... \wedge dx_n$ is the usual volume form)

$$= \omega_F(x)(v_1, ..., v_{n-1}) |\nabla \phi_n(x)|$$

Since $\omega_F(x)(v_1, ..., v_{n-1}) = (\Pi \circ \phi)^* \eta(x)(v_1, ..., v_{n-1})$ from Proposition 2, it follows that

$$\langle F(x), \nabla \phi_n(x) \rangle = g \circ (\Pi \circ \phi)(x) \det D\phi(x)$$

so

$$(\Pi \circ \phi)^* \eta = g \circ (\Pi \circ \phi) d\phi_1 \wedge ... \wedge d\phi_{n-1} = \frac{\langle F, \nabla \phi_n \rangle}{\det D\phi} d\phi_1 \wedge ... \wedge d\phi_{n-1}$$
$$= \langle F, n_S \rangle \frac{|\nabla \phi_n|}{\det D\phi} d\phi_1 \wedge ... \wedge d\phi_{n-1}$$

The proof is finished by appeal to

Lemma 4:

$$dS = \frac{|\nabla \phi_n|}{\det D\phi} d\phi_1 \wedge ... \wedge d\phi_{n-1}$$

is the usual volume element on S.

Proof: Ortho-normalize the gradients by Gram-Schmidt starting with $\nabla \phi_n$, use the identity

$$d\phi_i = \omega_{\nabla \phi_i}$$

and exterior algebra to show that $dS(x)(v_1, ..., v_{n-1}) = 1$ for any orthonormal basis of $T_x S$ belonging to the usual orientation.